

On Fuzzy Simplex and Fuzzy Convex Hull

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Abstract In this paper, we discuss fuzzy simplex and fuzzy convex hull, and give several representation theorems for fuzzy simplex and fuzzy convex hull. In addition, by giving a new characterization theorem of fuzzy convex hull, we improve some known results about fuzzy convex hull.

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1. Introduction

In many scientific and engineering applications, the fuzzy set concept plays an important role. The fuzziness appears when we need to perform, on manifold, calculations with imprecision variables. The fuzzy set theory was introduced initially by Zadeh [1] in 1965. In the theory and applications of fuzzy sets, convexity is a most useful concept. In fact, in the basic and classical paper [1], Zadeh paid special attention to the investigation of the convex fuzzy sets which covers nearly the second half of the space of the paper.

Following the seminal work of Zadeh on the definition of a convex fuzzy set, Ammar and Metz defined another type of convex fuzzy sets in [2]. A lot of scholars have discussed various aspects of the theory and applications of fuzzy convex analysis. In [3], by use of the relations between fuzzy points and fuzzy sets, Yuan and Lee gave some generalizations of convex fuzzy set. In [4], Wu and Cheng introduced and studied the concept of semi-strictly convex fuzzy sets, and presented the important connections between these convex fuzzy sets. In order to solve the open problem in fuzzy analysis that we proposed in [5], we introduced some new and more general definitions in the area of fuzzy starshapedness, and developed several theorems on the shadows of starshaped fuzzy sets [6], which generalize the important results obtained by Liu [7]. The

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aim of this paper is to discuss the fuzzy simplex which is the convex hull of a set of finite fuzzy points, and study the representations of fuzzy convex hull of a fuzzy set by fuzzy simplex. For the new progress about fuzzy convex analysis, one can refer to [8–15].

2. Preliminaries

A fuzzy set μ of Euclidean space R^n is characterized by a membership function $\mu(x)$ which associates with each point in R^n a real number in the interval $[0, 1]$, with the value of $\mu(x)$ at x representing the “grade of membership” of x in μ (see [1]). We denote the totality of fuzzy set of R^n by $F(R^n)$.

The ordinary subsets of R^n , sometimes called “crisp sets”, can be considered as a particular case of a fuzzy set with membership function which maps into $\{0, 1\}$.

Definition 1 ([16]) For a fuzzy set μ , its support set, denoted by $\text{supp}(\mu)$, is defined as

$$\text{supp}(\mu) = \{x \in R^n : \mu(x) > 0\}.$$

Definition 2 ([17]) For a fuzzy set μ and each $\alpha \in [0, 1]$, the α -cut of μ , denoted by $[\mu]^\alpha$, is defined as

$$[\mu]^\alpha = \begin{cases} \{x \in R^n : \mu(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1], \\ \text{supp}(\mu), & \text{if } \alpha = 0. \end{cases}$$

Definition 3 ([18]) A fuzzy set in R^n is called a fuzzy point if and only if it takes the value 0 for all $y \in R^n$ except one, say $x \in R^n$. If its value at x is r ($0 < r \leq 1$) we denote the fuzzy point by x_r , where the point x is called its support.

Definition 4 ([1]) A fuzzy set $\mu \in F(R^n)$ is said to be fuzzy convex if, for all $x, y \in R^n$,

$$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \quad 0 \leq \lambda \leq 1.$$

There are some well known results.

Lemma 1 ([17]) A fuzzy set $\mu \in F(R^n)$ is fuzzy convex if and only if its α -cuts are convex crisp sets in R^n .

Definition 5 ([18]) The fuzzy point x_λ is said to be contained in a fuzzy set μ , or to belong to μ , denoted by $x_\lambda \in \mu$, if and only if $\lambda \leq \mu(x)$.

