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A Quasilinear Parabolic System with Nonlocal Sources and Weighted Nonlocal Boundary Conditions

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Abstract In this paper, we investigate the blow-up properties of a quasilinear reaction-diffusion system with nonlocal nonlinear sources and weighted nonlocal Dirichlet boundary conditions. The critical exponent is determined under various situations of the weight functions. It is observed that the boundary weight functions play an important role in determining the blow-up conditions. In addition, the blow-up rate estimate of non-global solutions for a class of weight functions is also obtained, which is found to be independent of nonlinear diffusion parameters m and n.

Keywords quasilinear parabolic system; nonlocal boundary conditions; critical exponent; blow-up rate; weight functions.

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1. Introduction

In this paper, we study the following quasilinear parabolic system with coupled nonlocal sources and weighted nonlocal Dirichlet boundary conditions

$$\begin{cases} u_t = \Delta u^m + au^\alpha \displaystyle\int_\Omega v^p \mathrm{d}x,\ v_t = \Delta v^n + bv^\beta \displaystyle\int_\Omega u^q \mathrm{d}x, & (x,t) \in \Omega \times (0,T),\\ u = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y,\ v = \displaystyle\int_\Omega \psi(x,y)v(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $a,b,p,q > 0$, $\alpha,\beta \geq 0$, $m,n > 1$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The weight functions $\varphi(x,y), \psi(x,y)$ are continuous, nonnegative on $\partial\Omega \times \bar{\Omega}$ and satisfy $\int_\Omega \varphi(x,y)\mathrm{d}y, \int_\Omega \psi(x,y)\mathrm{d}y > 0$ on $\partial\Omega$. The initial data $u_0(x)$, $v_0(x) \in C^{2+\nu}$ with $0 < \nu < 1$ and satisfy

(H1) u_0, $v_0 > 0$ in Ω, $u_0 = (\int_\Omega \varphi(x,y)u_0\mathrm{d}y)$, $v_0 = (\int_\Omega \psi(x,y)v_0\mathrm{d}y)$ on $\partial\Omega$.

(H2) $\Delta u_0^m + au_0^\alpha \int_\Omega v_0^p\mathrm{d}x \geq 0$, $\Delta v_0^n + bv_0^\beta \int_\Omega u_0^q\mathrm{d}x \geq 0$ in Ω.

By the standard parabolic theory, there exist local nonnegative solutions to (1.1).

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Much effort has been contributed to the study of blow-up properties for nonlinear parabolic equations with nonlocal sources and homogeneous Dirichlet boundary conditions, see [1, 5, 6, 8–10, 12–15, 17] and references therein. In addition, there are some models equipped with nonlocal boundary conditions [3, 4].

Lin and Liu [11] considered the semilinear nonlocal problem with nonlocal boundary condition

$$\begin{cases} u_t - \Delta u = \displaystyle\int_\Omega g(u)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega K(x,y)u(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \bar{\Omega} \end{cases}$$

to investigate the local existence, the global and non-global existence, as well as the blow-up properties of solutions. Then, the coupled system case, i.e., $m = n = 1$ in (1.1), was studied in [16], where it was found that the boundary weight functions play substantial roles to determine whether the solutions are global or nonglobal. The scalar case of degenerate parabolic equation with nonlocal source and weighted nonlocal boundary condition was considered in [2].

This paper will extend the above results to the degenerate system (1.1). We will establish the critical exponents under various situations of the weight functions, as well as the blow-up rate of solutions for a class of weight functions.

2. Critical exponent

We deal with the critical exponents of (1.1) in this section. The discussion will be carried out via five cases with different combinations for $\int_\Omega \varphi(x,y)\mathrm{d}y$ and $\int_\Omega \psi(x,y)\mathrm{d}y$ being larger or smaller than one.

Theorem 2.1 *Assume that $\int_\Omega \varphi(x,y)\mathrm{d}y$, $\int_\Omega \psi(x,y)\mathrm{d}y > 1$ for all $x \in \partial\Omega$. If $pq > (1-\alpha)(1-\beta)$, or $\max\{\alpha,\beta\} > 1$, then the solutions of (1.1) blow up in finite time for any initial data.*

Proof Since $\int_\Omega \varphi(x,y)\mathrm{d}y$, $\int_\Omega \psi(x,y)\mathrm{d}y > 0$ on $\partial\Omega$, and the compatibility conditions $u_0 = \int_\Omega \varphi(x,y)u_0(y)\mathrm{d}y$, $v_0 = \int_\Omega \psi(x,y)v_0(y)\mathrm{d}y$ on $\partial\Omega$ with $u_0, v_0 > 0$ in Ω, we have u_0, $v_0 \geq \delta > 0$ on $\bar{\Omega}$.

