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Some Properties of the Full Function Space and the Substitution Space

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Abstract In this paper, an equivalent relation among the reflexivity, weak sequential completeness and bounded completeness in full function space is given. Some results on weakly sequential compactness of subset and the property (u) in substitution spaces are obtained.

Keywords full function space; substitution space; weakly sequentially complete; bounded complete; property (u).

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1. Preliminaries

In paper [1], Leonard gave some results on weakly sequentially complete set, compact set and relatively sequentially compact set in $l^p(X)$ $(1 \leq p < \infty)$. It is well known that substitution spaces $P_B B_s$ are more general than $l^p(X)$. In this paper, we discuss the equivalent relation among the reflexivity, weakly sequentially complete and bounded complete in full function spaces, and obtain some results on weakly sequential compactness of subset and the property (u) in substitution space $P_B B_s$ that are generalization and supplement of the results in $[1, 4, 5, 13]$.

Definition 1.1 ([2]) *Let S be an index set. A full function space B is a Banach space of (real or complex) function f on S such that for each f in B, each function g satisfying $|g(s)| \leq |f(s)|$ for each $s \in S$ is again in B and $\|g\| \leq \|f\|$.*

Definition 1.2 ([2]) *Let B be a full function space on S, and for each $s \in S$, let B_s be a normed linear space. Let $P_B B_s$ be the space of all those functions x on S such that*

(1) $x(s) \in B_s$ for each $s \in S$, and

(2) if $f(s) = \|x(s)\|$ for each $s \in S$, then $f \in B$.

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If for each $x \in P_B B_s$, define $\|x\| = \|f\|$, then $P_B B_s$ is said to be the substitution space of B_s in B.

Definition 1.3 ([3]) *Let B be a full function space on S and $f \in B$, $D \subset S$, $f_D(s) = \begin{cases} f(s), & s \in D \\ 0, & s \in S \setminus D \end{cases}$. Then $f_D \in B$. If for arbitrary $\varepsilon > 0$, there exists a finite set $D \subset S$, such that $\|f - f_D\| < \varepsilon$, we say that f can be finitely approximated.*

Remark 1.1 Let B be a full function space on S. $f \in B$ can be finitely approximated, then $\{s \in S : f(s) \neq 0\}$ is at most countable.

Let B be a full function space on S, where each function $f \in B$ can be finitely approximated. Set

$$Y = \{g : \forall\ f \in B, \sum_{s \in S} g(s)f(s) \text{ is absolutely convergent}, \sup_{f \in B, \|f\|=1} |\sum_{s \in S} g(s)f(s)| < \infty\},$$

then Y is a linear space. Define

$$\|g\| = \sup_{f \in B, \|f\|=1} |\sum_{s \in S} g(s)f(s)|, \quad g \in Y.$$

Y is also a full function space.

In the following, let B be a full function space on index set S, where each function $f \in B$ can be finitely approximated, and Y is given as above.

By using the standard argument, the following representation theorems, i.e., Lemmas 1.1 and 1.2, were proved in [3].

Lemma 1.1 ([3]) *$\Phi \in B^*$ if and only if there is a unique $g \in Y$ such that $\Phi(f) = \sum_{s \in S} g(s)f(s)$ for all $f \in B$, and $\|\Phi\| = \|g\|$.*

Lemma 1.2 ([3]) *$F \in (P_B B_s)^*$ if and only if there is a unique $x^* \in P_Y B_s^*$ such that $F(x) = \sum_{s \in S} x^*(s)(x(s))$ for all $x \in P_B B_s$, and $\|F\| = \|x^*\|$.*

Definition 1.4 *A Banach space X is said to be weakly sequentially complete if for any weak-Cauchy sequence $(x_n)_{n \geq 1} \subset X$, there is $x \in X$ such that $x_n \xrightarrow{w} x$ as $n \to \infty$.*

Definition 1.5 *A full function B is said to be bounded complete if f is a function on S with $\sup\{\|f_D\| : D \subset S,\ D \text{ is finite}\} < \infty$, then $f \in B$.*

Definition 1.6 ([14]) *A series $\sum_{n=1}^{\infty} x_n$ in Banach space X is said to be a weakly unconditional Cauchy (wuC for short) series if $\sup_{\Delta \in F} \|\sum_{n \in \Delta} x_n\| < \infty$, where $F = \{\Delta \subset \mathbb{N} : \Delta \text{ is finite set}\}$. A Banach space X is said to have property (u) if there is some wuC series $\sum_{n=1}^{\infty} y_n$ satisfying $x_n - \sum_{i=1}^{n} y_i \xrightarrow{w} 0$ as $n \to \infty$ for any weak Cauchy sequence $\{x_n\} \subset X$.*

There is an equivalent description of property (u) as follows: for any weak Cauchy sequence $\{x_n\} \subset X$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a wuC series $\sum_{n=1}^{\infty} z_n$ such that $x_{n_k} - \sum_{i=1}^{k} z_i \xrightarrow{w} 0$ as $k \to \infty$.

