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Local Derivations of a Matrix Algebra over a Commutative Ring

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Abstract Let R be a commutative ring with identity, $N_n(R)$ the matrix algebra consisting of all $n \times n$ strictly upper triangular matrices over R. Several types of proper local derivations of $N_n(R)$ $(n \leq 4)$ are constructed, based on which all local derivations of $N_n(R)$ $(n \leq 4)$ are characterized when R is a domain.

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1. Introduction

Let R be a commutative ring with identity, n a positive integer. We denote by R^* the set of nonzero elements in R. By $M_n(R)$ we denote the set of all $n \times n$ matrices over R, and by R^n we mean the set of all $1 \times n$ matrices over R. We denote by $N_n(R)$ (resp., $D_n(R)$) the set consisting of all $n \times n$ strictly upper triangular matrices (resp., diagonal matrices) over R. Let E denote the $n \times n$ identity matrix and E_{ij} $(1 \leq i, j \leq n)$ denote the standard matrix unit whose (i, j)-entry is 1 and whose other entries are 0. By definition, an algebra over R (or simply an R-algebra), is a set $\mathcal{A}$ with a ring structure and an R-module structure that share the same addition operation with the additional property that $(rA)B = r(AB) = A(rB)$ for any $r \in R$ and $A, B \in \mathcal{A}$. Recall that an R-linear map $\phi: \mathcal{A} \to \mathcal{A}$ is called a derivation if $\phi(A_1A_2) = \phi(A_1)A_2 + A_1\phi(A_2)$ for any $A_1, A_2 \in \mathcal{A}$. An R-linear map $\phi: \mathcal{A} \to \mathcal{A}$ is said to be a local derivation if for every $A \in \mathcal{A}$ there exists a derivation ϕ_A, depending on A, such that $\phi(A) = \phi_A(A)$. It is natural that

$$\text{derivations} \Rightarrow \text{local derivations.}$$

Larson [1] initially considered local maps in his examination of reflexivity and interpolation for subspaces of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. The notion local derivation was originally introduced by Larson and Sourour. A proper local derivation (means a local derivation which fails to be a derivation) on an operator algebra was found by Crist in [3]. Kadison [4] constructed an example of an algebra which has proper local derivations. Other work on the description of

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the local derivations on operator algebras can be found in [5–10]. In these articles all local derivations are actually global derivations. Concerning reports on derivations of matrix algebras and those of classic Lie algebras we refer to [11–15].

Certain special maps on $N_n(R)$ have been studied by several authors. For example, Cao [16–18] characterized all its automorphisms and Lie automorphisms and Ou [11] determined all its Lie derivations. In this article, we consider the local derivations of the matrix algebra $N_n(R)$ when $2 \leq n \leq 4$. Although, as stated in [2], it is somewhat difficult to construct proper local maps for an algebra system, yet three types of proper local derivations on $N_4(R)$ are constructed by us (see Section 2). We organize this article as follows. In Section 2, six types of standard local derivations of $N_n(R)$ $(n \leq 4)$ are constructed, for R an arbitrary commutative ring with identity. In Section 3, we characterize any local derivations of $N_n(R)$ $(n \leq 4)$ when R is a domain. The idea is to decompose each derivation into the sum of those standard derivations. Thus we can express all local derivations of $N_n(R)$ $(n \leq 4)$ in an explicit form.

Remark 1.1 We are regretful for leaving the general case that $n \geq 5$ unsolved. We suffer from the difficulty in verifying whether the standard maps are local derivations.

2. Construction of standard local derivations of $N_n(R)$ $(n \leq 4)$

We now construct several types of standard local derivations on $N_n(R)$, which will be used to generate all local derivations when $n \leq 4$.

(1) Inner derivations

Let $X \in N_n(R)$. Then the map $\operatorname{ad} X : N_n(R) \to N_n(R)$, defined by $Y \mapsto XY - YX$, is a derivation of $N_n(R)$, called an inner derivation of $N_n(R)$ induced by X.

(2) Diagonal derivations

Let $H \in D_n(R)$. Then the map $\eta_H : N_n(R) \to N_n(R)$, defined by $Y \mapsto HY - YH$, is a derivation of $N_n(R)$, called a diagonal derivation of $N_n(R)$ induced by H.

