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Maps Preserving Zero Lie Brackets on a Maximal Nilpotent Subalgebra of the Symplectic Algebra

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Abstract Let F be a field with char $F \neq 2$, l a maximal nilpotent subalgebra of the symplectic algebra $\mathrm{sp}(2m, F)$. In this paper, we characterize linear maps of l which preserve zero Lie brackets in both directions. It is shown that for $m \geq 4$, a map φ of l preserves zero Lie brackets in both directions if and only if $\varphi = \psi_c \sigma_{T_0} \lambda_\alpha \phi_d \eta_f$, where $\psi_c, \sigma_{T_0}, \lambda_\alpha, \phi_d, \eta_f$ are the standard maps preserving zero Lie brackets in both directions.

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1. Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem (LPP). The earliest paper on such a problem dates back to 1897 (see [1]), and a great deal of effort has been devoted to the study of this type of questions since then. One may consult the survey papers [2–4] for details. Linear preserver problem mainly includes the following three types. The first type of this question is concerned with the study of those linear maps preserving certain functions [5–7]. The second type is concerned with the study of linear maps which preserve certain subsets [8–11]. The third type is concerned with the study of linear maps preserving certain relations [12–22].

It is one of the linear preserver problems to classify commutativity preserving linear maps on matrix spaces or algebras. A linear map φ on an algebra or a matrix space $\mathcal{A}$ is said to be commutativity preserving in both directions when the condition $ab = ba$ holds if and only if

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$\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$. Commutativity preserving linear maps on spaces of matrices or operators have been considered by several authors [12–18]. There are several motivations to study this kind of maps. Problems concerning commutativity preserving maps are closely related to the study of Lie homomorphisms. Every algebra $\mathcal{A}$ becomes a Lie algebra if we introduce the Lie bracket $[a, b]$ by $[a, b] = ab - ba$ for $a, b \in \mathcal{A}$. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called Lie homomorphism if $\phi([a, b]) = [\phi(a), \phi(b)]$ for every pair $a, b \in \mathcal{A}$. It is clear that every Lie homomorphism preserves commutativity. The assumption of preserving commutativity can be reformulated as the assumption of preserving zero Lie brackets. Let $\mathcal{L}$ be a Lie algebra over a field, φ a linear map of $\mathcal{L}$. We say that φ preserves zero Lie brackets in both directions if for every pair $x, y \in \mathcal{L}$, we have $[x, y] = 0$ if and only if $[\varphi(x), \varphi(y)] = 0$. In this paper, we obtain three types of linear maps which preserve zero Lie brackets in both directions, but fail to preserve all Lie brackets.

Let F be a field with char F$\neq 2$ and F^* the group consisting of all non-zero elements of F. Let $F^{m\times n}$ denote the set of all $m \times n$ matrices over F, $E^{(n)}$ the $n \times n$ identity matrix ($E^{(m)}$ is abbreviated to E), $\mathrm{gl}(n, F)$ the general linear Lie algebra consisting of all $n \times n$ matrices over F with bracket: $[X, Y] = XY - YX$ for $X, Y \in \mathrm{gl}(n, F)$. For $A \in F^{n\times n}$, A' denotes the transpose of A. Let $T(n, F)$ (resp., $S(n, F)$) be the subalgebra of $\mathrm{gl}(n, F)$ consisting of all upper triangular (resp., strictly upper triangular) matrices, $T^*(n, F)$ the group consisting of all invertible elements in $T(n, F)$. Set $I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. The symplectic algebra $\mathrm{sp}(2m, F)$ is defined to be the subalgebra of $\mathrm{gl}(2m, F)$ consisting of all $X \in \mathrm{gl}(2m, R)$ satisfying $X'I = -IX$. The conditions for $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $(A, B, C, D \in F^{m\times m})$ to be symplectic are that $B' = B$, $C' = C$ and $D' = -A$. Let

$$l = \left\{ \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} | A \in S(m, F), B \in F^{m\times m}, B' = B \right\}.$$

It is a maximal nilpotent subalgebra of $\mathrm{sp}(2m, F)$. In this paper, by using the main theorem of [15], we shall describe all the linear maps of l which preserve zero Lie brackets in both directions for $m \geq 4$. The main idea of this paper is to reduce the problem on l to that on $S(m, F)$.

2. Preliminaries

For $1 \leq i \leq j \leq m$, let E_{ij} denote the $2m \times 2m$ matrix whose (i, j)-entry is 1 and all other entries are 0; $E_{i,-j}$ the $2m \times 2m$ matrix whose $(i, j + m)$-entry is 1 and all other entries are 0; $E_{-j,-i}$ the $2m \times 2m$ matrix whose $(j + m, i + m)$-entry is 1 and all other entries are 0. For $a \in F$, $1 \leq i < j \leq m$, set

$$T_{ij}(a) = a(E_{ij} - E_{-j,-i}), \quad T_{ij} = \{T_{ij}(a) | a \in F\};$$

$$T_{i,-j}(a) = a(E_{i,-j} + E_{j,-i}), \quad T_{i,-j} = \{T_{i,-j}(a) | a \in F\}.$$

For $1 \leq i \leq m$, set

$$T_{ii}(a) = a(E_{ii} - E_{-i,-i}), \quad T_{ii} = \{T_{ii}(a) | a \in F\};$$

$$T_{i,-i}(a) = aE_{i,-i}, \quad T_{i,-i} = \{T_{i,-i}(a) | a \in F\}.$$

Let $v = \{\begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \mid A \in S(m,F)\}$, $w = \{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in F^{m\times m}, B' = B\}$. Then $l = v + w$. The center of v is T_{1m} and the center of l is $T_{1,-1}$.

Denote by $\mathcal{T}$ the set of all linear maps of l that preserve zero Lie brackets in both directions and by $\mathcal{T}'$ the set of all bijections in $\mathcal{T}$. Denote by 1 the identity map on l. It is clear that for $\varphi \in \mathcal{T}$ and a linear function f from l to F, the map $\varphi + f : X \mapsto \varphi(X) + f(X)T_{1,-1}(1)$ is in $\mathcal{T}$.

Lemma 2.1

(i) If $\varphi \in \mathcal{T}$, then $\mathrm{Ker}\varphi \subseteq T_{1,-1}$.

(ii) $\varphi \in \mathcal{T}'$ if and only if $\varphi(T_{1,-1}(1)) \neq 0$.