Consider the ODE system

$$\begin{cases} w'(t) = a|\Omega|w^\alpha z^p, \ z'(t) = b|\Omega|w^q z^\beta, \\ w(0) = w_0, \ z(0) = z_0. \end{cases} \tag{2.1}$$

It is well known that the solution of (2.1) blows up in finite time whenever $pq > (1-\alpha)(1-\beta)$, or $\max\{\alpha,\beta\} > 1$. Choose $w_0 = z_0 = \delta$. Obviously, (w,z) is a subsolution of (1.1). □

Theorem 2.2 *Assume that $\int_\Omega \varphi(x,y)\mathrm{d}y > 1$, $\int_\Omega \psi(x,y)\mathrm{d}y \leq 1$ for all $x \in \partial\Omega$. If $\alpha > 1$, then the solutions of (1.1) blow up in finite time for any initial data.*

Proof Suppose $\alpha > 1$. Notice $u_0, v_0 \geq \delta$ on $\bar{\Omega}$. According to the assumption (H2) and the comparison principle, $u_t, v_t \ge 0$. Thus $u, v \ge \delta$ on $\bar\Omega \times [0,T)$. Then u satisfies

$$\begin{cases} u_t \ge \Delta u^m + a\delta^p|\Omega|u^\alpha & \text{in } \Omega\times(0,T),\\ u = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{on } \bar\Omega. \end{cases} \tag{2.2}$$

Let $s(t)$ solve the ODE problem

$$s'(t) = a\delta^p|\Omega|s^\alpha, \quad s(0)=\delta. \tag{2.3}$$

Since $\alpha > 1$, $s(t)$ blows up in finite time. On the other hand, obviously, $s(t)$ is a subsolution of (2.2) due to $\int_\Omega \varphi(x,y)\mathrm{d}y > 1$. $\square$

Theorem 2.3 *Assume that $\int_\Omega \varphi(x,y)\mathrm{d}y$, $\int_\Omega \psi(x,y)\mathrm{d}y < 1$ for all $x \in \partial\Omega$.*

(i) If $\alpha > m$ or $\beta > n$ or $pq > (m-\alpha)(n-\beta)$, then the solutions of (1.1) are global for small initial data, and non-global for large initial data.

(ii) If $\alpha < m$, $\beta < n$, with $pq < (m-\alpha)(n-\beta)$, then the solutions of (1.1) are global for any initial data.

(iii) If $\alpha < m$, $\beta < n$, with $pq = (m-\alpha)(n-\beta)$, then the solutions of (1.1) are globally bounded provided that a, b are small enough, and blow up in finite time provided that a, b are large enough.

Proof Let $\Phi(x)$, $\Psi(x)$, respectively, be the unique positive solutions of the linear elliptic problems

$$-\Delta\Phi = \varepsilon_1 \ \text{ in } \Omega, \quad \Phi = \int_\Omega \varphi(x,y)\mathrm{d}y \ \text{ on } \partial\Omega,$$

and

$$-\Delta\Psi = \varepsilon_2 \ \text{ in } \Omega, \quad \Psi = \int_\Omega \psi(x,y)\mathrm{d}y \ \text{ on } \partial\Omega.$$

Since $\int_\Omega \varphi(x,y)\mathrm{d}y$, $\int_\Omega \psi(x,y)\mathrm{d}y < 1$, we choose $\varepsilon_1, \varepsilon_2 > 0$ small enough such that $0 < \Phi(x) < 1$, $0 < \Psi(x) < 1$. Let

$$\max_{x\in\bar\Omega}\Phi(x) = \overline{K}_1, \ \min_{x\in\bar\Omega}\Phi(x) = \underline{K}_1, \ \max_{x\in\bar\Omega}\Psi(x) = \overline{K}_2, \ \min_{x\in\bar\Omega}\Psi(x) = \underline{K}_2.$$

(i) Define $\overline{u} = M^{l_1}\Phi^{\frac{1}{m}}$, $\overline{v} = M^{l_2}\Psi^{\frac{1}{n}}$, where l_1, l_2, $M > 0$ are to be determined. A direct computation shows

$$\Delta\overline{u}^m + a\overline{u}^\alpha\int_\Omega \overline{v}^p\mathrm{d}x \le -\varepsilon_1 M^{ml_1} + a|\Omega|M^{\alpha l_1 + pl_2}\overline{K}_1^{\frac{\alpha}{m}}\overline{K}_2^{\frac{p}{n}},$$

$$\Delta\overline{v}^n + b\overline{v}^\beta\int_\Omega \overline{u}^q\mathrm{d}x \le -\varepsilon_2 M^{nl_2} + b|\Omega|M^{\beta l_2 + ql_1}\overline{K}_2^{\frac{\beta}{n}}\overline{K}_1^{\frac{q}{m}}.$$

Since $\alpha > m$ or $\beta > n$ or $pq > (m-\alpha)(n-\beta)$, we can choose $l_1, l_2 > 0$ such that $\alpha l_1 + pl_2 > ml_1$, $ql_1 + \beta l_2 > nl_2$. Let

$$M = \min\Big\{(\varepsilon_1^{-1}a|\Omega|\overline{K}_1^{\frac{\alpha}{m}}\overline{K}_2^{\frac{p}{n}})^{\frac{1}{ml_1-\alpha l_1 - pl_2}}, \ (\varepsilon_2^{-1}b|\Omega|\overline{K}_2^{\frac{\beta}{n}}\overline{K}_1^{\frac{q}{m}})^{\frac{1}{nl_2 - ql_1 - \beta l_2}}\Big\}.$$