(3) Central derivations

Assume that $n \geq 3$. For $\alpha = (c_2, c_3, \ldots, c_{n-2}) \in R^{n-3}$, the map $\mu_\alpha : N_n(R) \to N_n(R)$, defined by $\sum_{1 \leq i < j \leq n} a_{ij} E_{ij} \mapsto (\sum_{k=2}^{n-2} a_{k,k+1} c_k) E_{1n}$ is a derivation of $N_n(R)$, called a central derivation of $N_n(R)$ induced by α.

In [11] we have known these types of Lie derivations for $N_n(R)$ and a description for any Lie derivation of $N_n(R)$. Since any derivation on $N_n(R)$ is a Lie derivation on $N_n(R)$, and another two types of standard Lie derivation on $N_n(R)$ (defined in [11]) are not derivations of $N_n(R)$, we can easily get any derivation on $N_n(R)$.

Lemma 2.1 (following from the main theorem of [11]) *Let ρ be a derivation of $N_n(R)$.*

(i) When $n = 2$, then $\rho = \eta_H$ with $H \in D_n(R)$.

(ii) When $n = 3$, then $\rho = \eta_H + \operatorname{ad} X$ with $H \in D_n(R)$, $X \in N_n(R)$.

(iii) When $n > 3$, then $\rho = \eta_H + \operatorname{ad} X + \mu_\alpha$ with $H \in D_n(R)$, $\alpha \in R^{n-3}$, $X \in N_n(R)$.

(4) Extensible local derivations of $N_n(R)$ ($n = 3$ or $n = 4$).

Definition 2.2 *Let $v \in R$. If any nonzero $r \in R$ is a factor of v, that is $v = ra$ for $a \in R$, then v is called divisible in R. The set of all divisible elements in R is denoted by $V(R)$.*

Let $v \in V(R)$ be divisible. We define $\psi_v : N_n(R) \to N_n(R)$, by

$$\sum_{1 \le i < j \le n} a_{ij} E_{ij} \mapsto v a_{1n} E_{1n}.$$

Then ψ_v is an R-linear map of $N_n(R)$ to itself.

Lemma 2.3 *Suppose $n = 3$ or $n = 4$. Then ψ_v is a local derivation of $N_n(R)$. It is a derivation of $N_n(R)$ if and only if $v = 0$.*

Proof We only consider the case when $n = 4$. Since the proof is very easy, we here omit the proof when $n = 3$. For a given $A = \sum_{1 \le i < j \le 4} a_{ij} E_{ij} \in N_4(R)$, we intend to show that the action of ψ_v on A exactly agrees with that of a derivation on it. If $a_{12} \neq 0$, suppose that $v = a_{12} b_{12}$ for $b_{12} \in R$ (by assumption on v). Then $\psi_v(A) = [-\mathrm{ad}(b_{12} a_{14} E_{24})](A)$, as desired. Similarly, if $a_{34} \neq 0$, we suppose that $v = a_{34} b_{34}$ for $b_{34} \in R$, then $\psi_v(A) = [\mathrm{ad}(b_{34} a_{14} E_{13})](A)$. If $a_{23} \neq 0$, suppose that $v = a_{23} b_{23}$ for $b_{23} \in R$, then $\psi_v(A) = \mu_{b_{23} a_{14}}(A)$. Now we suppose that $a_{12} = a_{23} = a_{34} = 0$, then $\psi_v(A) = \eta_H(A)$, where $H = \mathrm{diag}\{v, 0, v, 0\}$. It has been shown that ψ_v is a local derivation of $N_4(R)$. If ψ_v is a derivation of $N_4(R)$, then ψ_v must be the zero map (note that ψ_v maps E_{12}, E_{23}, E_{34} to zero, respectively), which leads to $v = 0$. On the contrary, if $v = 0$, then ψ_v is the zero map and is naturally a derivation of $N_4(R)$. □

ψ_v is called an extensible local derivation of $N_n(R)$ induced by $v \in V(R)$.

(5) Contractible local derivations of $N_4(R)$

Definition 2.4 *Let $V(R)$ be as above. $s \in V(R)$ is said to be strongly divisible, if for any given $a_{12}, a_{23}, a_{34} \in R^*$ and $a_{13}, a_{24} \in R$, the system of linear equations*

$$\begin{cases} a_{23} x_{12} - a_{12} x_{23} = s a_{13}, \\ a_{34} x_{23} - a_{23} x_{34} = s a_{24}, \\ a_{24} x_{12} + a_{34} x_{13} - a_{13} x_{34} - a_{12} x_{24} + a_{23} x_{14} = 0, \end{cases} \tag{2.1}$$

on variables: $\{x_{12}, x_{23}, x_{34}, x_{13}, x_{24}, x_{14}\}$ has at least one solution in R^6. The set of all strongly divisible elements in R is denoted by $S(R)$.