(iii) If $\varphi \in \mathcal{T}'$, then $\varphi(T_{1,-1}(1)) = T_{1,-1}(c)$ for some $c \in F^*$.

Proof (i) If $X \in l$ such that $\varphi(X) = 0$, then for any $Y \in l$ we have $[\varphi(X), \varphi(Y)] = 0$. So $[X,Y] = 0$, i.e., X is in the center $T_{1,-1}$ of l.

(ii) Clearly, if φ is bijective, we have $\varphi(T_{1,-1}(1)) \neq 0$. Conversely, if $\varphi(T_{1,-1}(1)) \neq 0$ and φ is not bijective, then there exists some non-zero $X \in l$ such that $\varphi(X) = 0$. By (i) we have $X = T_{1,-1}(c)$ with some $c \in F^*$. It follows that $\varphi(T_{1,-1}(1)) = 0$, a contradiction.

(iii) The assertion follows from the fact that $T_{1,-1}$ is the center of l and (ii). □

3. Standard maps of v

It is obvious that v is isomorphic to $S(m,F)$. Cao et al. [15] have described the linear maps preserving commutativity in both directions on $S(m,F)$. We now transfer them to v for later use. v has the following standard maps that preserve zero Lie brackets in both directions.

(a) $\psi_{v,c} : X \mapsto cX$, where c is a constant in F^*.

(b) $\sigma_{v,P} : X \mapsto P^{-1}XP$, where $P = \begin{pmatrix} A & 0 \\ 0 & A'^{-1} \end{pmatrix}$ with $A \in T^*(m,F)$.

(c) $\eta_{v,f} : X \mapsto X + f(X)T_{1m}(1)$, where $f : v \to F$ is a linear function from l to F.

(d) $\omega = 1$ or ω: $X = \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \mapsto \begin{pmatrix} -RA'R & 0 \\ 0 & RAR \end{pmatrix}$ where $R = E_{1m} + E_{2,m-1} + \cdots + E_{m-1,2} + E_{m1}$.

(e) $\mu_b^{(ij)}$ for $b \in F$, $i = 1, m$ and $j = 1, 2$, are defined by

$$\mu_b^{(11)} : X = \Sigma_{1\le i<j\le m} T_{ij}(a_{ij}) \mapsto X + T_{2m}(ba_{12});$$
$$\mu_b^{(m1)} : X = \Sigma_{1\le i<j\le m} T_{ij}(a_{ij}) \mapsto X + T_{1,m-1}(ba_{m-1,m});$$
$$\mu_b^{(12)} : X = \Sigma_{1\le i<j\le m} T_{ij}(a_{ij}) \mapsto X + T_{2m}(ba_{13}) + T_{3m}(ba_{12});$$
$$\mu_b^{(m2)} : X = \Sigma_{1\le i<j\le m} T_{ij}(a_{ij}) \mapsto X + T_{1,m-2}(ba_{m-1,m}) + T_{1,m-1}(ba_{m-2,m}).$$

We call the linear maps of types (a)–(e) defined above standard maps. By Lemmas 2.2, 2.3 and Theorem 1.1 (see [15]), we have the following theorem.

Theorem 3.1 *Let $m \geq 4$. Then a linear map φ of v preserves commutativity in both directions* *if and only if* φ *is of the form*

$$\varphi=\psi_{v,c}\sigma_{v,T}\omega\mu_{b_4}^{(m2)}\mu_{b_3}^{(12)}\mu_{b_2}^{(m1)}\mu_{b_1}^{(11)}\eta_{v,f},$$

where $\psi_{v,c}$, $\sigma_{v,T}$, ω, $\mu_{b_4}^{(m2)}$, $\mu_{b_3}^{(12)}$, $\mu_{b_2}^{(m1)}$, $\mu_{b_1}^{(11)}$, $\eta_{v,f}$ *are the standard maps of* v.

4. Standard maps of l

We now define some standard maps of l which preserve zero Lie brackets in both directions, then we use them to prove the main theorem of this paper. It is easy to check that the following linear maps of l are all in $\mathcal{T}$ when $m\ge 4$.

(i) $\psi_c: X\mapsto cX$, where c is a constant in F^*.

(ii) $\sigma_T: X\mapsto T^{-1}XT$, where $T=\begin{pmatrix} A & AB\\ 0 & A'^{-1}\end{pmatrix}$ with $A\in T^*(m,F)$ and $B=B'$.

(iii) $\eta_f: X\mapsto X+f(X)T_{1,-1}(1)$, where f is a linear function from l to F.

(iv) Let $\alpha=(a_1\ a_2\ a_3)\in F^{1\times 3}$. $\lambda_\alpha: X=\sum_{1\le i<j\le m}T_{ij}(a_{ij})+\sum_{1\le k\le l\le m}T_{k,-l}(b_{kl})\mapsto X+T_{2,-2}(a_1a_{12}+a_2a_{13}+a_3a_{23})+T_{2,-3}(a_2a_{12})$.

(v) $\phi_d: X=\begin{pmatrix} A & B\\ 0 & -A'\end{pmatrix}\mapsto\begin{pmatrix} A & dB\\ 0 & -A'\end{pmatrix}$ for $d\in F^*$.

It is clear that σ_T is a Lie automorphism of l, which is called the *inner automorphism* induced by T. If f is a linear function satisfying the additional conditions: $f([X,Y])=0$ for any $X,Y\in l$ and $1+f(T_{1,-1}(1))\ne 0$, then η_f is also a Lie automorphism of l, called the center automorphism. If $d=r^2$ for some $r\in F^*$, then ϕ_d is the inner automorphism of l induced by $\begin{pmatrix} r^{-1}E & 0\\ 0 & rE\end{pmatrix}$. If $d\notin(F^*)^2$, then ϕ_d is also an automorphism but not an inner automorphism. ψ_c and λ_α are all nonsingular linear maps preserving zero Lie brackets in both directions, but generally they are neither Lie automorphisms nor Lie anti-automorphisms.

5. The main results and their proofs

Throughout this section, we assume without loss of generality that φ is bijective and $m\ge 4$. In fact, if φ is not bijective, we have $\varphi(T_{1,-1}(1))=0$ by Lemma 2.1 (ii). Let f be a linear function from l to F such that $f(T_{1,-1}(1))\ne 0$. Then $\varphi+f\in\mathcal{T}'$ again by Lemma 2.1 (ii). Thus φ can be replaced with $\varphi+f$. For $s\subseteq l$, we denote by $C(s)$ the centralizer of s in l, i.e., $C(s)=\{Y\in l\mid [X,Y]=0,\forall X\in s\}$. In order to prove the main result in this paper, we need to give some lemmas first.