Lemma 2 ([3]) A fuzzy set $\mu \in F(R^n)$ is fuzzy convex if and only if for any two fuzzy points x_λ, y_γ ,

$$x_\lambda, y_\gamma \in \mu, \alpha \in [0, 1] \Rightarrow \alpha x_\lambda + (1 - \alpha)y_\gamma \in \mu.$$

Definition 6 (Extension Principle [17]) Let X, Y be two non-empty sets of R^n , $f : X \rightarrow Y$ and μ belong to $F(X)$. Then $f(\mu)$ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset, \quad y \in Y. \end{cases}$$

Lemma 3 Let x_a^1, x_b^2 be two fuzzy points in R^n . Then, for any $\lambda \in [0, 1]$, we have

$$\lambda x_a^1 + (1 - \lambda)x_b^2 = (\lambda x^1 + (1 - \lambda)x^2)_{\min\{a,b\}}.$$

Proof Define the mapping $f : R^n \times R^n \rightarrow R^n$ as $f(x^1, x^2) = x^1 + x^2$, and define $g : R^n \rightarrow R^n$ as $g(x) = kx$, where $k \in R$ is a constant. By Extension Principle, we have that for fuzzy points $x_a^1, x_b^2, x_a^1 + x_b^2 = (x^1 + x^2)_{\min\{a,b\}}, kx_a^1 = (kx^1)_a$ and $kx_b^2 = (kx^2)_b$ for any $k \in R$. Consequently, we have that

$$\lambda x_a^1 + (1 - \lambda)x_b^2 = (\lambda x^1 + (1 - \lambda)x^2)_{\min\{a,b\}}$$

for each $\lambda \in [0, 1]$. \square

Remark 1 By this statement we have for $k, h \in R, kx_r + hx_r = (kx + hx)_r = (k + h)x_r$ and for fuzzy points x_a^1, x_b^2 , if $a < b$, then $0x_a^1 + x_b^2 = x_a^2 \neq x_b^2$.

Lemma 4 A fuzzy set $\mu \in F(R^n)$ is fuzzy convex if and only if for any two fuzzy points x_λ, y_γ and $\alpha \in [0, 1]$,

$$x_\lambda, y_\gamma \in \mu \text{ and } \lambda = \mu(x), \gamma = \mu(y) \Rightarrow \alpha x_\lambda + (1 - \alpha)y_\gamma \in \mu.$$

Proof Suppose μ is fuzzy convex. By Lemma 2 we have that for any two fuzzy points x_λ, y_γ and $\alpha \in [0, 1]$, if x_λ, y_γ belong to μ and $\lambda = \mu(x)$, then $\gamma = \mu(y)$ implies $\alpha x_\lambda + (1 - \alpha)y_\gamma \in \mu$. Conversely, let x_λ, y_γ belong to μ , that is, $\lambda \leq \mu(x)$ and $\gamma \leq \mu(y)$. Thus $\min\{\lambda, \gamma\} \leq \min\{\mu(x), \mu(y)\}$. Since fuzzy points $x_{\mu(x)}, y_{\mu(y)}$ belong to μ , by the hypothesis we have $\alpha x_{\mu(x)} + (1 - \alpha)y_{\mu(y)} \in \mu$, for each $\alpha \in [0, 1]$. Consequently, by Lemma 3 we have $\alpha x_\lambda + (1 - \alpha)y_\gamma \in \mu$. \square

Definition 7 ([19]) If μ is a fuzzy set, then its convex hull is defined as

$$\text{conv}(\mu) = \inf\{\nu : \mu \subseteq \nu, \nu \text{ is convex}\} = \text{the smallest convex fuzzy set containing } \mu.$$

As usual, I will be used to denote the unit interval. For any $x \in R^n$ and $p \in \mathbb{N}$, put

$$C(x, p) = \{\{x_1, \dots, x_p\} \subset R^n : \text{there exist } \alpha_i \in I, x = \sum_{i=1}^p \alpha_i x_i, \sum_{i=1}^p \alpha_i = 1\},$$

and put

$$D(x, n + 1) = \{\{y_1, \dots, y_k\} \subset R^n : \text{there exist } \alpha_i \in I, x = \sum_{i=1}^{n+1} \alpha_i x_i, \sum_{i=1}^{n+1} \alpha_i = 1, x_i \in \{y_1, \dots, y_k\}, k \leq n + 1\}.$$

In the sequel, we will use Proposition 6.3 in [19].