Let $s \in S(R)$ be strongly divisible. We define $\theta_s : N_4(R) \to N_4(R)$, by

$$\sum_{1 \le i < j \le 4} a_{ij} E_{ij} \mapsto s a_{13} E_{13} + s a_{24} E_{24}.$$

Then θ_s is an R-linear map of $N_4(R)$ to itself.

Lemma 2.5 *θ_s is a local derivation of $N_4(R)$. It is a derivation of $N_4(R)$ if and only if $s = 0$.*

Proof For any given $A = \sum_{1 \le i < j \le 4} a_{ij} E_{ij} \in N_4(R)$, we intend to show that the action of θ_s on A exactly agrees with that of a derivation on it.

Case 1 $a_{23} = 0$.

In this case, if $a_{34} \neq 0$, assume that $s = a_{34}b_{34}$ (by assumption on s), then

$$\theta_s(A) = [-\mathrm{ad}(b_{34}a_{14}E_{13}) + \eta_H](A), \text{ where } H = \mathrm{diag}\{s, s, 0, 0\} \in D_4(R).$$

Similarly, if $a_{12} \neq 0$, assume that $s = a_{12}b_{12}$, then

$$\theta_s(A) = [\mathrm{ad}(b_{12}a_{14}E_{24}) + \eta_H](A), \text{ where } H = \mathrm{diag}\{s, s, 0, 0\} \in D_4(R).$$

If $a_{12} = a_{34} = 0$, then

$$\theta_s(A) = \eta_{H_1}(A), \text{ where } H_1 = \mathrm{diag}\{s, 2s, 0, s\} \in D_4(R).$$

Case 2 $a_{23} \neq 0$.

If $a_{12} = a_{34} = 0$, assume that $-2s = a_{23}b_{23}$ (note that $-2s \in V(R)$). Then

$$\theta_s(A) = [\mu_{a_{14}b_{23}} + \eta_H](A), \text{ where } H = \mathrm{diag}\{s, 0, 0, -s\} \in D_4(R).$$

If $a_{12} = 0$ but $a_{34} \neq 0$, assume that $s = a_{34}b_{34}$ and $-s = a_{23}b_{23}$. Then

$$\theta_s(A) = [\mu_{a_{14}b_{23}} + \eta_{H_1} + \mathrm{ad}(b_{34}a_{24}E_{23})](A), \text{ where } H_1 = \mathrm{diag}\{s, 0, 0, 0\} \in D_4(R).$$

If $a_{12} \neq 0$ but $a_{34} = 0$, assume that $s = a_{12}b_{12}$ and $-s = a_{23}b_{23}$. Then

$$\theta_s(A) = [\mu_{a_{14}b_{23}} + \eta_{H_2} - \mathrm{ad}(b_{12}a_{13}E_{23})](A), \text{ where } H_2 = \mathrm{diag}\{0, 0, 0, -s\} \in D_4(R).$$

If a_{12}, a_{23}, a_{34} are all nonzero, since Equation (2.1) has at least one solution in R^6. Say

$$\begin{cases} x_{12} = r_{12}; & x_{13} = r_{13}; \\ x_{23} = r_{23}; & x_{24} = r_{24}; \\ x_{34} = r_{34}; & x_{14} = r_{14} \end{cases}$$

is a solution. Set $Y = \sum_{1 \leq i < j \leq 4} r_{ij}E_{ij}$. Then one can verify that

$$\theta_s(A) = [\mu_{r_{14}} + \mathrm{ad}\, Y](A).$$

These show that θ_s is a local derivation of $N_4(R)$. If $s = 0$, then obviously $\theta_s = 0$. If $s \neq 0$, since

$$\theta_s(E_{12}E_{23}) = \theta_s(E_{13}) = sE_{13} \neq \theta_s(E_{12})E_{23} + E_{12}\theta_s(E_{23}) = 0,$$

we see that θ_s is not a derivation of $N_4(R)$. □

θ_s is called a contractible local derivation of $N_4(R)$ induced by $s \in S(R)$.