Lemma 5.1 *Let* $\varphi\in\mathcal{T}'$. *Then* w *defined in Section 2 leaves stable under* φ.

Proof Let $p=\sum_{\substack{1\le i\le j\le m\\ i+j\le m+1}}T_{i,-j}$. If we can prove that p leaves stable under φ, then w, being the centralizer of p in l, also leaves stable under φ. So for our goal, it suffices to prove that p is invariant under φ. It is clear that the set

$$\mathcal{B}=\{T_{i,-j}(1)\mid 1\le i\le j\le m, i+j\le m+1\}$$

is the canonical basis of p. So we only need to show that $\varphi(X) \in p$ for any $X \in \mathcal{B}$. It is not difficult to check that $\dim C(X) \geq \frac{1}{2}m(m+1) + \frac{1}{2}m(m-1) - (m-1)$ for any $X \in \mathcal{B}$. Since φ is bijective and preserves zero Lie brackets in both directions, we have $\dim C(\varphi(X)) = \dim C(X)$, so

$$\dim C(\varphi(X)) \geq \frac{1}{2}m(m+1) + \frac{1}{2}m(m-1) - (m-1) \text{ for any } X \in \mathcal{B}. \tag{1}$$

In the following, we first prove that for any $X \in \mathcal{B}$, $\varphi(X)$ must be in w.

If there exists some $X \in \mathcal{B}$ such that $\varphi(X) \notin w$, then we can assume that $\varphi(X) = \sum_{1 \leq i < j \leq m} T_{ij}(a_{ij}) + W$ with some $a_{st} \neq 0$ for $1 \leq s < t \leq m$ and $W \in w$. Let i_0, j_0 be such that $a_{i_0 j_0} \neq 0$ and $a_{i_0,k} = 0$ for all $k < j_0$ and $a_{k,j_0} = 0$ for all $k > i_0$. Set

$$M_1 = E^{(2m)} - \sum_{k=1}^{m-j_0} T_{j_0,j_0+k}(a_{i_0 j_0}^{-1} a_{i_0,j_0+k}),$$

$$M_2 = E^{(2m)} + \sum_{k=1}^{i_0-1} T_{k,i_0}(a_{i_0 j_0}^{-1} a_{k,j_0}),$$

$$v_0 = \sum_{k=1}^{m-j_0} T_{j_0,j_0+k} + \sum_{l=1}^{j_0} T_{l,-j_0} + \sum_{h=1}^{m-j_0} T_{j_0,-(j_0+h)}.$$

Then v_0 is a subspace of l, and $v_0 \cap C(\sigma_{M_2}\sigma_{M_1}\varphi(X)) = \{0\}$. Obviously, $l \supseteq v_0 \oplus C(\sigma_{M_2}\sigma_{M_1}\varphi(X))$ and $\dim v_0 \geq m$. So

$$\dim C(\varphi(X)) = \dim C(\sigma_{M_2}\sigma_{M_1}\varphi(X)) \leq \frac{1}{2}m(m+1) + \frac{1}{2}m(m-1) - m. \tag{2}$$

This contradicts (1). So $\varphi(X) \in w$ for any $X \in \mathcal{B}$. That is to say $\varphi(p) \subseteq w$.

It is easy to see $C(p) = w$, $C(w) = w$. It is not difficult to check that $\varphi(C(p)) = C(\varphi(p))$. So

$$\varphi(w) = \varphi(C(p)) = C(\varphi(p)) \supseteq C(w) = w.$$

Since φ is a bijective linear map and preserves zero Lie brackets in both directions, we have $\varphi(w) = w$. $\square$

Let $\varphi \in \mathcal{T}'$. Since w is stable under φ, φ induces a linear map $\overline{\varphi}$ of l/w by $\overline{\varphi}(\overline{Y}) = \overline{\varphi(Y)}$ for $Y \in l$. Now we prove that $\overline{\varphi}$ is a bijective linear map and preserves zero Lie brackets in both directions by the following lemma.

Lemma 5.2 *If $\varphi \in \mathcal{T}'$, then $\overline{\varphi}$ defined above is a bijective linear map and preserves zero Lie brackets in both directions.*

Proof It is clear that $\overline{\varphi}$ is a linear map. By Lemma 5.1, we have $\varphi(w) = w$. If $\overline{\varphi}(\overline{X}) = \overline{0}$, i.e., $\varphi(X) \in w$, then $X \in w$. So $\overline{X} = \overline{0}$. This implies that $\overline{\varphi}$ is bijective. For any $X = \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} \in l$ and $Y = \begin{pmatrix} C & D \\ 0 & -C' \end{pmatrix} \in l$, let $X_1 = \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix}$, $W_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, $Y_1 = \begin{pmatrix} C & 0 \\ 0 & -C' \end{pmatrix}$, $W_2 = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$. If $[\overline{X}, \overline{Y}] = \overline{0}$, i.e., $[X, Y] \in w$, then $[X_1, Y_1] = 0$. We have $[\varphi(X_1), \varphi(Y_1)] = 0$.

So

$$\begin{aligned}[\varphi(X),\varphi(Y)] &= [\varphi(X_1+W_1),\varphi(Y_1+W_2)]\\ &= [\varphi(X_1),\varphi(Y_1)]+[\varphi(X_1),\varphi(W_2)]+[\varphi(W_1),\varphi(Y_1)]+[\varphi(W_1),\varphi(W_2)]\\ &= [\varphi(X_1),\varphi(W_2)]+[\varphi(W_1),\varphi(Y_1)]\in w.\end{aligned}$$

Thus

$$[\overline{\varphi}(\overline{X}),\overline{\varphi}(\overline{Y})]=[\overline{\varphi(X)},\overline{\varphi(Y)}]=\overline{[\varphi(X),\varphi(Y)]}=\overline{0}.$$

Conversely, $\varphi^{-1}\in\mathcal{T}'$. So φ^{-1} can also induce a linear map $\overline{\varphi^{-1}}$ of l/w by $\overline{\varphi^{-1}}(\overline{Y})=\overline{\varphi^{-1}(Y)}$. Similarly to the above, we can also get if $[\overline{X},\overline{Y}]=0$, then $[\overline{\varphi^{-1}}(\overline{X}),\overline{\varphi^{-1}}(\overline{Y})]=0$. So if $[\overline{\varphi}(\overline{X}),\overline{\varphi}(\overline{Y})]=0$, then $[\overline{\varphi^{-1}}(\overline{\varphi}(\overline{X})),\overline{\varphi^{-1}}(\overline{\varphi}(\overline{Y}))]=[\overline{X},\overline{Y}]=\overline{0}$. That is to say $\overline{\varphi}$ preserves zero Lie brackets in both directions. □

Now we give the main result of this paper.