Then

$$0 = \overline{u}_t \ge \Delta\overline{u}^m + a\overline{u}^\alpha\int_\Omega \overline{v}^p\mathrm{d}x, \quad 0 = \overline{v}_t \ge \Delta\overline{v}^n + b\overline{v}^\beta\int_\Omega \overline{u}^q\mathrm{d}x$$

for $(x,t)\in\Omega\times\mathbb{R}^+$. Moreover,

$$\begin{aligned}\overline{u}(x,t)|_{x\in\partial\Omega}&=M^{l_1}\Phi^{\frac{1}{m}}|_{x\in\partial\Omega}=M^{l_1}\Big(\int_\Omega\varphi(x,y)\mathrm{d}y\Big)^{\frac{1}{m}}\ge M^{l_1}\int_\Omega\varphi(x,y)\mathrm{d}y\\&\ge M^{l_1}\int_\Omega\varphi(x,y)\Phi(y)^{\frac{1}{m}}\mathrm{d}y=\int_\Omega\varphi(x,y)\overline{u}(y,t)\mathrm{d}y,\end{aligned}$$

and similarly,

$$\bar{v}(x,t)\mid_{x\in\partial\Omega}>\int_\Omega\psi(x,y)\bar{v}(y,t)\mathrm{d}y.$$

Let u_0,v_0 be small enough such that $\overline{u}=M^{l_1}\Phi^{\frac{1}{m}}\ge u_0,\ \ \overline{v}=M^{l_2}\Psi^{\frac{1}{n}}\ge v_0$ to ensure $(\overline{u},\overline{v})$ is a positive bounded supersolution of (1.1).

On the other hand, let $(\underline{u},\underline{v})$ be the solution of

$$\begin{cases}u_t=\Delta u^m+au^\alpha\displaystyle\int_\Omega v^p\mathrm{d}x,\ v_t=\Delta v^n+bv^\beta\int_\Omega u^q\mathrm{d}x & \text{in } \Omega\times(0,T),\\ u(x,t)=v(x,t)=0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x) & \text{on } \bar{\Omega},\end{cases}\tag{2.4}$$

and obviously a subsolution of (1.1). It is known that $(\underline{u},\underline{v})$ blows up for large initial data if $pq>(m-\alpha)(n-\beta)$ or $\alpha>m$ or $\beta>n$ (see [17]).

(ii) Similarly to the arguments for (i), let

$$\overline{u}=M^{l_1}\Phi^{\frac{1}{m}},\ \ \overline{v}=M^{l_2}\Psi^{\frac{1}{n}},$$

where l_1, l_2, $M>0$ are to be determined. By $\alpha<m$, $\beta<n$ and $pq<(m-\alpha)(n-\beta)$, choose $l_1,l_2>0$ such that $\alpha l_1+pl_2<ml_1$, $ql_1+\beta l_2<nl_2$. Let

$$\begin{aligned}M=\max\Big\{&(\varepsilon_1^{-1}a|\Omega|\overline{K}_1^{\frac{\alpha}{m}}\overline{K}_2^{\frac{p}{n}})^{\frac{1}{ml_1-\alpha l_1-pl_2}},\ (\varepsilon_2^{-1}b|\Omega|\overline{K}_2^{\frac{\beta}{n}}\overline{K}_1^{\frac{q}{m}})^{\frac{1}{nl_2-ql_1-\beta l_2}},\\&(\underline{K}_1^{-\frac{1}{m}}\max_{\bar{\Omega}}u_0)^{\frac{1}{l_1}},\ (\underline{K}_2^{-\frac{1}{n}}\max_{\bar{\Omega}}v_0)^{\frac{1}{l_2}}\Big\}.\end{aligned}$$

We have

$$0=\overline{u}_t\ge\Delta\overline{u}^m+a\overline{u}^\alpha\int_\Omega\overline{v}^p\mathrm{d}x,\ \ 0=\overline{v}_t\ge\Delta\overline{v}^n+b\overline{v}^\beta\int_\Omega\overline{u}^q\mathrm{d}x$$

for $(x,t)\in\Omega\times\mathbb{R}^+$. So, $(\overline{u},\overline{v})$ is a bounded supersolution of (1.1) provided

$$\overline{u}(x,0)=M^{l_1}\Phi^{\frac{1}{m}}\ge u_0,\ \overline{v}(x,0)=M^{l_2}\Psi^{\frac{1}{n}}\ge v_0.$$