(6) Local central derivations of $N_4(R)$

Definition 2.6 *Let $w \in R$. If for any $a \in R^*$ there exist $b, c \in R$ such that $w = ab + c$ and $ac = 0$, then w is said to be generalized divisible.*

It is obvious that all such (generalized divisible) elements in R form an ideal of R. We denote it by $W(R)$. It is clear that $V(R)$ and $S(R)$ also are ideals of R and $S(R) \subseteq V(R) \subseteq W(R)$. Let $w_1, w_2 \in W(R)$ both be generalized divisible. Define $\phi_{w_1,w_2} : N_4(R) \to N_4(R)$, by

$$\sum_{1 \leq i < j \leq 4} a_{ij}E_{ij} \mapsto (w_1a_{13} + w_2a_{24})E_{14}.$$

Then ϕ_{w_1,w_2} is an R-linear map of $N_4(R)$ to itself.

Lemma 2.7 *ϕ_{w_1,w_2} is a local derivation of $N_4(R)$. It is a derivation of $N_4(R)$ if and only if $w_1 = w_2 = 0$.*

Proof We start by proving that $\phi_{w_1,0}$ is a local derivation. For a given $A = \sum_{1\le i<j\le 4} a_{ij}E_{ij} \in N_4(R)$, if $a_{23} = 0$, then the action of $\phi_{w_1,0}$ on A agrees with that of the inner derivation $-\mathrm{ad}(w_1E_{34})$. If $a_{23} \neq 0$, then by assumption on w, there exist $q, r \in R$ such that $w_1 = a_{23}q + r$ and $ra_{23} = 0$. Then it is not difficult to verify that

$$\phi_{w_1,0}(A) = [\mu_{a_{13}q} - \mathrm{ad}(rE_{34})](A).$$

So $\phi_{w_1,0}$ is a local derivation of $N_4(R)$. Similarly, ϕ_{0,w_2} is a local derivation of $N_4(R)$. Then so is ϕ_{w_1,w_2}, since $\phi_{w_1,w_2} = \phi_{w_1,0}, + \phi_{0,w_2}$. If $w_1 = w_2 = 0$, ϕ_{w_1,w_2} is the zero map. If w_1, w_2 are not both zero, say $w_1 \neq 0$. Since each of the generators $\{E_{12}, E_{23}, E_{34}\}$ of $N_4(R)$ is mapped to zero by ϕ_{w_1,w_2}, we see that ϕ_{w_1,w_2} fails to be a derivation of $N_4(R)$. Otherwise ϕ_{w_1,w_2} should map each element in $N_4(R)$ to zero, in contradiction with $\phi_{w_1,w_2}(E_{13}) = w_1E_{14}$. $\square$

ϕ_{w_1,w_2} is called a local central derivation of $N_4(R)$ induced by $(w_1, w_2) \in W(R) \bigoplus W(R)$.

3. Local derivations of $N_n(R)$

We start this section by giving several lemmas, then we make use of them to prove the main theorem.

Lemma 3.1 *Let ϕ be a local derivation of an R-algebra $\mathcal{A}$ to itself. If $A^2 = 0$, then $A\phi(A) + \phi(A)A = 0$.*

Proof If $A^2 = 0$, then

$$A\phi(A) + \phi(A)A = A\phi_A(A) + \phi_A(A)A = \phi_A(A^2) = \phi_A(0) = 0,$$

where ϕ_A is a derivation of $\mathcal{A}$ corresponding to A. $\square$

Let $P_{n-1}(R) = \sum_{j-i\ge 2} RE_{ij}$, $P_{n-2}(R) = \sum_{j-i\ge 3} RE_{ij}, \dots, P_3(R) = \sum_{j-i\ge n-2} RE_{ij} = RE_{1,n-1} + RE_{2,n} + RE_{1,n}$, $P_2(R) = \sum_{j-i\ge n-1} RE_{ij} = RE_{1,n}$. By definition of local derivations one easily sees that:

Lemma 3.2 *If ϕ is a local derivation of $P_n(R)$, then $P_{n-1}(R)$, $P_{n-2}(R), \dots, P_3(R)$, $P_2(R)$ all are stable under ϕ.*

Lemma 3.3 *Let ϕ be a local derivation of $N_n(R)$. Then there exists a derivation ρ of $N_n(R)$ such that $\rho + \phi$ maps each of the generators $\{E_{12}, E_{23}, \dots, E_{n-1,n}\}$ of $N_n(R)$ to zero.*