Theorem 5.1 *Let $m\geq 4$. A linear map φ of l preserves zero Lie brackets in both directions if and only if φ is of the form*

$$\varphi=\psi_c\sigma_{T_0}\lambda_\alpha\phi_d\eta_f,$$

where ψ_c, σ_{T_0}, λ_α, ϕ_d, η_f are the standard maps preserving zero Lie brackets in both directions.

The "if" part of the theorem is clear. For the "only if" part, we will prove it for the case $m\geq 5$ and the case $m=4$, respectively.

Proof Since l/w is isomorphic to v, we may directly view l/w as v. So by Theorem 3.1, $\overline{\varphi}$ can be written in the form:

$$\overline{\varphi}=\psi_{v,c}\sigma_{v,T}\omega\mu_{b_4}^{(m2)}\mu_{b_3}^{(12)}\mu_{b_2}^{(m1)}\mu_{b_1}^{(11)}\eta_{v,f},$$

where $\psi_{v,c}$, $\sigma_{v,T}$, ω, $\mu_{b_4}^{(m2)}$, $\mu_{b_3}^{(12)}$, $\mu_{b_2}^{(m1)}$, $\mu_{b_1}^{(11)}$, $\eta_{v,f}$ are the standard maps of v. It is easy to see that $\psi_{v,c}=\overline{\psi_c}$, $\sigma_{v,T}=\overline{\sigma_T}$. So $\overline{\sigma_T^{-1}\psi_c^{-1}\varphi}=\omega\mu_{b_4}^{(m2)}\mu_{b_3}^{(12)}\mu_{b_2}^{(m1)}\mu_{b_1}^{(11)}\eta_{v,f}$. Denote $\sigma_T^{-1}\psi_c^{-1}\varphi$ by φ_1. In the following, we assume that $m\geq 5$ and we will give the proof step by step.

Step 1. There exist $T_1=E^{(2m)}+S_1$ and $T_2=E^{(2m)}+S_2$ with $S_1,S_2\in v$ such that

$$\sigma_{T_2}\sigma_{T_1}\varphi_1(T_{ij}(1))\equiv T_{ij}(1)\ (\operatorname{mod} w)\ \text{for}\ 1\leq i<j\leq m.$$

It is easy to see that $\varphi_1(T_{1m}(1))=T_{1m}(c)+W_0$ for some $c\in F^*$ and $W_0\in w$. Since $T_{1,-m}(1)\in p$, we may write $\varphi_1(T_{1,-m}(1))$ as

$$\varphi_1(T_{1,-m}(1))=\sum_{\substack{1\leq i\leq j\leq m\\ i+j\leq m+1}}T_{i,-j}(x_{ij}).$$

It follows from $[\varphi_1(T_{1m}(1)),\varphi_1(T_{1,-m}(1))]\neq 0$ that $x_{1m}\neq 0$. If $\omega\neq 1$, then $\varphi_1(T_{1,m-1}(1))=-f(T_{1,m-1}(1))T_{1m}(1)-T_{2m}(1)+W$ for some $W\in w$. By applying φ_1 on $[T_{1,m-1}(1),T_{1,-m}(1)]=0$, we have that $x_{1m}=0$, a contradiction. So $\omega=1$. That is to say

$$\overline{\varphi_1}=\mu_{b_4}^{(m2)}\mu_{b_3}^{(12)}\mu_{b_2}^{(m1)}\mu_{b_1}^{(11)}\eta_{v,f}.$$

For $2 \le i < j \le m-1$, we can assume that $\varphi_1(T_{ij}(1)) = T_{ij}(1) + f(T_{ij}(1))T_{1m}(1) + W_{ij}$ for some $W_{ij} \in w$. By considering the action of φ_1 on $[T_{ij}(1), T_{1,-m}(1)] = 0$, we have that $f(T_{ij}(1))x_{1m} = 0$ and $x_{kj} = x_{jl} = 0$ for $1 \le k \le j$, $j+1 \le l \le m$. That is to say

$$\begin{aligned}\varphi_1(T_{1,-m}(1)) &= T_{1,-1}(x_{11}) + T_{1,-2}(x_{12}) + T_{2,-2}(x_{22}) + T_{1,-m}(x_{1m}),\\ \varphi_1(T_{ij}(1)) &\equiv T_{ij}(1) \pmod{w} \text{ for } 2 \le i < j \le m-1.\end{aligned}$$

For $4 \le k \le m-1$, by operating φ_1 on $[T_{1k}(1), T_{1,-m}(1)] = 0$, we get that $f(T_{1k}(1))x_{1m} = 0$, which means that $f(T_{1k}(1)) = 0$. By applying φ_1 on $[T_{13}(1), T_{1,-m}(1)] = 0$, we have that $f(T_{13}(1))x_{1m} = 0$ and $b_3 x_{1m} = 0$, which means that $f(T_{13}(1)) = b_3 = 0$.

Suppose $\varphi_1(T_{3,-m}(1)) = \sum_{1\le i\le j\le m} T_{i,-j}(y_{ij})$. By applying φ_1 on $[T_{1k}(1), T_{3,-m}(1)] = 0$ for $4 \le k \le m-1$, we get $y_{sk} = y_{kl} = 0$ for $1 \le s \le k$ and $k+1 \le l \le m$. So $\varphi_1(T_{3,-m}(1)) = \sum_{1\le i\le j\le 3} T_{i,-j}(y_{ij}) + T_{1,-m}(y_{1m}) + T_{2,-m}(y_{2m}) + T_{3,-m}(y_{3m}) + T_{m,-m}(y_{mm})$. Since $T_{3,-m}(1) \notin p$ and $\varphi_1(p) = p$, we have that y_{2m}, y_{3m} and y_{mm} cannot be zero simultaneously. By considering the action of φ_1 on $[T_{12}(1), T_{3,-m}(1)] = 0$, we have $b_1 y_{2m} = b_1 y_{3m} = b_1 y_{mm} = 0$. So $b_1 = 0$.