(iii) Let $\overline{u}=M^{l_1}\Phi^{\frac{1}{m}},\ \ \overline{v}=M^{l_2}\Psi^{\frac{1}{n}}$, where l_1, l_2, $M>0$ are to be determined. Since $\alpha<m$, $\beta<n$, $pq=(m-\alpha)(n-\beta)$, choose $l_1,l_2>0$ such that $\alpha l_1+pl_2=ml_1$, $ql_1+\beta l_2=nl_2$. Let a,b be small so that

$$a\le\varepsilon_1(|\Omega|\overline{K}_1^{\frac{\alpha}{m}}\overline{K}_2^{\frac{p}{n}})^{-1},\ b\le\varepsilon_2(|\Omega|\overline{K}_2^{\frac{\beta}{n}}\overline{K}_1^{\frac{q}{m}})^{-1}.$$

We have

$$0=\overline{u}_t\ge\Delta\overline{u}^m+a\overline{u}^\alpha\int_\Omega\overline{v}^p\mathrm{d}x,\ \ 0=\overline{v}_t\ge\Delta\overline{v}^n+b\overline{v}^\beta\int_\Omega\overline{u}^q\mathrm{d}x$$

for $(x,t)\in\Omega\times\mathbb{R}^+$. Now choose M large enough to satisfy

$$M^{l_1}\Phi^{\frac{1}{m}}\ge\|u_0(x)\|_\infty,\ M^{l_2}\Psi^{\frac{1}{n}}\ge\|v_0(x)\|_\infty\ \text{ on } \bar{\Omega}.$$

Then $(\bar{u}, \bar{v})$ is a time-independent supersolution of (1.1).

On the other hand, suppose that $h(t)$ solves the ODE problem

$$h'(t) = C_0 h^s(t), \quad h(0) = h_0,$$

with

$$C_0 = \min\Big\{ \Big(l_1 \overline{K}_1^{\frac{1}{m}}\Big)^{-1} \Big(a|\Omega| \underline{K}_1^{\frac{\alpha}{m}} \underline{K}_2^{\frac{p}{n}} - \varepsilon_1\Big),\ \Big(l_2 \overline{K}_2^{\frac{1}{n}}\Big)^{-1} \Big(b|\Omega| \underline{K}_2^{\frac{\beta}{n}} \underline{K}_1^{\frac{q}{m}} - \varepsilon_2\Big)\Big\},$$
$$s = \min\{(m-1)l_1 + 1, (n-1)l_2 + 1\},\ h_0 = \min\Big\{ (\delta \overline{k}_1^{\frac{-1}{m}})^{\frac{1}{l_1}}, (\delta \overline{k}_2^{\frac{-1}{n}})^{\frac{1}{l_2}} \Big\}.$$

Clearly, $h(t)$ blows up in finite time $T_1 > 0$, whenever $a > \varepsilon_1(|\Omega| \underline{K}_1^{\frac{\alpha}{m}} \underline{K}_2^{\frac{p}{n}})^{-1}$, $b > \varepsilon_2(|\Omega| \underline{K}_2^{\frac{\beta}{n}} \underline{K}_1^{\frac{q}{m}})^{-1}$. Set

$$\underline{u} = h^{l_1}(t) \Phi^{\frac{1}{m}}(x),\ \underline{v} = h^{l_2}(t) \Psi^{\frac{1}{n}}(x),$$

with l_1, $l_2 > 0$ satisfying $\alpha l_1 + p l_2 = m l_1$, $q l_1 + \beta l_2 = n l_2$. A direct computation shows

$$\begin{aligned} \Delta \underline{u}^m + a \underline{u}^\alpha \int_\Omega \underline{v}^p \mathrm{d}x &\geq \Big(-\varepsilon_1 + a|\Omega| \underline{K}_1^{\frac{\alpha}{m}} \underline{K}_2^{\frac{p}{n}} \Big) h^{m l_1}(t) \\ &\geq l_1 h^{l_1 - 1}(t) h'(t) \Phi(x)^{\frac{1}{m}} = \underline{u}_t, \end{aligned}$$

and similarly,

$$\Delta \underline{v}^n + b \underline{v}^\beta \int_\Omega \underline{u}^q \mathrm{d}x \geq \underline{v}_t.$$

Moreover,

$$u(x,0) = u_0(x) \geq h_0^{l_1} \Phi^{\frac{1}{m}}(x),\ v(x,0) = v_0(x) \geq h_0^{l_2} \Psi^{\frac{1}{n}}(x) \quad \text{on } \bar{\Omega}.$$

Obviously, $(\underline{u}, \underline{v})$ is a blow-up subsolution to (1.1). $\square$

Theorem 2.4 *Assume that* $\int_\Omega \varphi(x,y)\mathrm{d}y = \int_\Omega \psi(x,y)\mathrm{d}y = 1$ *for all* $x \in \partial\Omega$.