Proposition 6.3 in [19] is as follows:

The convex hull of a fuzzy set μ is given by

$$\text{conv}\mu(x) = \sup_{p \in \mathbb{N}} \sup_{A \in C(x,p)} \inf\{\mu(y) : y \in A\}.$$

Definition 8 The convex hull $\tilde{\Delta}_n$ of a finite union set of $n + 1$ fuzzy points $x_{r_1}^1, x_{r_2}^2, \dots, x_{r_{n+1}}^{n+1}$ in R^m is called an n -dimensional fuzzy simplex if the flat of minimal dimension containing the

support set $\{x^1, x^2, \dots, x^{n+1}\}$ has dimension n . The points $x_{r_i}^i$ ($i = 1, \dots, n + 1$) are called fuzzy vertices.

3. Fuzzy simplex and convex hull

In this section, we will give the main results about fuzzy simplex and convex hull.

Theorem 1 If $\tilde{\Delta}_n$ is an n -dimensional fuzzy simplex in R^m with fuzzy vertices $x_{r_i}^i$ ($i = 1, \dots, n + 1$), then $\tilde{\Delta}_n$ consists of all fuzzy points x_r in R^m for which constants $\alpha_j \geq 0$ ($j = 1, \dots, n + 1$) exist such that

$$x_r = \sum_{j=1}^{n+1} \alpha_j y_{r_j}^j, \sum_{j=1}^{n+1} \alpha_j = 1, y_{r_j}^j \in \{x_{r_i}^i : i = 1, \dots, n + 1\}. \tag{1}$$

Proof The theorem is clearly true for $n = 0$. Let the set of fuzzy point x_r for which (1) holds be denoted by $L(\tilde{\Delta}_n)$.

Firstly, we show that $L(\tilde{\Delta}_n)$ is a fuzzy convex set. Suppose $n = 1$. If $x_r \in L(\tilde{\Delta}_1)$, $z_l \in L(\tilde{\Delta}_1)$, then, without loss of generality, let $x_r = \sum_{j=1}^2 \alpha_j y_{r_j}^j$, $z_l = \sum_{k=1}^2 \beta_k y_{r_k}^k$, $\sum_{j=1}^2 \alpha_j = 1$, $\sum_{k=1}^2 \beta_k = 1$, $y_{r_j}^j, y_{r_k}^k \in \{x_{r_i}^i : i = 1, 2\}$.

Now there are several cases. We only prove three cases of them, and the others can be proved similarly.

Case 1 If $x_r = \alpha_1 x_{r_1}^1 + (1 - \alpha_1) x_{r_2}^2$ and $z_l = \beta_1 x_{r_1}^1 + (1 - \beta_1) x_{r_2}^2$. Then for each $\lambda \in [0, 1]$, we have

$$\lambda x_r + (1 - \lambda) z_l = (\lambda \alpha_1 + (1 - \lambda) \beta_1) x_{r_1}^1 + (\lambda(1 - \alpha_1) + (1 - \lambda)(1 - \beta_1)) x_{r_2}^2,$$

and $(\lambda \alpha_1 + (1 - \lambda) \beta_1) + (\lambda(1 - \alpha_1) + (1 - \lambda)(1 - \beta_1)) = 1$ which implies $\lambda x_r + (1 - \lambda) z_l \in L(\tilde{\Delta}_1)$.

Case 2 If $x_r = \alpha_1 x_{r_2}^2 + (1 - \alpha_1) x_{r_1}^1$ and $z_l = \beta_1 x_{r_1}^1 + (1 - \beta_1) x_{r_2}^2$. Then for each $\lambda \in [0, 1]$, we have

$$\lambda x_r + (1 - \lambda) z_l = (\lambda(1 - \alpha_1) + (1 - \lambda) \beta_1) x_{r_1}^1 + (\lambda \alpha_1 + (1 - \lambda)(1 - \beta_1)) x_{r_2}^2,$$

and $(\lambda(1 - \alpha_1) + (1 - \lambda) \beta_1) + (\lambda \alpha_1 + (1 - \lambda)(1 - \beta_1)) = 1$ which implies $\lambda x_r + (1 - \lambda) z_l \in L(\tilde{\Delta}_1)$.