Let X be a Banach space, $\pi(X) = \{(y_n) : \sum_{n=1}^{\infty} y_n$ is a wuC series in $X\}$ and $\beta(X) = \{v \in$

$X^{**} : v$ is a point of w^*-limit of some sequence in $X\}$. Then $\beta(X)$ is a closed subspace of X^{**} (see [15]), and $(\pi(X), |||\cdot|||)$ is a Banach space, where $|||(y_n)||| = \sup_{\Delta \in F} \|\sum_{n\in\Delta} y_n\|$ (see [16]). Define

$$T : \pi(X) \longrightarrow \beta(X), T((y_n)) = w^* - \lim_{n\to\infty} \sum_{i=1}^{n} y_i,$$

$$N(T) = \{(y_n) \in \pi(X) : T((y_n)) = 0\}$$

and

$$\hat{T} : \pi(X)/N(T) \longrightarrow \beta(X),\ \hat{T}([(y_n)]) = T((y_n)).$$

Then T is a bounded linear operator and $\|T\| = 1$. Moreover, if X has property (u), then $\hat{T}$ is one-to-one onto operator with $\|\hat{T}\| = 1$ and $u = \|\hat{T}^{-1}\|$ is called (u)-model of X. Obviously, $u \geq 1$ (see [13]).

Lemma 1.3 ([13]) *Suppose Banach space X has property (u) and u is (u)-model of X. Let $\{x_n\} \subset X$ be a weak Cauchy sequence. Then for any $\varepsilon > 0$, there exists wuC series $\sum_{n=1}^{\infty} y_n$ such that*

(1) $x_n - \sum_{i=1}^{n} y_i \xrightarrow{w} 0$ as $n \to \infty$ and

(2) $\sup_{\Delta\in F} \|\sum_{i\in\Delta} y_i\| \leq u \overline{\lim}_{n\to\infty} \|x_n\| + \varepsilon$, where $F = \{\Delta \subset \mathbb{N} : \Delta$ is a finite set$\}$.

In the following we introduce three special types of substitution spaces.

Let X be a Banach space with a basis $\{x_n\}$. $\{x_n\}$ is called a hyperorthogonal basis of X if $\sum_{n=1}^{\infty} \alpha_n x_n$ is in X, then for $|\beta_n| \leq |\alpha_n|$, $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} \beta_n x_n$ is in X and $\|\sum_{n=1}^{\infty} \beta_n x_n\| \leq \|\sum_{n=1}^{\infty} \alpha_n x_n\|$. Let $\{X_n\}$ be a sequence of Banach spaces. Set $Y = P_X X_n = \{y = (y_1, y_2, \ldots) : y_n \in X_n, n = 1, 2, \ldots, \sum_{n=1}^{\infty} \|y_n\| x_n \in X\}$ and $\|y\| = \|\sum_{n=1}^{\infty} \|y_n\| x_n\|$. By [5], we know that Banach space X with the hyperorthogonal basis is a full function space, $Y = P_X X_n$ is a special substitution space of X_n in X, and each element x in X can be finitely approximated.

Let $\{X_n\}$ be a sequence of Banach spaces. For each $p, 1 \leq p < \infty$, the direct sum of those spaces in the sense of l^p is defined as follows

$$l^p(X_n) = \left\{x = (x_1, x_2, \ldots, x_n, \ldots) : x_n \in X_n, \sum_{n=1}^{\infty} \|x_n\|^p < \infty\right\}$$

and

$$\|x\| = (\sum_{n=1}^{\infty} \|x_n\|^p)^{\frac{1}{p}}.$$

$l^p(X_n)$ $(1 \leq p \leq \infty)$ is a special substitution space, and their full function spaces are l^p. $l^p(X)$ is a special case of the $l^P(X_n)$ when $X_n = X, n = 1, 2, \ldots$.