(i) If $pq > (1-\alpha)(1-\beta)$ *or* $\alpha > 1$ *or* $\beta > 1$, *then the solutions of (1.1) blow up in finite time for any initial data.*

(ii) If $pq \leq (1-\alpha)(1-\beta)$, $\alpha < 1$ *and* $\beta < 1$, *then the solutions of (1.1) are globally bounded.*

Proof Notice that $u_0, v_0 \geq \delta$ on $\bar{\Omega}$, and the solution (w, z) of (2.1) blows up for any initial data (w_0, z_0) for $pq > (1-\alpha)(1-\beta)$ or $\alpha > 1$ or $\beta > 1$. Moreover, (w, z) is a subsolution to (1.1) by letting (w_0, z_0) be so small that $w_0 \leq u_0,\ z_0 \leq v_0$. On the other hand, (w, z) is global bounded for any (w_0, z_0) if $pq \leq (1-\alpha)(1-\beta)$, $\alpha < 1$ and $\beta < 1$. Choose (w_0, z_0) so large that $w_0 \geq u_0,\ z_0 \geq v_0$, then (w, z) is a supersolution of (1.1). $\square$

Theorem 2.5 *Assume that* $\int_\Omega \varphi(x,y)\mathrm{d}y = 1,\ \int_\Omega \psi(x,y)\mathrm{d}y < 1$ *for all* $x \in \partial\Omega$.

(i) If $\alpha > 1$, *then the solutions of (1.1) blow up in finite time for any initial data.*

(ii) If $\alpha \leq 1$ *and* $\beta > n$ *or* $pq > (m-\alpha)(n-\beta)$, *then the solutions of (1.1) blow up in finite time for large initial data.*

Proof The proofs for the two cases are similar to those of Theorems 2.2 and 2.3 (i), respectively. We omit the detail. $\square$

3. Blow-up rate

To obtain the estimate, we introduce the following transformation

$$u^m = U, \quad v^n = \Big(\frac{m}{n}\Big)^{\frac{n}{n-1}} V, \quad \tau = mt.$$

Then (1.1) becomes

$$\begin{cases} U_\tau = U^{r_1}\Big(\Delta U + a_1 U^{\alpha_1}\displaystyle\int_\Omega V^{p_1}\mathrm{d}x\Big),\ V_\tau = V^{r_2}\Big(\Delta V + b_1 V^{\beta_1}\displaystyle\int_\Omega U^{q_1}\mathrm{d}x\Big) & \text{in } \Omega\times(0,T_1),\\ U(x,\tau) = \Big(\displaystyle\int_\Omega \varphi(x,y)U^{\frac{1}{m}}(y,\tau)\mathrm{d}y\Big)^m,\ V(x,\tau) = \Big(\displaystyle\int_\Omega \psi(x,y)V^{\frac{1}{n}}(y,\tau)\mathrm{d}y\Big)^n & \text{on } \partial\Omega\times(0,T_1),\\ U(x,0) = U_0(x),\ V(x,0) = V_0(x) & \text{on } \bar\Omega, \end{cases} \tag{3.1}$$

where

$$r_1 = 1-\frac{1}{m},\ r_2 = 1-\frac{1}{n},\ \alpha_1 = \frac{\alpha}{m},\ p_1 = \frac{p}{n},\ \beta_1 = \frac{\beta}{n},\ q_1 = \frac{q}{n}, \tag{3.2}$$

$$U_0 = u_0^m,\ V_0 = \Big(\frac{n}{m}\Big)^{\frac{n}{n-1}} v_0^n,\ a_1 = \Big(\frac{n}{m}\Big)^{\frac{p}{n-1}} a,\ b_1 = \Big(\frac{n}{m}\Big)^{\frac{\beta-n}{n-1}} b. \tag{3.3}$$

For convenience, a special algebraic characteristic system is introduced.

$$\begin{pmatrix} \alpha_1 + r_1 - 1 & p_1 \\ q_1 & \beta_1 + r_2 - 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2\end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.4}$$

namely

$$k_1 = \frac{1+p_1-r_2-\beta_1}{p_1q_1-(1-r_1-\alpha_1)(1-r_2-\beta_1)}, \quad k_2 = \frac{1+q_1-r_1-\alpha_1}{p_1q_1-(1-r_1-\alpha_1)(1-r_2-q_1)}.$$

Assumptions (H1)–(H2) become

(H3) $U_0, V_0 > 0$ in Ω, $U_0 = (\int_\Omega \varphi(x,y)U_0^{\frac{1}{m}}\mathrm{d}y)^m$, $V_0 = (\int_\Omega \psi(x,y)V_0^{\frac{1}{n}}\mathrm{d}y)^n$ on $\partial\Omega$.

(H4) $\Delta U_0 + a_1U_0^{\alpha_1}\int_\Omega V_0^{p_1}\mathrm{d}x$, $\Delta V_0 + b_1V_0^{\beta_1}\int_\Omega U_0^{q_1}\mathrm{d}x \ge 0$ in Ω.

We also need additional assumptions on the initial data U_0, V_0.