Proof For our purpose, we only need to prove that if ϕ maps $E_{01}, E_{12}, E_{23}, \dots, E_{k-1,k}$ $(1 \le k < n$, by E_{01} we mean 0) to zero, respectively, then we can choose a derivation ψ of $N_n(R)$ such that $\psi+\phi$ maps $E_{01}, E_{12}, E_{23}, \dots, E_{k-1,k}$ and $E_{k,k+1}$ to zero, respectively. Assume that $\phi(E_{i-1,i}) = 0$, $i = 1, 2, \dots, k$, and consider the action of ϕ on $E_{k,k+1}$. By definition of local derivations, the action of ϕ on $E_{k,k+1}$ agrees with that of a derivation on it. So there exist a diagonal matrix $H = \mathrm{diag}\{d_1, d_2, \dots, d_n\} \in D_n(R)$, an upper triangular matrix $X = \sum_{1\le i<j\le n} a_{ij}E_{ij}$ and

$\alpha = (c_2, c_3, \ldots, c_{n-2}) \in R^{n-3}$ such that

$$\phi(E_{k,k+1}) = [\mathrm{ad}\, X + \eta_H + \mu_\alpha](E_{k,k+1}), \text{ when } 1 < k < n-1;$$

$$\phi(E_{k,k+1}) = [\mathrm{ad}\, X + \eta_H](E_{k,k+1}), \text{ when } k = 1 \text{ or } n-1.$$

When the first case occurs,

$$\phi(E_{k,k+1}) = (d_k - d_{k+1})E_{k,k+1} + (XE_{k,k+1} - E_{k,k+1}X) + c_k E_{1n}.$$

Choose $\alpha_k = (0, 0, \ldots, 0, -c_k, 0, \ldots, 0) \in R^{n-3}$ with $-c_k$ in the $(k-1)$-th position. Set

$$H_k = \mathrm{diag}\{0, \ldots, 0, d_k - d_{k+1}, 0, \ldots, 0\}$$

with $d_k - d_{k+1}$ in the $(k+1)$-th position. Let $\psi_1 = \mu_{\alpha_k} + \eta_{H_k}$. Then $\psi_1 + \phi$, denoted by ϕ_1, maps $E_{k,k+1}$ to $XE_{k,k+1} - E_{k,k+1}X$, and $\phi_1(E_{i,i+1}) = 0$ for $i = 1, 2, \ldots, k-1$.

Rewrite X as a block matrix: $X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}$, where $X_1 \in N_k(R), X_3 \in N_{n-k}(R)$. Denote by $E^{(k)}_{ik}$ the $k \times k$ matrix unit; by $E^{(k)}$ the $k \times k$ identity matrix. Then

$$\phi_1(E_{k,k+1}) = XE_{k,k+1} - E_{k,k+1}X = \begin{pmatrix} X_1E^{(k)}_{kk} & 0 \\ 0 & 0 \end{pmatrix} E_{k,k+1} - E_{k,k+1} \begin{pmatrix} 0 & 0 \\ 0 & E^{(n-k)}_{11}X_3 \end{pmatrix}.$$

For $1 \le i \le k-2$, since $E_{i,i+1} + E_{k,k+1}$ and $E_{k,k+1}$ are square nilpotent, by Lemma 3.1 we have

$$\phi_1(E_{k,k+1})E_{i,i+1} + E_{i,i+1}\phi_1(E_{k,k+1}) = 0.$$

This shows that $a_{i+1,k} = 0$ for $i = 1, 2, \ldots, k-2$. Set $Y_1 = \sum_{i=1}^{k-1} a_{i,k}E^{(k)}_{i,k}$ and set $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & X_3 \end{pmatrix}$. It can be verified that

$$\phi_1(E_{k,k+1}) = \begin{pmatrix} Y_1E^{(k)}_{kk} & 0 \\ 0 & 0 \end{pmatrix} E_{k,k+1} - E_{k,k+1} \begin{pmatrix} 0 & 0 \\ 0 & E^{(n-k)}_{11}X_3 \end{pmatrix} = YE_{k,k+1} - E_{k,k+1}Y.$$

Let $\psi_2 = -\mathrm{ad}\, Y$. Then $\psi_2 + \phi_1$ maps $E_{k,k+1}$ to zero. Simultaneously, $\psi_2 + \phi_1$ maps $E_{12}, E_{23}, \ldots, E_{k-1,k}$ to zero, respectively (recall that $a_{i,k} = 0$ for $i = 2, 3, \ldots, k-1$). This means that if we choose $\psi = -\mathrm{ad}\, Y + \mu_{\alpha_k} + \eta_{H_k}$, then $\psi + \phi$ maps $E_{12}, E_{23}, \ldots, E_{k,k+1}$ to zero, respectively, as desired.