By operating φ_1 on $[T_{1k}(1), T_{1,-(m-1)}(1)] = 0$ for $k = 2, 3, \ldots, m-2, m$, we know that there exist $z_{11}, z_{1,m-1} \in F$ such that

$$\varphi_1(T_{1,-(m-1)}(1)) = T_{1,-1}(z_{11}) + T_{1,-(m-1)}(z_{1,m-1}).$$

It is clear that $z_{1,m-1} \ne 0$. Considering the action of φ_1 on $[T_{m-2,m}(1), T_{1,-(m-1)}(1)] = 0$, we get $b_4 z_{1,m-1} = 0$, which implies that $b_4 = 0$. By considering the action of φ_1 on $[T_{m-1,m}(1), T_{1,-(m-1)}(1)] = 0$, we obtain $b_2 z_{1,m-1} = 0$, which means that $b_2 = 0$.

Since $[T_{km}(1) - T_{k,m-1}(1), T_{1,-m}(1) + T_{1,-(m-1)}(1)] = 0$ for $2 \le k \le m-2$, by applying φ_1 on the two sides of the above equation, we have $x_{1m} - z_{1,m-1} = 0$ and $f(T_{k,m}(1))x_{1m} = 0$. So $f(T_{km}(1)) = 0$. Now considering the action of φ_1 on $[T_{1m}(1) - T_{1,m-1}(1), T_{1,-m}(1) + T_{1,-(m-1)}(1)] = 0$, we have $x_{1m} - z_{1,m-1} + f(T_{1m}(1))x_{1m} = 0$. So $f(T_{1m}(1)) = 0$. Let

$$\begin{aligned}T_1 &= E^{(2m)} + f(T_{12}(1))T_{2m}(1),\\ T_2 &= E^{(2m)} - f(T_{m-1,m}(1))T_{1,m-1}(1).\end{aligned}$$

Then

$$\sigma_{T_2}^{-1}\sigma_{T_1}^{-1}\varphi_1(T_{ij}(1)) \equiv T_{ij}(1) \pmod{w} \text{ for } 1 \le i < j \le m.$$

Denote $\sigma_{T_2}^{-1}\sigma_{T_1}^{-1}\varphi_1$ by φ_2.

Step 2. There exist $\alpha = (a_1\ a_2\ a_3) \in F^{1\times 3}$ and $T_3 = E^{(2m)} + W$ with $W \in w$ such that

$$\lambda_\alpha^{-1}\sigma_{T_3}^{-1}\varphi_2(T_{ij}(1)) \equiv T_{ij}(1) \pmod{T_{1,-1}} \text{ for } 1 \le i < j \le m.$$

Suppose that

$$\varphi_2(T_{i,i+1}(1)) = T_{i,i+1}(1) + \sum_{1\le k\le l\le m} T_{k,-l}(a_{kl}^{(i)}), \quad 1 \le i \le m-1. \tag{3}$$

By applying φ_2 on $[T_{12}(1), T_{3t}(1)] = 0$, $4 \le t \le m$, and $[T_{i,i+1}(1), T_{1s}(1)] = 0$, $2 \le i \le m-1$,

$2 \le s \le m$ and $s \ne i$, we get that

$$\varphi_2(T_{12}(1)) = T_{12}(1) + \sum_{i=1}^{m} T_{1,-i}(a_{1i}^{(1)}) + T_{2,-2}(a_{22}^{(1)}) + T_{2,-3}(a_{23}^{(1)}) + T_{3,-3}(a_{33}^{(1)}),$$
$$\varphi_2(T_{i,i+1}(1)) = T_{i,i+1}(1) + \sum_{k=1}^{i} T_{k,-i}(a_{ki}^{(i)}) + \sum_{l=i+1}^{m} T_{i,-l}(a_{il}^{(i)}) + T_{1,-1}(a_{11}^{(i)}).$$

For $j \ne i-1, i+1$, by considering the action of φ_2 on $[T_{i,i+1}(1), T_{j,j+1}(1)] = 0$, we get $a_{i,j+1}^{(i)} = a_{i+1,j}^{(j)}$. Choose

$$T_3 = E^{(2m)} + \sum_{k=2}^{m} T_{1,-k}(a_{1,k-1}^{(k-1)}) + \sum_{2 \le i \le j \le m} T_{i,-j}(a_{i-1,j}^{(i-1)}).$$

Then

$$\sigma_{T_3}^{-1}\varphi_2(T_{12}(1)) = T_{12}(1) + T_{1,-1}(-a_{11}^{(1)}) + T_{2,-2}(a_{22}^{(1)}) + T_{2,-3}(a_{23}^{(1)}) + T_{3,-3}(a_{33}^{(1)}), \tag{4}$$
$$\sigma_{T_3}^{-1}\varphi_2(T_{i,i+1}(1)) = T_{i,i+1}(1) + T_{1,-1}(a_{11}^{(i)}) + T_{i,-i}(a_{ii}^{(i)} - 2a_{i-1,i+1}^{(i-1)}). \tag{5}$$

Denote $\sigma_{T_3}^{-1}\varphi_2$ by φ_3. For $1 \le k < l \le m$ and $l - k \ne 1$, suppose that

$$\varphi_3(T_{kl}(1)) = T_{kl}(1) + \sum_{1 \le s \le t \le m} T_{s,-t}(b_{st}^{(kl)}).$$

For $1 \le i \le m-1$ and $i \ne l, k-1$, by applying φ_3 on $[T_{kl}(1), T_{i,i+1}(1)] = 0$, we have that the entries in the $(i+1)$-row of $\varphi_3(T_{kl}(1))$ are all zero except $b_{22}^{(13)}$.