(H5) $\Delta U_0 + a_1U_0^{\alpha_1}\int_\Omega V_0^{p_1}\mathrm{d}x \ge \delta U_0^{\frac{1}{k_1}+1-r_1}$, $\Delta V_0 + b_1V_0^{\beta_1}\int_\Omega U_0^{q_1}\mathrm{d}x \ge \delta V_0^{\frac{1}{k_2}+1-r_2}$ in Ω,

with

$$\delta = \max\Big\{\delta_1,\ \delta_2,\ 2k_1\tilde C_0^{-\frac{1}{k_1(1+q_1-r_1-\alpha_1)}},\ 2k_2\tilde C_0^{-\frac{1}{k_2(1+p_1-r_2-\beta_1)}}\Big\},$$

$$\delta_1 = a_1(k_2r_1)^{-1}|\Omega|\Big(\tfrac{k_2(1+k_1)}{k_1(k_2p_1+1)}\Big)^{k_2p_1+1},\ \delta_2 = b_1(k_1r_2)^{-1}|\Omega|\Big(\tfrac{k_1(1+k_2)}{k_2(k_1q_1+1)}\Big)^{k_1q_1+1}$$

and $\tilde C_0$ defined by the sequel (3.6).

We will use parameters k_1 and k_2 to describe the blow-up rate for (3.1), which give the blow-up rate of u and v near the blow-up time immediately.

Theorem 3.1 *Under the assumptions (H3)–(H4), suppose* $p_1 > \max\{\beta_1 + r_2 - 1, 1\}$, $q_1 > \max\{\alpha_1 + r_1 - 1, 1\}$, *with* $\int_\Omega \varphi(x,y)\mathrm{d}y$, $\int_\Omega \psi(x,y)\mathrm{d}y \le 1$. *Let* (U,V) *be the solution of (3.1) with blow-up time* T_1. *Then*

$$C_3(T_1-\tau)^{-k_1} \le \max_{\bar\Omega} U(\cdot,\tau) \le C_1(T_1-\tau)^{-k_1},$$

$$C_4(T_1-\tau)^{-k_2}\leq \max_{\bar\Omega} V(\cdot,\tau)\leq C_2(T_1-\tau)^{-k_2}$$

with $C_i>0$ $(i=1,\dots,4)$ *independent of* t.

Proof Denote $M(\tau)=\max_{\bar\Omega}U(\cdot,\tau)$, $N(\tau)=\max_{\bar\Omega}V(\cdot,\tau)$. Then

$$M_\tau\leq a_1|\Omega|M^{r_1+\alpha_1}N^{p_1},\quad N_\tau\leq b_1|\Omega|N^{r_2+\beta_1}M^{q_1}.$$

By the Young inequality with $1+p_1-r_2-\beta_1, 1+q_1-r_1-\alpha_1>0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}(M^{1+q_1-r_1-\alpha_1}+N^{1+p_1-r_2-\beta_1})\\ &\quad\leq [a_1(1+q_1-r_1-\alpha_1)+b_1(1+p_1-r_2-\beta_1)]\,|\Omega|M^{q_1}N^{p_1}\\ &\quad\leq K_1(M^{1+q_1-r_1-\alpha_1}+N^{1+p_1-r_2-\beta_1})^{\frac{q_1(1+p_1-r_2-\beta_1)+p_1(1+q_1-r_1-\alpha_1)}{(1+p_1-r_2-\beta_1)(1+q_1-r_1-\alpha_1)}},\end{aligned}$$

where

$$K_1=[a_1(1+q_1-r_1-\alpha_1)+b_1(1+p_1-r_2-\beta_1)]|\Omega|K_0^{\frac{q_1(1+p_1-r_2-\beta_1)+p_1(1+q_1-r_1-\alpha_1)}{(1+p_1-r_2-\beta_1)(1+q_1-r_1-\alpha_1)}},$$

$$K_0=\max\Big\{\frac{q_1(1+p_1-r_2-\beta_1)}{q_1(1+p_1-r_2-\beta_1)+p_1(1+q_1-r_1-\alpha_1)},\frac{p_1(1+q_1-r_1-\alpha_1)}{q_1(1+p_1-r_2-\beta_1)+p_1(1+q_1-r_1-\alpha_1)}\Big\}.$$

Integrating the above inequality from τ to T_1, we obtain

$$M^{1+q_1-r_1-\alpha_1}+N^{1+p_1-r_2-\beta_1}\geq \tilde C_0\,(T_1-\tau)^{-\frac{(1+p_1-r_2-\beta_1)(1+q_2-r_1-\alpha_1)}{p_1q_1-(1-r_1-\alpha_1)(1-r_2-q_1)}} \tag{3.5}$$

with

$$\tilde C_0=\Big[\frac{p_1q_1-(1-r_1-\alpha_1)(1-r_2-\beta_1)}{(1+p_1-r_2-\beta_1)(1+q_1-r_1-\alpha_1)}K_1\Big]^{-\frac{(1+p_1-r_2-\beta_1)(1+q_1-r_1-\alpha_1)}{p_1q_1-(1-r_1-\alpha_1)(1-r_2-\beta_1)}}. \tag{3.6}$$