When the latter occurs, Y, H_k are selected as above, and let $\psi = \eta_{H_k} - \mathrm{ad}\, Y$. Then the assertion also holds. □

Lemma 3.4 *Let ϕ be a local derivation of $N_4(R)$. Suppose $\phi(E_{13}) = \sum_{1 \le i < j \le 4} a_{ij}E_{ij}$ and $\phi(E_{24}) = \sum_{1 \le i < j \le 4} b_{ij}E_{ij}$. If $\phi(E_{i,i+1}) = 0$ for $i = 1, 2, 3$, then*

(i) $a_{i,i+1} = a_{24} = b_{i,i+1} = b_{13} = 0$ for $i = 1, 2, 3$;

(ii) $a_{13} = b_{24}$, and a_{13} is divisible when R is a domain;

(iii) Both a_{14} and b_{14} are generalized divisible.

Proof By Lemma 3.2, we know that $a_{i,i+1} = b_{i,i+1} = 0$ for $i = 1, 2, 3$. Since $E_{12} + E_{13}$ and E_{13}

are square nilpotent, by Lemma 3.1 we have

$$E_{12}\phi(E_{13})+\phi(E_{13})E_{12}=0.$$

This results in $a_{24}=0$. Similarly, $b_{13}=0$. This completes (i).

Now we consider the action of ϕ on $E_{12}+E_{13}-E_{34}+E_{24}$. On the one hand, the result is $a_{13}E_{13}+b_{24}E_{24} \pmod{RE_{14}}$. On the other hand, this action agrees with a derivation of $N_4(R)$ on it. Thus by Lemma 2.1, there exist $D=\mathrm{diag}\{d_1,d_2,d_3,d_4\}\in D_4(R)$ and $X=\sum_{1\le i<j\le 4} r_{ij}E_{ij}\in N_4(R)$ such that

$$\begin{aligned}&\phi(E_{12}+E_{13}-E_{34}+E_{24})\\ &\quad=(D+X)(E_{12}+E_{13}-E_{34}+E_{24})-(E_{12}+E_{13}-E_{34}+E_{24})(D+X)\\ &\quad\equiv(d_1-d_2)E_{12}+(d_1-d_3-r_{23})E_{13}-(d_3-d_4)E_{34}+(d_2-d_4-r_{23})E_{24} \pmod{RE_{14}}.\end{aligned}$$

By comparing the two results, we have that $d_1=d_2$, $d_3=d_4$. Then we further get

$$a_{13}=d_1-d_3-r_{23}=d_2-d_4-r_{23}=b_{24}.$$

Now we go on proving that a_{13} is divisible. For any $a\in R^*$, consider the action of ϕ on $aE_{12}+aE_{23}+E_{34}+E_{13}$. On the one hand,

$$\phi(aE_{12}+aE_{23}+E_{34}+E_{13})\equiv a_{13}E_{13} \pmod{RE_{24}+RE_{14}}.$$

On the other hand, by Lemma 2.1, there exist

$$C=\mathrm{diag}\{c_1,c_2,c_3,c_4\}\in D_4(R),\quad Y=\sum_{1\le i<j\le 4} s_{ij}E_{ij}\in N_4(R)$$

such that

$$\begin{aligned}\phi(aE_{12}+aE_{23}+E_{34}+E_{13})\equiv&(C+Y)(aE_{12}+aE_{23}+E_{34}+E_{13})-\\ &(aE_{12}+aE_{23}+E_{34}+E_{13})(C+Y) \pmod{RE_{24}+RE_{14}}\\ \equiv&a(c_1-c_2)E_{12}+a(c_2-c_3)E_{23}+(c_3-c_4)E_{34}+\\ &(c_1-c_3+as_{12}-as_{23})E_{13}(\mathrm{mod}\, RE_{24}+RE_{14}).\end{aligned}$$

By comparing the two results, we have that $c_1=c_2=c_3=c_4$ and $a_{13}=a(s_{12}-s_{23})$. This means that any nonzero element a in R is a factor of a_{13}, forcing $a_{13}\in V(R)$. This completes (ii).