Case 3 If $x_r = \alpha_1 x_{r_1}^1 + (1 - \alpha_1) x_{r_1}^1 = x_{r_1}^1$ and $z_l = \beta_1 x_{r_1}^1 + (1 - \beta_1) x_{r_2}^2$. Then for each $\lambda \in [0, 1]$, we have

$$\lambda x_r + (1 - \lambda) z_l = (\lambda + (1 - \lambda) \beta_1) x_{r_1}^1 + (1 - \lambda)(1 - \beta_1) x_{r_2}^2,$$

and $(\lambda + (1 - \lambda) \beta_1) + (1 - \lambda)(1 - \beta_1) = 1$ which implies $\lambda x_r + (1 - \lambda) z_l \in L(\tilde{\Delta}_1)$.

All in all, by Lemma 4 we have $L(\tilde{\Delta}_1)$ is fuzzy convex. Also $x_{r_1}^1, x_{r_2}^2 \in L(\tilde{\Delta}_1)$. Hence $L(\tilde{\Delta}_1) \supseteq \tilde{\Delta}_1$. Similarly, we can prove $L(\tilde{\Delta}_n)$ is fuzzy convex and $L(\tilde{\Delta}_n) \supseteq \tilde{\Delta}_n$.

In fact, by Remark 1, the condition (1) is equivalent to that there are constants $\alpha_j \geq 0$ ($j = 1, \dots, k; k \leq n + 1$) such that

$$x_r = \sum_{j=1}^k \alpha_j y_{r_j}^j, \sum_{j=1}^k \alpha_j = 1, \{y_{r_j}^j : j = 1, \dots, k\} \subseteq \{x_{r_i}^i : i = 1, \dots, n + 1\}. \tag{2}$$

We prove $L(\tilde{\Delta}_n) \subseteq \tilde{\Delta}_n$ by induction. It is clearly true for $n = 0$. Assume it is true for $1, \dots, n - 1$. Let $x_r \in L(\tilde{\Delta}_n)$ and $x_r = \sum_{j=1}^k \alpha_j y_{r_j}^j$. Rearranging if necessary, we may suppose

$0 \leq \alpha_1 < 1$ and $y_{r_1}^1 = x_{r_1}^1$. Then $x_r = \alpha_1 x_{r_1}^1 + \sum_{j=2}^k \alpha_j y_{r_j}^j \equiv \alpha_1 x_{r_1}^1 + (1 - \alpha_1) z_\beta$, where

$$z_\beta = \sum_{j=2}^k \alpha_j y_{r_j}^j / (1 - \alpha_1), \quad \sum_{j=2}^k \alpha_j / (1 - \alpha_1) = 1$$

and $\{y_{r_j}^j : j = 2, \dots, k\} \subseteq \{x_{r_i}^i : i = 2, \dots, n + 1\}$. By our induction assumption, $z_\beta \in \text{conv}(\bigcup_{j=2}^k y_{r_j}^j)$. Since $\text{conv}(\bigcup_{j=2}^k y_{r_j}^j) \subseteq \tilde{\Delta}_n$, we have $x_r = \alpha_1 x_{r_1}^1 + (1 - \alpha_1) z_\beta \in \tilde{\Delta}_n$, so that $L(\tilde{\Delta}_n) \subseteq \tilde{\Delta}_n$. This completes the whole proof that $L(\tilde{\Delta}_n) = \tilde{\Delta}_n$. \square

Corollary 1 *If $\tilde{\Delta}_n$ is an n -dimensional fuzzy simplex in R^m with fuzzy vertices $x_{r_i}^i$ ($i = 1, \dots, n + 1$), then $\tilde{\Delta}_n$ consists of all points x_r in R^m for which constants $\alpha_j \geq 0$ ($j = 1, \dots, m + 1$) exist such that*

$$x_r = \sum_{j=1}^{m+1} \alpha_j y_{r_j}^j, \quad \sum_{j=1}^{m+1} \alpha_j = 1, \quad y_{r_j}^j \in \{x_{r_i}^i : i = 1, \dots, n + 1\}. \quad (3)$$