Set

$$J_1(x,\tau)=U_\tau-\delta U^{\frac{1}{k_1}+1},\quad J_2(x,\tau)=V_\tau-\delta V^{\frac{1}{k_2}+1}.$$

We know from (H4) that $U_\tau,V_\tau\geq 0$ on $\bar\Omega_T$. It follows from $p_2>\max\{\alpha_1+r_1-1,1\}$, $p_1>\max\{\beta_1+r_2-1,1\}$ that $k_1,k_2>0$. A direct computation shows

$$\begin{aligned}&J_{1\tau}-U^{r_1}\Delta J_1-2r_1\delta U^{\frac{1}{k_1}}J_1-a_1p_1U^{\alpha_1+r_1}\int_\Omega V^{p_1-1}J_2\mathrm{d}x\\ &\quad=r_1U^{-1}J_1^2+r_1\delta^2U^{\frac{2}{k_1}+1}+\delta\frac{k_1+1}{k_1^2}U^{r_1+\frac{1}{k_1}-1}|\nabla U|^2+a_1\delta p_1U^{\alpha_1+r_1}\int_\Omega V^{p_1+\frac{1}{k_2}}\mathrm{d}x+\\ &\qquad a_1\alpha_1U_\tau U^{\alpha_1+r_1-1}\int_\Omega V^{p_1}\mathrm{d}x-a_1\delta(\tfrac{1}{k_1}+1)U^{\alpha_1+r_1+\frac{1}{k_1}}\int_\Omega V^{p_1}\mathrm{d}x\\ &\quad\geq r_1\delta^2U^{\frac{2}{k_1}+1}+a_1\delta p_1U^{\alpha_1+r_1}\int_\Omega V^{p_1+\frac{1}{k_2}}\mathrm{d}x-a_1\delta(\tfrac{1}{k_1}+1)U^{\alpha_1+r_1+\frac{1}{k_1}}\int_\Omega V^{p_1}\mathrm{d}x.\end{aligned}$$

Since $\frac{1}{2+k_1(1-\alpha_1-r_1)}+\frac{p_1k_2}{1+p_1k_2}=1$, by the Hölder inequality and the Young inequality, we have

$$\begin{aligned}U^{\frac{1}{k_1}}\int_\Omega V^{p_1}\mathrm{d}x&\leq|\Omega|^{\frac{1}{1+p_1k_2}}U^{\frac{1}{k_1}}\Big(\int_\Omega V^{p_1+\frac{1}{k_2}}\mathrm{d}x\Big)^{\frac{p_1k_2}{1+p_1k_2}}\\ &\leq|\Omega|^{\frac{1}{1+p_1k_2}}\tfrac{p_1k_2}{1+p_1k_2}\theta^{-\frac{1+p_1k_2}{p_1k_2}}\int_\Omega V^{p_1+\frac{1}{k_2}}\mathrm{d}x+\\ &\qquad|\Omega|^{\frac{1}{1+p_1k_2}}\tfrac{1}{2+k_1(1-\alpha_1-r_1)}\big(\theta U^{\frac{1}{k_1}}\big)^{2+k_1(1-\alpha_1-r_1)},\end{aligned}$$

where $\theta=|\Omega|^{\frac{k_2p_1}{(1+p_1k_2)^2}}(\frac{k_2(1+k_1)}{k_1(1+p_1k_2)})^{\frac{p_1k_2}{1+p_1k_2}}$. From the above inequality, we have

$$\begin{aligned}&J_{1\tau}-U^{r_1}\Delta J_1-2r_1\delta U^{\frac{1}{k_1}}J_1-a_1p_1U^{\alpha_1+r_1}\int_\Omega V^{p_1-1}J_2\mathrm{d}x\\&\geq r_1\delta^2U^{\frac{2}{k_1}+1}-a_1\delta|\Omega|^{\frac{1}{1+p_1k_2}\frac{1+k_1}{k_1(2+k_1(1-\alpha_1-r_1))}}\theta^{2+k_1(1-\alpha_1-r_1)}U^{\frac{2}{k_1}+1}\\&\geq r_1\delta(\delta-\delta_1)U^{\frac{2}{k_1}+1}\geq 0,\end{aligned}$$

and similarly,

$$J_{2\tau}-V^{r_2}\Delta J_2-2r_2\delta V^{\frac{1}{k_2}}J_2-b_1q_1V^{\beta_1+r_2}\int_\Omega U^{q_1-1}J_1\mathrm{d}x\geq 0.$$

We have for $(x,\tau)\in\partial\Omega\times(0,T_1)$ that

$$\begin{aligned}J_1(x,\tau)&=U_\tau-\delta U^{\frac{1}{k_1}+1}\\&=\Big(\int_\Omega\varphi(x,y)u(y,\tau)\mathrm{d}y\Big)^{m-1}\Big(\int_\Omega\varphi(x,y)u_\tau(y,\tau)\mathrm{d}y-\delta\Big(\int_\Omega\varphi(x,y)u(y,\tau)\mathrm{d}y\Big)^{\frac{m}{k_1}+1}\Big).\end{aligned}$$