Recently, Saito, Takahashi, et al. [6–12] introduced and studied the ψ-direction sum of $X_1, X_2, \ldots, X_n$, while $X_1, X_2, \ldots, X_n$ are Banach spaces. Let Δ_n be the $(n-1)$-simplex: $\{(s_1, s_2 \ldots, s_{n-1}) \in R_+^{n-1} : s_1 + s_2 + \cdots + s_{n-1} \leq 1\}$. Let Ψ_n be a set of all continuous convex functions on Δ_n, and $\psi \in \Psi_n$, which satisfies the following conditions:

(a) $\psi(0, 0, \ldots, 0) = \psi(0, 1, \ldots, 0) = \cdots = \psi(0, 0, \ldots, 1) = 1$;

(b) $\psi(s_1, s_2, \ldots, s_{n-1}) \geq (s_1 + s_2 + \cdots + s_{n-1}) \times \psi(\frac{s_1}{s_1+s_2+\cdots+s_{n-1}}, \ldots, \frac{s_{n-1}}{s_1+s_2+\cdots+s_{n-1}})$;

(c) $\psi(s_1, s_2, \ldots, s_{n-1}) \geq (1-s_1) \times \psi(0, \frac{s_2}{1-s_1}, \ldots, \frac{s_{n-1}}{1-s_1})$;

(d) $\psi(s_1, s_2, \ldots, s_{n-1}) \geq (1-s_{n-1}) \times \psi(\frac{s_1}{1-s_{n-1}}, \ldots, \frac{s_{n-2}}{1-s_{n-1}}, 0)$.

The ψ-direct sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is the direct sum of $X_1, X_2, \ldots, X_n$ equipped with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_\psi = \|(\|x_1\|, \|x_2\|, \ldots, \|x_n\|)\|_\psi$$

with $x_i \in X_i, 1 \leq i \leq n$. By [6–12], we know that ψ-direct sum is a very important concept by which many important geometrical properties in Banach space were obtained, and the usual l^p-sum $X \oplus_p Y$, $1 \leq p < \infty$, is its special case. In fact, let

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\}, & \text{if } p = \infty. \end{cases}$$

Then ψ_p-direct sum $X \oplus_{\psi_p} Y$ is just the l_p-sum $X \oplus_p Y$, namely

$$\|(x,y)\|_\psi = \|(x,y)\|_p = \begin{cases} \{\|x\|^p + \|y\|^p\}^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max\{\|x\|, \|y\|\}, & \text{if } p = \infty, \end{cases}$$

for $(x,y) \in X \oplus_p Y$. Dowling [6] pointed out that ψ-direct sum is a special substitution space and its full function space is $\mathbb{C}^n$.

2. Results and proofs

Let B be a full function space on index set S. We always assume that for every $s \in S$ there is $f \in B$ such that $f(s) \neq 0$. Thus $\chi_D \in B$ for any finite subset $D \subset S$. Further valuation function $\delta_s \in B^*$ and $\|\delta_s\| \leq \frac{1}{\|\chi_{\{s\}}\|}$, where $\delta_s = f(s)$, $\forall f \in B$.

The following lemma has been proved in [3]. We again give its proof for the sake of completeness.

Lemma 2.1 ([3]) *Let B be a full function space and each function $f \in B$ can be finitely approximated, and linear spanning of valuation functionals* span$\{\delta_s : s \in S\}$ *be dense in B^*. If x is in $P_B B_s$ and $\{x^n\}$ is a sequence in $P_B B_s$, then $x^n \xrightarrow{w} x$ as $n \to \infty$ if and only if*

(1) $\{\|x^n\|\}$ is bounded, and

(2) $x^n(s) \xrightarrow{w} x(s)$ as $n \to \infty$, $s \in S$.

Proof For fixed s, $p_s(x) = x(s), x \in P_B B_S$. Then

$$\|p_s(x)\| = \|x(s)\| = \frac{\|x(s)\| \, \|\chi_{\{s\}}\|}{\|\chi_{\{s\}}\|} \leq \frac{\|x\|}{\|\chi_{\{s\}}\|}.$$

Hence, "only if" part is true.