Proof If $n = m$, then we have nothing to prove. Assume $n < m$. By Theorem 1, there are constants $\alpha_j \geq 0$ ($j = 1, \dots, n + 1$) such that

$$x_r = \sum_{j=1}^{n+1} \alpha_j y_{r_j}^j, \quad \sum_{j=1}^{n+1} \alpha_j = 1, \quad y_{r_j}^j \in \{x_{r_i}^i : i = 1, \dots, n + 1\}.$$

Let $\alpha_{n+2} = \dots = \alpha_{m+1} = 0$ and $y_{r_{n+2}}^{n+2} = \dots = y_{r_{m+1}}^{m+1} = y_{r_{n+1}}^{n+1}$. Then by Remark 1, we get the desired result. \square

Let $F_c(R^m)$ denote the collection of all fuzzy sets with the following properties:

- 1) μ is upper semi-continuous;
- 2) the closure of $\text{supp}(\mu)$ is compact.

Theorem 2 *Let $\mu \in F_c(R^m)$. Then $x_r \in \text{conv}(\mu)$ if and only if x_r is contained in a finite-dimensional fuzzy simplex $\tilde{\Delta}$ whose vertices belong to μ .*

Proof The union of all n -dimensional fuzzy simplices whose vertices belong to μ is denoted by $\bigcup_{n=1}^\infty \tilde{A}_n$ and define $K(\mu) = \bigcup_{n=1}^\infty \tilde{A}_n$. Then $K(\mu)$ is fuzzy convex. To prove this, choose $x_a, y_b \in K(\mu)$, $s_{\min\{a,b\}} \equiv \lambda x_a + (1 - \lambda) y_b$, $\lambda \in [0, 1]$.

Firstly we prove there exists a fuzzy simplex $\tilde{\Delta}_h \subseteq K(\mu)$ ($h \leq m$) such that x_a belongs to $\tilde{\Delta}_h$. If $a < K(\mu)(x)$, then by the definition of $K(\mu)$, there exists a fuzzy simplex $\tilde{\Delta}_h \subseteq K(\mu)$ such that $\tilde{\Delta}_h(x) > a$ which implies $\tilde{\Delta}_h$ is the desired simplex. If $a = K(\mu)(x)$. Then for any positive integer n , there exists an $i(n)$ -dimensional fuzzy simplex $\tilde{\Delta}_{i(n)}$ with vertices $x_{r_1(n)}^{1(n)}, x_{r_2(n)}^{2(n)}, \dots, x_{r_{i(n)+1(n)}}^{(i(n)+1)(n)}$ such that $\tilde{\Delta}_{i(n)}(x) \geq (a - 1/n)$. By Corollary 1, there are constants $\alpha(n)_j \geq 0$ ($j = 1, \dots, m + 1$) such that

$$x_{\tilde{\Delta}_{i(n)}(x)} = \sum_{j=1}^{m+1} \alpha(n)_j y(n)^j, \quad \sum_{j=1}^{m+1} \alpha(n)_j = 1, \quad y(n)^j \in \{x_{r_l(n)}^{l(n)} : l = 1, \dots, i(n) + 1\}.$$

Since the closure of $\text{supp}(\mu)$ is compact and $\alpha(n)_j$ belongs to $[0, 1]$ ($j = 1, \dots, m + 1, n = 1, 2, 3, \dots$), passing to subsequence if necessary, we may assume that the sequences $\{\alpha(n)_j y(n)^j\}_{n=1}^\infty$

$(j = 1, \dots, m + 1)$ converge to the fuzzy points $\beta_j z_{r_j}^j$ ($j = 1, \dots, m + 1$) such that

$$x_a = \sum_{j=1}^{m+1} \beta_j z_{r_j}^j, \quad \sum_{j=1}^{m+1} \beta_j = 1.$$

For the fuzzy points $z_{r_j}^j$ ($j = 1, \dots, m + 1$), without loss of generality, denote the set of all distinct fuzzy points by $B = \{z_{r_j}^j : j = 1, \dots, k + 1, k \leq m\}$. Evidently, $\text{conv}(B)$ is a fuzzy simplex $\tilde{\Delta}_h$ whose vertices belong to B , so that $x_a \in \tilde{\Delta}_h$.