For $2 \le k < l \le m-1$, by applying φ_3 on $[T_{k,l+1}(1), T_{1l}(1)] = 0$ and $[T_{kl}(1), T_{1,l+1}(1)] = 0$, respectively, we get $b_{14}^{(13)} = b_{44}^{(13)} = 0$, $b_{1,l+1}^{(1l)} = b_{l+1,l+1}^{(1l)} = 0$ and $b_{1,l+1}^{(kl)} = b_{k,l+1}^{(kl)} = b_{l+1,l+1}^{(kl)} = 0$. By applying φ_3 on $[T_{12}(1), T_{13}(1)] = 0$, we get $a_{33}^{(1)} = 0$ and $a_{23}^{(1)} = b_{22}^{(13)}$. By operating φ_3 on $[T_{1k}(1) + T_{1l}(1), T_{k,l+1}(1) - T_{l,l+1}(1)] = 0$ for $2 \le k < l \le m-1$, we get $b_{1k}^{(k,l+1)} = b_{kk}^{(k,l+1)} = a_{ll}^{(l)} - 2a_{l-1,l+1}^{(l-1)} = 0$. So $\varphi(T_{kl}(1))$ can be rewritten as

$$\begin{aligned}
\varphi_3(T_{12}(1)) &= T_{12}(1) + T_{1,-1}(-a_{11}^{(1)}) + T_{2,-2}(a_{22}^{(1)}) + T_{2,-3}(a_{23}^{(1)}),\\
\varphi_3(T_{23}(1)) &= T_{23}(1) + T_{1,-1}(a_{11}^{(2)}) + T_{2,-2}(a_{22}^{(2)} - 2a_{13}^{(1)}),\\
\varphi_3(T_{13}(1)) &= T_{13}(1) + T_{1,-1}(b_{11}^{(13)}) + T_{2,-2}(b_{22}^{(13)}),\\
\varphi_3(T_{i,i+1}(1)) &= T_{i,i+1}(1) + T_{1,-1}(a_{11}^{(i)}) \ \text{ for } 3 \le i \le m-1,\\
\varphi_3(T_{kl}(1)) &= T_{kl}(1) + T_{1,-1}(b_{11}^{(kl)}) \ \text{ for } 2 \le k < l \le m \text{ and } l \ne k+1,\\
\varphi_3(T_{1l}(1)) &= T_{1l}(1) + T_{1,-1}(b_{11}^{(1l)}) \ \text{ for } 4 \le l \le m.
\end{aligned}$$

Set $\alpha = (a_1\ a_2\ a_3)$ with $a_1 = a_{22}^{(1)},\ a_2 = a_{23}^{(1)}, a_3 = a_{22}^{(2)} - 2a_{13}^{(1)}$, then

$$\begin{aligned}
\lambda_\alpha^{-1}\varphi_3(T_{12}(1)) &= T_{12}(1) + T_{1,-1}(-a_{11}^{(1)}),\\
\lambda_\alpha^{-1}\varphi_3(T_{i,i+1}(1)) &= T_{i,i+1}(1) + T_{1,-1}(a_{11}^{(i)}) \ \text{ for } 2 \le i \le m-1,\\
\lambda_\alpha^{-1}\varphi_3(T_{kl}(1)) &= T_{kl}(1) + T_{1,-1}(b_{11}^{(kl)}) \ \text{ for } 1 \le k < l \le m \text{ and } l \ne k+1.
\end{aligned}$$

That is to say

$$\lambda_\alpha^{-1}\sigma_{T_3}^{-1}\varphi_2(T_{ij}(1)) \equiv T_{ij}(1) \ (\mathrm{mod}\, T_{1,-1}) \text{ for } 1 \le i < j \le m.$$

Denote $\lambda_a^{-1}\varphi_3$ by φ_4.

Step 3. There exist some $b, d \in F^*$, $T_4 = E^{(2m)} + T_{1m}(bd^{-1})$ and a linear function from l to F such that

$$\phi_d^{-1}\sigma_{T_4}\varphi_4 = \eta_f.$$

Suppose $\varphi_4(T_{k,-l}(1)) = \sum_{1\le s\le t\le m} T_{s,-t}(c_{st}^{(kl)})$. Since $T_{1,-1}$ is the center of l, we have $\varphi_4(T_{1,-1}(1)) = T_{1,-1}(c_{11}^{(11)})$ with $c_{11}^{(11)} \in F^*$. For $2 \le s \le m$, $1 \le k \le l \le m$ and $s \ne k, l$, by applying φ_4 on $[T_{1s}(1), T_{k,-l}(1)] = 0$, we get that the entries of $\varphi_4(T_{k,-l}(1))$ in the s-row and $(s+m)$-column are all zero. That is to say

$$\begin{aligned}\varphi_4(T_{1,-l}(1)) =& T_{1,-1}(c_{11}^{(1l)}) + T_{1,-l}(c_{1l}^{(1l)}) + T_{l,-l}(c_{ll}^{(1l)}) \text{ for } 2 \le l \le m, && (6)\\ \varphi_4(T_{k,-l}(1)) =& T_{1,-1}(c_{11}^{(kl)}) + T_{1,-k}(c_{1k}^{(kl)}) + T_{1,-l}(c_{1l}^{(kl)}) + T_{k,-l}(c_{kl}^{(kl)}) +\\ & T_{k,-k}(c_{kk}^{(kl)}) + T_{l,-l}(c_{ll}^{(kl)}) \text{ for } 2 \le k < l \le m, && (7)\\ \varphi_4(T_{k,-k}(1)) =& T_{1,-1}(c_{11}^{(kk)}) + T_{1,-k}(c_{1k}^{(kk)}) + T_{k,-k}(c_{kk}^{(kk)}) \text{ for } 2 \le k \le m. && (8)\end{aligned}$$

For $3 \le l \le m$, by applying φ_4 on $[T_{12}(1) - T_{1l}(1), T_{1,-2}(1) + T_{1,-l}(1)] = 0$, we get $c_{12}^{(12)} = c_{1l}^{(1l)}$, $c_{22}^{(12)} = c_{ll}^{(1l)} = 0$.

For $2 \le k < l \le m-1$, by operating φ_4 on $[T_{1k}(1) - T_{l,l+1}(1), T_{k,-l}(1) + T_{1,-(l+1)}(1)] = 0$ and $[T_{1l}(1) - T_{k,l+1}(1), T_{k,-l}(1) + T_{1,-(l+1)}(1)] = 0$, respectively, we have that $c_{1k}^{(kl)} = c_{kk}^{(kl)} = c_{1l}^{(kl)} = c_{ll}^{(kl)} = 0$ and $c_{kl}^{(kl)} = c_{1,l+1}^{(l,l+1)}$.