The left task of this lemma is to show that a_{14} and b_{14} are generalized divisible. For any $a\in R^*$, consider the action of ϕ on $E_{13}+aE_{23}$. On the one hand, the result is $a_{13}E_{13}+a_{14}E_{14}$. On the other hand, there exist certain $H=\mathrm{diag}\{h_1,h_2,h_3,h_4\}\in D_4(R)$, $c\in R$ and $Z=\sum_{1\le i<j\le 4} t_{ij}E_{ij}\in N_4(R)$ such that

$$\begin{aligned}\phi(E_{13}+aE_{23})&=(H+Z)(E_{13}+aE_{23})-(E_{13}+aE_{23})(H+Z)+acE_{14}\\ &=(h_1-h_3+at_{12})E_{13}+a(h_2-h_3)E_{23}-at_{34}E_{24}+(ac-t_{34})E_{14}.\end{aligned}$$

By comparing, we have that $at_{34}=0$ and $a_{14}=ac-t_{34}$. Set $q=c$ and $r=-t_{34}$, we see that $a_{14}=aq+r$ and $ar=0$. Therefore, a_{14} is generalized divisible. Similarly, $b_{14}\in W(R)$. This completes (iii). □

Lemma 3.5 *Let R be a domain and ϕ a local derivation of $N_4(R)$ satisfying $\phi(E_{i,i+1}) = 0$ for $i = 1, 2, 3$. If $\phi(E_{13}) = sE_{13}$, $\phi(E_{24}) = sE_{24}$ with $s \in R$, then s is strongly divisible.*

Proof By Lemma 3.4, we have known that s is divisible. We now only need to prove that, for any given $a_{12}, a_{23}, a_{34} \in R^*$ and $a_{13}, a_{24} \in R$, Equation (2.1) on variables: $\{x_{12}, x_{23}, x_{34}, x_{13}, x_{24}, x_{14}\}$ has at least one solution in R^6. For our purpose, we consider the action of ϕ on $A = \sum_{i=1}^3 a_{i,i+1}E_{i,i+1}$ $+a_{13}E_{13} + a_{24}E_{24}$. The result, by assumption on ϕ, is $sa_{13}E_{13} + sa_{24}E_{24}$. On the other hand, the action of ϕ on A agrees with that of a derivation on it, thus there exist $c \in R$, $X = \sum_{1\le i<j\le 4} u_{ij}E_{ij} \in N_4(R)$ and $H = \mathrm{diag}\{d_1, d_2, d_3, d_4\} \in D_4(R)$ such that $\phi(A) = (\eta_H + \mathrm{ad}\ X + \mu_c)(A)$. The result of this action should also be

$$\begin{aligned}\phi(A) = &\sum_{i=1}^3 a_{i,i+1}(d_i - d_{i+1})E_{i,i+1} + \\ &(d_1a_{13} - d_3a_{13} + a_{23}u_{12} - a_{12}u_{23})E_{13} + (d_2a_{24} - d_4a_{24} + a_{34}u_{23} - a_{23}u_{34})E_{24} + \\ &(a_{24}u_{12} + a_{34}u_{13} - a_{13}u_{34} - a_{12}u_{24} + a_{23}c)E_{14}.\end{aligned}$$

By comparing the two results, we firstly have that $d_1 = d_2 = d_3 = d_4$, and then we further get

$$\begin{cases} a_{23}u_{12} - a_{12}u_{23} = sa_{13}, \\ a_{34}u_{23} - a_{23}u_{34} = sa_{24}, \\ a_{24}u_{12} + a_{34}u_{13} - a_{13}u_{34} - a_{12}u_{24} + a_{23}c = 0. \end{cases}$$

This shows that Equation (2.1) has a solution

$$\begin{cases} x_{12} = u_{12}; \quad x_{13} = u_{13}; \\ x_{23} = u_{23}; \quad x_{24} = u_{24}; \\ x_{34} = u_{34}; \quad x_{14} = c, \end{cases}$$

which implies that $s \in S(R)$. □

The following is the main theorem of this article.

Theorem 3.6 *Let R be a domain and ϕ an R-linear map of $N_n(R)$ $(2 \le n \le 4)$ to itself. Then ϕ is a local derivation of $N_n(R)$ if and only if that*

(i) When $n = 2$, $\phi = \eta_H$;

(ii) When $n = 3$, $\phi = \mathrm{ad}\, X + \eta_H + \psi_v$;

(iii) When $n = 4$, $\phi = \mathrm{ad}\, X + \eta_H + \mu_c + \phi_{w_1,w_2} + \psi_v + \theta_s$,

where $\mathrm{ad}\, X$ is an inner derivation induced by $X \in N_n(R)$; η_H is a diagonal derivation induced by $H \in D_n(R)$; μ_c is a central derivation induced by $c \in R$; ψ_v is an extensible local derivation induced by $v \in V(R)$; ϕ_{w_1,w_2} is a local central derivation induced by $w_1, w_2 \in W(R)$ and θ_s is a contractible local derivation induced by $s \in S(R)$.