Similarly, there exists a fuzzy simplex $\tilde{\Delta}_l \subseteq K(\mu)$ ($l \leq m$) with fuzzy vertices $w_{c_i}^i$ ($i = 1, \dots, l + 1$) such that $y_b \in \tilde{\Delta}_l$.

Now for $s_{\min\{a,b\}} \equiv \lambda x_a + (1 - \lambda)y_b$, one can prove by induction that $s_{\min\{a,b\}}$ is contained in a fuzzy simplex whose vertices belong to the set of fuzzy points $\{z_{r_j}^j : j = 1, \dots, k + 1, \} \cup \{w_{c_i}^i : i = 1, \dots, l + 1\}$. Hence, $s_{\min\{a,b\}} \in K(\mu)$, so that $K(\mu)$ is fuzzy convex. Consequently, $\text{conv}(\mu) \subseteq K(\mu)$. Since $\tilde{A}_n \subseteq \text{conv}(\mu)$ for each n , we have $K(\mu) \subseteq \text{conv}(\mu)$. Thus, we have $K(\mu) = \text{conv}(\mu)$. \square

Proposition 6.10 in [19] is as follows:

For any fuzzy set μ and any $\alpha \in I$ we have

$$(\text{conv } \mu)^{-1}[\alpha, 1] = \text{conv}(\mu^{-1}[\alpha, 1])$$

and

$$(\text{conv } \mu)^{-1}]\alpha, 1] = \text{conv}(\mu^{-1}]\alpha, 1]).$$

However, as shown in the following counterexample, the first equation is not true in general.

Counterexample 1 Define a fuzzy set μ of R as

$$\mu(x) = \begin{cases} 1 + x, & x \in [-1, 0), \\ 1 - x, & x \in (0, 1], \\ 0.5, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\text{conv}\mu(x) = \begin{cases} 1 + x, & x \in [-1, 0], \\ 1 - x, & x \in (0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, by a simple calculation we have $(\text{conv } \mu)^{-1}[1, 1] = (\text{conv } \mu)^{-1}\{1\} = \{0\}$, but $\text{conv}(\mu^{-1}[1, 1]) = \text{conv}(\mu^{-1}\{1\}) = \emptyset$, which contradicts Proposition 6.10 in [19].

In order to correct Proposition 6.10 in [19], we give a lemma which improves Proposition 6.3 in [19].

Lemma 5 *The convex hull of a fuzzy set of n -dimensional Euclidean space R^n is given by*

$$\text{conv}\mu(x) = \sup_{A \in D(x, n+1)} \inf\{\mu(y) : y \in A\}.$$

Proof Put

$$\widehat{\mu}(x) = \sup_{p \in \{1, \dots, n+1\}} \sup_{A \in C(x,p)} \inf\{\mu(y) : y \in A\}, \quad \hat{\mu}(x) = \sup_{A \in D(x,n+1)} \inf\{\mu(y) : y \in A\}.$$

Firstly, we prove $\widehat{\mu}(x) = \hat{\mu}(x)$. Indeed for any $A = \{x_1, \dots, x_p\} \in C(x,p)$, $p \leq n+1$, let $y_j = x_j$, $j = 1, \dots, p$. Then there are

$$\alpha_i \in I, \quad x = \sum_{i=1}^{n+1} \alpha_i x_i, \quad \sum_{i=1}^{n+1} \alpha_i = 1,$$

where $\alpha_{p+1} = \dots = \alpha_{n+1} = 0$ and $x_p = x_{p+1}, \dots, x_{n+1} = y_p$. Thence $A \in D(x, n+1)$. Conversely, for any $A = \{y_1, \dots, y_k\} \in D(x, n+1)$, there exist

$$\alpha_i \in I, \quad x = \sum_{i=1}^{n+1} \alpha_i x_i, \quad \sum_{i=1}^{n+1} \alpha_i = 1, \quad x_i \in \{y_1, \dots, y_k\}, \quad k \leq n+1.$$

Rearranging those coefficients according to y_i , we can get that there exist

$$\beta_i \in I, \quad x = \sum_{i=1}^k \beta_i y_i, \quad \sum_{i=1}^k \beta_i = 1, \quad k \leq n+1,$$

which implies $A \in C(x, k)$.