For $2 \le k \le m-2$, by applying φ_4 on $[T_{1m}(1) - T_{k.m-1}(1), T_{k,-m}(1) + T_{1,-(m-1)}(1)] = 0$ and $[T_{1,m-1}(1) - T_{1k}(1), T_{m-1,-m}(1) + T_{k,-m}(1)] = 0$, respectively, we have $c_{1m}^{(km)} = c_{mm}^{(km)} = 0$, $c_{km}^{(km)} = c_{1,m-1}^{(1,m-1)}$, $c_{1,m-1}^{(m-1,m)} = c_{1k}^{(km)}$,$c_{m-1,m}^{(m-1,m)} = c_{km}^{(km)}$ and $c_{m-1,-(m-1)}^{(m-1,m)} = c_{kk}^{(km)} = 0$.

By operating φ_4 on $[T_{1m}(1) - T_{12}(1), T_{m-1,-m}(1) + T_{2,-(m-1)}(1)] = 0$, we get $c_{1m}^{(m-1,m)} = c_{mm}^{(m-1,m)} = 0$, $c_{m-1,m}^{(m-1,m)} = c_{2,m-1}^{(2,m-1)}$.

For $2 \le k \le m-1$, by applying φ_4 on $[T_{1k}(1) - T_{k,k+1}(1), T_{k,-k}(1) + T_{1,-(k+1)}(1)] = 0$, we have that $c_{1k}^{(kk)} = 0$ and $c_{kk}^{(kk)} = c_{1,k+1}^{(1,k+1)}$.

By operating φ_4 on $[T_{1,m}(1) - T_{12}(1) - T_{23}(1), T_{m,-m}(1) + T_{2,-m}(1) + T_{1,-3}(1)] = 0$, we get that $c_{mm}^{(mm)} = c_{2m}^{(2m)} = c_{13}^{(13)}$ and $c_{1m}^{(mm)} = c_{12}^{(2m)}$.

Let $b = c_{12}^{(2m)}$, $d = c_{12}^{(12)}$. Then

$$\begin{aligned}&\varphi_4(T_{k,-l}(1)) = T_{1,-1}(c_{11}^{(kl)}) + T_{k,-l}(d) \quad \text{for } 1 \le k \le l \le m-1 \text{ and } (k,l) \ne (1,1),\\ &\varphi_4(T_{1,-m}(1)) = T_{1,-1}(c_{11}^{(1m)}) + T_{1,-m}(d),\\ &\varphi_4(T_{k,-m}(1)) = T_{1,-1}(c_{11}^{(km)}) + T_{k,-m}(d) + T_{1,-k}(b) \text{ for } 2 \le k \le m.\end{aligned}$$

Let $T_4 = E^{(2m)} + T_{1m}(bd^{-1})$. Then

$$\begin{aligned}\phi_d^{-1}\sigma_{T_4}\varphi_4(T_{1,-m}(1)) &= T_{1,-1}(d^{-1}(c_{11}^{(kl)} - 2b)) + T_{1,-m}(1),\\ \phi_d^{-1}\sigma_{T_4}\varphi_4(T_{k,-l}(1)) &= T_{1,-1}(d^{-1}c_{11}^{(kl)}) + T_{k,-l}(1),\end{aligned}$$

where $1 \le k \le l \le m$ and $(k,l) \ne (1,1), (1,m)$.

Let $f(T_{12}(1)) = -d^{-1}a_{11}^{(1)}$, $f(T_{i,i+1}(1)) = d^{-1}a_{11}^{(i)}$ for $2 \le i \le m-1$, $f(T_{kl}(1)) = d^{-1}b_{11}^{(kl)}$ for $1 \le k < l \le m$ and $l \ne k+1$, $f(T_{1,-1}(1)) = d^{-1}c_{11}^{(11)} - 1$, $f(T_{1,-m}(1)) = d^{-1}(c_{11}^{(1m)} - 2b)$,

$f(T_{s,-t}(1)) = d^{-1}c_{11}^{(st)}$ for $1 \le s \le t \le m$ and $(s,t) \ne (1,1),(1,m)$. Then $\phi_d^{-1}\sigma_{T_4}\varphi_4 = \eta_f$.

Above discussion shows that

$$\phi_d^{-1}\sigma_{T_4}\lambda_\alpha^{-1}\sigma_{T_3}^{-1}\sigma_{T_2}^{-1}\sigma_{T_1}^{-1}\sigma_T^{-1}\psi_c^{-1}\varphi = \eta_f.$$

Let $T_0 = TT_1T_2T_3T_4^{-1}$. Then

$$\varphi = \psi_c\sigma_{T_0}\lambda_\alpha\phi_d\eta_f.$$

Now we prove the theorem for the case $m = 4$. Suppose that

$$\varphi_1(T_{1,-4}(1)) = \sum_{\substack{1\le i\le j\le 4\\ i+j\le 5}} T_{i,-j}(x_{ij}),$$

then similarly to Step 1, we can prove that $x_{14} \ne 0$. By applying φ_1 on $[T_{23}(1), T_{1,-4}(1)] = 0$, we have $f(T_{23}(1))x_{14} = 0$ and $x_{13} = x_{23} = 0$. So $f(T_{23}(1)) = 0$. If $\omega \ne 1$, then

$$\varphi_1(T_{13}(1)) = -T_{13}(b_3) - T_{24}(1) - f(T_{13}(1))T_{14}(1) + W \text{ for some } W \in w.$$

By operating φ_1 on $[T_{13}(1), T_{1,-4}(1)] = 0$, we get $x_{14} = 0$. This contradiction means that $\omega = 1$. Again by applying φ_1 on $[T_{13}(1), T_{1,-4}(1)] = 0$, we have $f(T_{13}(1))x_{14} = b_3x_{14} = 0$, which implies that $f(T_{13}(1)) = b_3 = 0$.

Since $T_{1,-3}(1) \in p$, we may write

$$\varphi_1(T_{1,-3}(1)) = \sum_{\substack{1\le i\le j\le 4\\ i+j\le 5}} T_{i,-j}(y_{ij}).$$

By considering the action of φ_1 on $[T_{14}(1), T_{1,-3}(1)] = 0$, we have that $y_{14} = 0$. Since $[T_{13}(1), T_{1,-3}(1)] \ne 0$, we have that y_{13} and y_{23} cannot be zero simultaneously. By operating φ_1 on $[T_{24}(1), T_{1,-3}(1)] = 0$, we get that $y_{13}b_4 = y_{23}b_4 = 0$. So $b_4 = 0$.