Proof The sufficiency is obvious by Section 2. For the necessity, we give the proof in three cases.

Case 1 $n = 2$.

It is clear that $\phi(E_{12}) = dE_{12}$ for $d \in R$. Then we see that $\phi = \eta_H$, where $H = \mathrm{diag}\{d, 0\}$.

Case 2 $n = 3$.

By Lemma 3.3, we can choose $H \in D_3(R), X \in N_3(R)$ such that $\eta_H + \mathrm{ad}\, X + \phi$ maps E_{12}, E_{23} to zero, respectively. Denote $\eta_H + \mathrm{ad}\, X + \phi$ by ϕ_1 and suppose $\phi_1(E_{13}) = vE_{13}$ (using Lemma 3.2). For any $a \in R^*$, consider the action of ϕ_1 on $A = aE_{12} + aE_{23} + E_{13}$, we have that

$$\phi_1(A) = vE_{13}.$$

On the other hand, the action of ϕ_1 on A agrees with that of a derivation of $N_3(R)$ on it. Thus there exist $D = \mathrm{diag}\{d_1, d_2, d_3\} \in D_3(R)$, $Y = \sum_{1 \le i < j \le 3} a_{ij} E_{ij}^{(3)} \in N_3(R)$ such that

$$\phi_1(A) = (\eta_D + \mathrm{ad}\ Y)(A) = a(d_1 - d_2)E_{12} + a(d_2 - d_3)E_{23} + (d_1 - d_3 + aa_{12} - aa_{23})E_{13},$$

By comparing, we see that $d_1 = d_2 = d_3$. We further get $v = a(a_{12} - a_{23})$. Therefore v is divisible. Using $v \in V(R)$, we construct the extensible local derivation ψ_v of $N_3(R)$. It is easy to see that ϕ_1 is exactly ψ_v. So $\phi = -\mathrm{ad}\, X - \eta_H + \psi_v$, as desired.

Case 3 $n = 4$.

By Lemma 3.3, we can choose $H \in D_4(R), X \in N_4(R)$ and $c \in R$ such that $\mu_c + \eta_H + \mathrm{ad}\, X + \phi$ maps E_{12}, E_{23}, E_{34} to zero, respectively. Denote $\mu_c + \eta_H + \mathrm{ad}\, X + \phi$ by ϕ_1. By Lemma 3.4, we may assume that $\phi_1(E_{13}) = sE_{13} + cE_{14}$ and $\phi_1(E_{24}) = sE_{24} + dE_{14}$, where $c, d \in W(R)$. Set $w_1 = -c, w_2 = -d$. Then w_1, w_2 also are generalized divisible. Using w_1, w_2, we construct the local central derivation ϕ_{w_1,w_2} of $N_4(R)$. Denote $\phi_{w_1,w_2} + \phi_1$ by ϕ_2. Then one can verify that $\phi_2(E_{13}) = sE_{13}$ and $\phi_2(E_{24}) = sE_{24}$. Suppose that $\phi_2(E_{14}) = vE_{14}$. As in case 2 we can prove that $v \in V(R)$ (the similar process is omitted). Using $v \in V(R)$, we construct the extensible local derivation ψ_v of $N_4(R)$, and denote $-\psi_v + \phi_2$ by ϕ_3. Then $\phi_3(E_{i,i+1}) = 0$ for $i = 1, 2, 3$, $\phi_3(E_{14}) = 0$ and ϕ_3 maps E_{13} to sE_{13}, maps E_{24} to sE_{24}, respectively. By Lemma 3.5, we know that s is strongly divisible. We use s to construct the contractible local derivation θ_s of $N_4(R)$. It is easy to check that ϕ_3 is exactly θ_s. In the end we obtain

$$\phi = -\mathrm{ad}\ X - \eta_H - \mu_c - \phi_{w_1,w_2} + \psi_v + \theta_s.$$

This completes the proof. □

Remark 3.7 It is easy to see that the decomposition of a local derivation ϕ on $N_n(R)$ ($n \le 4$) into the sum of those standard ones (as in Theorem 3.6) is unique. In Theorem 3.6, R is assumed to be a domain. We conjecture that Theorem 3.6 also holds when this assumption is removed.

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