Since $U_\tau(x,\tau)=J_1(x,\tau)+\delta U^{\frac{1}{k_1}+1}$, we have

$$\begin{aligned}&\int_\Omega\varphi(x,y)u_\tau(y,\tau)\mathrm{d}y-\delta\Big(\int_\Omega\varphi(x,y)u(y,\tau)\mathrm{d}y\Big)^{\frac{m}{k_1}+1}\\&=\int_\Omega\varphi(x,y)U^{\frac{1-m}{m}}J_1(y,\tau)\mathrm{d}y+\delta\Big(\int_\Omega\varphi(x,y)U^{\frac{1}{m}(\frac{m}{k_1}+1)}(y,\tau)\mathrm{d}y-\Big(\int_\Omega\varphi(x,y)U^{\frac{1}{m}}(y,\tau)\mathrm{d}y\Big)^{\frac{m}{k_1}+1}\Big).\end{aligned}$$

Noticing that $0<F(x)=\int_\Omega\varphi(x,y)\mathrm{d}y\leq 1$ on $\in\partial\Omega$ with $\frac{m}{k_1}+1>1$, we can apply the Jensen inequality to the last integral in the above inequality to get

$$\begin{aligned}&\int_\Omega\varphi(x,y)U^{\frac{1}{m}(\frac{m}{k_1}+1)}(y,\tau)\mathrm{d}y-\Big(\int_\Omega\varphi(x,y)U^{\frac{1}{m}}(y,\tau)\mathrm{d}y\Big)^{\frac{m}{k_1}+1}\\&\geq F(x)\Big(\int_\Omega\varphi(x,y)U^{\frac{1}{m}}(y,\tau)\frac{\mathrm{d}y}{F(x)}\Big)^{\frac{m}{k_1}+1}-\Big(\int_\Omega\varphi(x,y)U^{\frac{1}{m}}(y,\tau)\mathrm{d}y\Big)^{\frac{m}{k_1}+1}\\&\geq 0,\end{aligned}$$

since $\frac{m}{k_1}+1>1$, $0<F(x)\leq 1$. Hence,

$$J_1(x,\tau)\geq\Big[\int_\Omega\varphi(x,y)U^{\frac{1}{m}}(y,\tau)\mathrm{d}y\Big]^{m-1}\int_\Omega\varphi(x,y)U^{\frac{1-m}{m}}(y,\tau)J_1(y,\tau)\mathrm{d}y$$

for $(x,\tau)\in\partial\Omega\times(0,T_1)$, and similarly,

$$J_2(x,\tau)\geq\Big[\int_\Omega\psi(x,y)V^{\frac{1}{n}}(y,\tau)\mathrm{d}y\Big]^{n-1}\int_\Omega\psi(x,y)V^{\frac{1-n}{n}}(y,\tau)J_2(y,\tau)\mathrm{d}y$$

on $\partial\Omega\times(0,T_1)$. Moreover, $J_1(x,0),J_2(x,0)\geq 0$ on $\bar\Omega$. By the comparison principle, we get $J_1,J_2\geq 0$ on $\bar\Omega\times(0,T_1)$, namely,

$$U_\tau\geq\delta U^{\frac{1}{k_1}+1},\quad V_\tau\geq\delta V^{\frac{1}{k_2}+1}\quad\text{on }\bar\Omega\times(0,T_1).$$

Integrating from τ to T_1, we conclude that

$$U(x,\tau)\leq C_1(T_1-\tau)^{-k_1},\quad V(x,\tau)\leq C_2(T_1-\tau)^{-k_2},\tag{3.7}$$

with $C_1=(\delta k_1^{-1})^{-k_1}$, $C_2=(\delta k_2^{-1})^{-k_2}$.

Combining (3.7) with (3.5) yields the lower bounds of the blow-up rate. The proof is completed. □

Finally, let us go back to consider the blow-up rate of (1.1). Introduce a new algebraic system

$$\begin{pmatrix} \alpha-1 & p \\ q & \beta-1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

namely

$$\eta_1 = \frac{p-\beta+1}{pq-(1-\alpha)(1-\beta)}, \quad \eta_2 = \frac{q-\alpha+1}{pq-(1-\alpha)(1-\beta)}.$$

From Theorem 3.1 with (3.2), (3.3), we obtain the blow-up rate theorem for (1.1) immediately.

Theorem 3.2 *Under the conditions of Theorem 3.1, let (u, v) be the solution of (1.1) with blow-up time T. Then there exist $C_i^* > 0$ $(i = 1, \dots, 4)$ such that*

$$C_3^*(T-t)^{-\eta_1} \le \max_{\bar\Omega} u(\cdot, t) \le C_1^*(T-t)^{-\eta_1},$$
$$C_4^*(T-t)^{-\eta_2} \le \max_{\bar\Omega} v(\cdot, t) \le C_2^*(T-t)^{-\eta_2}.$$

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