By applying φ_1 on $[T_{24}(1) - T_{23}(1), T_{1,-4}(1) + T_{1,-3}(1)] = 0$, we get that $y_{23} = 0$ and $x_{14} - y_{13} = f(T_{24}(1))x_{14} = 0$. So $f(T_{24}(1)) = 0$. By operating φ_1 on $[T_{14}(1) - T_{13}(1), T_{1,-4}(1) + T_{1,-3}(1)] = 0$, we have that $x_{14} - y_{13} + f(T_{14}(1))x_{14} = 0$. So $f(T_{14}(1)) = 0$. By applying φ_1 on $[T_{34}(1), T_{1,-3}(1)] = 0$, we get that $y_{13}b_2 = 0$. So $b_2 = 0$.

Suppose that $\varphi_1(T_{3,-4}(1)) = \sum_{1\le i\le j\le 4} T_{i,-j}(z_{ij})$. Since $T_{3,-4}(1) \notin p$, we have z_{24}, z_{34} and z_{44} cannot be zero simultaneously. By applying φ_1 on $[T_{12}(1), T_{3,-4}(1)] = 0$, we have that $b_1z_{24} = b_1z_{34} = b_1z_{44} = 0$. So $b_1 = 0$.

Let

$$T_1 = E^{(8)} + f(T_{12}(1))T_{24}(1), \quad T_2 = E^{(8)} - f(T_{34}(1))T_{13}(1).$$

Then

$$\sigma_{T_2}^{-1}\sigma_{T_1}^{-1}\varphi_1(T_{ij}(1)) \equiv T_{ij}(1) \pmod{w} \text{ for } 1 \le i < j \le 4.$$

Denote $\sigma_{T_2}^{-1}\sigma_{T_1}^{-1}\varphi_1$ by φ_2. For this φ_2, we can prove that there exist some $\alpha \in F^{1\times 3}$, $T_3 = E^{(8)} + W$ with $W \in w$, b, $d \in F^*$ $T_4 = E^{(8)} + T_{14}(bd^{-1})$ and a linear function f from l to F such that $\phi_d^{-1}\sigma_{T_4}\lambda_\alpha^{-1}\sigma_{T_3}^{-1}\varphi_2 = \eta_f$ in the same way as Steps 2-3 in the case $m \ge 5$. We omit the repeated arguments. Now we have proved that $\phi_d^{-1}\sigma_{T_4}\lambda_\alpha^{-1}\sigma_{T_3}^{-1}\sigma_{T_2}^{-1}\sigma_{T_1}^{-1}\sigma_T^{-1}\psi_c^{-1}\varphi=\eta_f$. Also let $T_0=TT_1T_2T_3T_4^{-1}$. Then

$$\varphi=\psi_c\sigma_{T_0}\lambda_\alpha phi_d\eta_f. \quad \square$$

References

[1] FROBENIUS G. *Uber die darstellung der endlichen gruppen durch lineare substitutioen* [J]. Sitzungsber Deutsch Akad Wiss Berlin, 1897, 994–1015.

[2] LI C K, TSING N K. *Linear preserver problems: a brief introduction and some special techniques* [J]. Linear Algebra Appl., 1992, **162/164**: 217–235.

[3] PIERCE S, et al. *A survey of linear preserver problems* [J]. Linear and Multilinear Algebra, 1992, **33**: 1–192.

[4] LI C K, PIERCE S. *Linear preserver problems* [J]. Amer. Math. Monthly, 2001, **108**(7): 591–605.

[5] DOLINAR G, ŠEMRL P. *Determinant preserving maps on matrix algebras* [J]. Linear Algebra Appl., 2002, **348**: 189–192.

[6] HLADNIK M, OMLADIČ M, RADJAVI H. *Trace-preserving homomorphisms of semigroups* [J]. J. Funct. Anal., 2003, **204**(2): 269–292.

[7] BELL J, SOUROUR A R. *Additive rank-one preserving mappings on triangular matrix algebras* [J]. Linear Algebra Appl., 2000, **312**(1-3): 13–33.

[8] ŠEMRL P. *Linear mappings preserving square-zero matrices* [J]. Bull. Austral. Math. Soc., 1993, **48**(3): 365–370.

[9] CHEBOTAR M A, KE W F, LEE P H. *On maps preserving square-zero matrices* [J]. J. Algebra, 2005, **289**(2): 421–445.

[10] DOLINAR G. *Maps on matrix algebras preserving idempotents* [J]. Linear Algebra Appl., 2003, **371**: 287–300.

[11] CAO Chongguang, ZHANG Xian. *Additive operators preserving idempotent matrices over fields and applications* [J]. Linear Algebra Appl., 1996, **248**: 327–338.

[12] CHOI M D, JAFARIAN A A, RADJAVI H. *Linear maps preserving commutativity* [J]. Linear Algebra Appl., 1987, **87**: 227–241.

[13] PETEK T. *Spectrum and commutativity preserving mappings on triangular matrices* [J]. Linear Algebra Appl., 2002, **357**: 107–122.

[14] MARCOUX L W, SOUROUR A R. *Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras* [J]. Linear Algebra Appl., 1999, **288**(1-3): 89–104.

[15] CAO You'an, CHEN Zhonghu, HUANG Chong. *Commutativity preserving linear maps and Lie automorphisms of strictly triangular matrix space* [J]. Linear Algebra Appl., 2002, **350**: 41–66.

[16] BREŠAR M, ŠEMRL P. *Commutativity preserving linear maps on central simple algebras* [J]. J. Algebra, 2005, **284**(1): 102–110.

[17] ŠEMRL P. *Commutativity preserving maps* [J]. Linear Algebra Appl., 2008, **429**(5-6): 1051–1070.

[18] BREŠAR M, ŠEMRL P. *On bilinear maps on matrices with applications to commutativity preservers* [J]. J. Algebra, 2006, **301**(2): 803–837.

[19] JI Guoxing. *Similarity-preserving linear maps on $\mathcal{B}(\mathcal{H})$* [J]. Linear Algebra Appl., 2003, **360**: 249–257.

[20] FANG Li, JI Guoxing, PANG Yongfeng. *Maps preserving the idempotency of products of operators* [J]. Linear Algebra Appl., 2007, **426**(1): 40–52.

[21] LI C K, ŠEMRL P, SZE N S. *Maps preserving the nilpotency of products of operators* [J]. Linear Algebra Appl., 2007, **424**(1): 222–239.

[22] TANG Xiaomin. *Linear operators preserving adjoint matrix between matrix spaces* [J]. Linear Algebra Appl., 2003, **372**: 287–293.