Now we prove $\widehat{\mu}(x) = \text{conv}\mu(x)$. Evidently, $\widehat{\mu}(x) \leq \text{conv}\mu(x)$. Conversely, for any $A = \{x_1, \dots, x_p\} \in C(x,p)$ with $p \geq n+2$, there are $\alpha_i \in I$ such that

$$x = \sum_{i=1}^p \alpha_i x_i, \quad \sum_{i=1}^p \alpha_i = 1.$$

By Caratheodory Theorem [20] there exist $n+1$ points in A such that the other points are the convex combinations of those points. Without loss of generality, suppose they are x_1, \dots, x_{n+1} . Thus there are $b_i(j) \in I$, $j = n+2, \dots, p$, $i = 1, \dots, n+1$ such that

$$x_j = \sum_{i=1}^{n+1} b_i(j) x_i, \quad \sum_{i=1}^{n+1} b_i(j) = 1.$$

Consequently, we have

$$x = \sum_{i=1}^{n+1} \alpha_i x_i + \sum_{j=n+2}^p \alpha_j \sum_{i=1}^{n+1} b_i(j) x_i = \sum_{i=1}^{n+1} \left(\alpha_i + \sum_{j=n+2}^p b_i(j) \alpha_j \right) x_i,$$

and

$$\sum_{i=1}^{n+1} \left(\alpha_i + \sum_{j=n+2}^p b_i(j) \alpha_j \right) = 1,$$

which implies $\{x_1, \dots, x_{n+1}\} \in C(x, n+1)$. Since $\inf\{\mu(x_i) : i = 1, \dots, p\} \leq \inf\{\mu(x_i) : i = 1, \dots, n+1\}$ and by the arbitrariness of A we have $\widehat{\mu}(x) \geq \text{conv}\mu(x)$ which completes the whole proof. \square

Now, we state the correct form of Proposition 6.10 in [19] as follows.

Theorem 3 For any fuzzy set μ and any $\alpha \in I$ we have

$$(\text{conv}\mu)^{-1}[\alpha, 1] = \text{conv}(\mu^{-1}[\alpha, 1]).$$

Furthermore, if μ belongs to $F_c(R^m)$, then $(\text{conv } \mu)^{-1}[\alpha, 1] = \text{conv}(\mu^{-1}[\alpha, 1])$.

Proof Since R. Lowen did not give the detailed proof of Proposition 6.10 in [19], we will prove the whole modified statement.

On the one hand, since $\text{conv } \mu$ is a convex fuzzy set containing μ , we have that

$$(\text{conv } \mu)^{-1}[\alpha, 1] \supseteq \text{conv}(\mu^{-1}[\alpha, 1]) \text{ for } \alpha \in I.$$

On the other hand, for any given $\alpha \in I$, let x belong to $(\text{conv } \mu)^{-1}[\alpha, 1]$. Then by Proposition 6.3 in [19] there exist $\{x_1, \dots, x_p\} \in C(x, p)$ such that

$$\inf\{\mu(x_i) : i = 1, \dots, p\} > \alpha,$$

and there are real numbers $\alpha_i \in I$, such that

$$x = \sum_{i=1}^p \alpha_i x_i, \quad \sum_{i=1}^p \alpha_i = 1,$$

which implies $x_1, \dots, x_p \in \mu^{-1}[\alpha, 1]$ and $x \in \text{conv}(\mu^{-1}[\alpha, 1])$.

Obviously, for $\alpha = 0$, $(\text{conv } \mu)^{-1}[0, 1] = \text{conv}(\mu^{-1}[0, 1]) = R^n$. Thus we will suppose $\alpha > 0$. Similarly, we can get $(\text{conv } \mu)^{-1}[\alpha, 1] \supseteq \text{conv}(\mu^{-1}[\alpha, 1])$ for $\alpha \in I$. For the converse part, let x belong to $(\text{conv } \mu)^{-1}[\alpha, 1]$. Then by Lemma 5 we have that there exist $n+1$ sequences $\{x_i^m\}_{m=1}^\infty$, $i = 1, \dots, n+1$ such that

$$x = \sum_{i=1}^{n+1} \alpha_i^m x_i^m, \quad \sum_{i=1}^{n+1} \alpha_i^m = 1,$$

and $\inf\{\mu(x_i^m) : i = 1, \dots, n+1\} \geq \alpha - 1/m$. Since the support set $\overline{\mu^{-1}[0, 1]}$ is compact and the sequences $\{\alpha_i^m\}_{m=1}^\infty$, $i = 1, \dots, n+1$ are included in $[0, 1]$, passing to subsequence if necessary, we may assume that the sequences $\{x_i^m\}_{m=1}^\infty$, $i = 1, \dots, n+1$ converge to the point x_{i0} , $i = 1, \dots, n+1$ and the sequences $\{\alpha_i^m\}_{m=1}^\infty$, $i = 1, \dots, n+1$ converge to the point α_{i0} , $i = 1, \dots, n+1$. Since μ is upper semi-continuous, we get that $\inf\{\mu(x_{i0} : i = 1, \dots, n+1)\} \geq \alpha$. Moreover, we have

$$x = \sum_{i=1}^{n+1} \alpha_{i0} x_{i0}, \quad \sum_{i=1}^{n+1} \alpha_{i0} = 1,$$

which implies $x \in \text{conv}(\mu^{-1}[\alpha, 1])$. We complete the whole proof here. \square

Remark 2 Counterexample 1 has shown that Condition (i) is indispensable and the following example will show Condition (ii) is also necessary.

Counterexample 2 Define a fuzzy set μ of R as

$$\mu(x) = \begin{cases} 1 - e^{-x}, & x \in [0, 1) \cup (1, +\infty), \\ 1, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\text{conv}\mu(x) = \begin{cases} 1 - e^{-x}, & x \in [0, 1), \\ 1, & x \in [1, +\infty), \\ 0, & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, by a simple calculation we have $(\text{conv}\mu)^{-1}[1, 1] = (\text{conv}\mu)^{-1}\{1\} = [1, +\infty)$, but $\text{conv}(\mu^{-1}[1, 1]) = \text{conv}(\mu^{-1}\{1\}) = \{1\}$.

Corollary 2 *If fuzzy set μ is closed and $\mu \in F_c(R^m)$, then $\text{conv}\mu$ is closed.*

Proof Analogous to the proof of Theorem 3. \square

Remark 3 In general, it is not true that if μ is closed, then $\text{conv}\mu$ is closed. There exist simple counterexamples for ordinary sets.

Corollary 3 *If $\tilde{\Delta}_n$ is an n -dimensional fuzzy simplex in R^m with fuzzy vertices $x_{r_i}^i$ ($i = 1, \dots, n+1$). Then for each $\alpha \in (0, 1]$, the α -cut $[\tilde{\Delta}_n]^\alpha$ is an ordinary simplex in R^m ; furthermore, for $0 < \alpha \leq \beta \leq 1$, $\dim([\tilde{\Delta}_n]^\alpha) \geq \dim([\tilde{\Delta}_n]^\beta)$, where $\dim([\tilde{\Delta}_n]^\alpha)$ denotes the dimension of the simplex $[\tilde{\Delta}_n]^\alpha$.*

Proof Since the finite union set of $n + 1$ fuzzy points $x_{r_1}^1, x_{r_2}^2, \dots, x_{r_{n+1}}^{n+1}$ belongs to $F_c(R^m)$, the conclusion follows immediately from Theorem 3 because $(\text{conv}\mu)^{-1}[\alpha, 1] = [\text{conv}\mu]^\alpha$, and $\text{conv}(\mu^{-1}[\alpha, 1]) = \text{conv}[\mu]^\alpha$ for all $\alpha \in (0, 1]$. \square

4. Conclusions

In this present investigation, fuzzy simplex has been defined and discussed. The main theorems (Theorems 1 and 2) completely characterize the fuzzy simplex and fuzzy convex hull in Euclidean space R^n , and also extend the counterparts in classical mathematics. Furthermore, we have improved the characterization theorem in [19] by Lemma 5 and have corrected Proposition 6.10 in [19] by Theorem 3. We hope that our results of the fuzzy simplex may lead to significant, new and innovative results in those related fields such as image representation and pattern recognition [21, 22].

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