For the "if" part, let $x^* \in P_Y B_S^*$, D be a finite subset of S, and $g(s) = \|x^*(s)\|$ for each $s \in S$. Then $g \in Y$ (where Y is given as Lemma 1.1) and $\|x^* - x_D^*\| = \|x_{S \setminus D}^*\| = \|g_{S \setminus D}\|$, where $x_D^*(s) = x^*(s), s \in D$, and $x_D^*(s) = 0$, $s \in S \setminus D$. Since span $\{\delta_s : s \in S\}$ is dense in B^*, equivalently span $\{\chi_{\{s\}} : s \in S\}$ is dense in Y, then for $g \in Y$ and any $\varepsilon > 0$, there is a finite set

$D_s \subset S$ such that $\|g - \sum_{s\in D_s} \alpha(s)\chi_{\{s\}}\| < \varepsilon$. Also Y is a full function space, so

$$\|g_{S\setminus D_s}\| \le \|g - \sum_{s\in D_s} \alpha(s)\chi_{\{s\}}\| < \varepsilon.$$

Therefore, we have $\|x^* - x^*_{D_s}\| < \varepsilon$.

Let D be an arbitrary finite subset of S. Evidently, $x^*_D = \sum_{s\in D_s} \alpha(s)\chi_{\{s\}}$. By Lemma 1.2, we obtain that

$$x^*_D(x^{(n)} - x) = \sum_{s\in D} x^*(s)(x^{(n)}(s) - x(s)).$$

By (2) we have $x^*_D(x^{(n)} - x) \to 0$ as $n \to \infty$. Since $\{x^*_D : x^* \in P_Y B^*_s, D$ is an arbitrary finite subset of $S\}$ is dense in $P_Y B^*_s$, it follows from (1) that $x^{(n)} \xrightarrow{w} x$. □

Lemma 2.2 *Let B be a full function space and each function $f \in B$ can be finitely approximated, and* $\operatorname{span}\{\delta_s : s \in S\}$ *be dense in B^*. $(f^n)_{n\ge1} \subset B$ is bounded. If $\{f^n(s)\}$ is a Cauchy number sequence for each $s \in S$, then $\{f^n\}_{n\ge1}$ is a weak-Cauchy sequence.*

Proof Let $\Phi \in B^*$. Since $\operatorname{span}\{\delta_s : s \in S\}$ is dense in B^*, for each $\varepsilon > 0$, there exists $\sum_{i=1}^k \alpha_i\delta_{s_i}$ such that $\|\Phi - \sum_{i=1}^k \alpha_i\delta_{s_i}\| < \frac{\varepsilon}{3M}$, where $M = \sup_n \|f^n\|$.

Because $\{f^n(s)\}_{n\ge1}$ $(s \in S)$ is a Cauchy number sequence, we may take a large enough $N \in \mathbb{N}$ such that

$$\Big|(\sum_{i=1}^k \alpha_i\delta_{s_i})(f^m) - (\sum_{i=1}^k \alpha_i\delta_{s_i})(f^n)\Big| < \frac{\varepsilon}{3}, \quad m, n > N.$$

Therefore

$$\begin{aligned}|\Phi(f^m) - \Phi(f^n)| \le&\Big|\Phi(f^m) - (\sum_{i=1}^k \alpha_i\delta_{s_i})(f^m)\Big| + \\ &\Big|(\sum_{i=1}^k \alpha_i\delta_{s_i})(f^m) - (\sum_{i=1}^k \alpha_i\delta_{s_i})(f^n)\Big| + \Big|(\sum_{i=1}^k \alpha_i\delta_{s_i})(f^n) - \Phi(f^n)\Big| \\ \le&\|\Phi - \sum_{i=1}^k \alpha_i\delta_{s_i}\| \cdot \|f^m\| + |\sum_{i=1}^k \alpha_i(f^m(s_i) - f^n(s_i))| + \|\Phi - \sum_{i=1}^k \alpha_i\delta_{s_i}\| \cdot \|f^n\| \\ \le&\frac{\varepsilon}{3M}\sup_n \|f^n\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}\sup_n \|f^n\| = \varepsilon.\end{aligned}$$

By the arbitrariness of ε and Φ, $(f^n)_{n\ge1}$ is a w-Cauchy sequence. □

Theorem 2.1 *Let B be a full function space and each function $f \in B$ can be finitely approximated. If B is bounded complete, then B is also weakly sequentially complete.*

Proof Let $\{f^n\}_{n\ge1} \subset B$ be a w-Cauchy sequence. Then $\{\Phi(f^n)\}_{n\ge1}$ is a Cauchy number sequence for each $\Phi \in B^*$. Specially, $\{f^n(s)\}_{n\ge1}$ $(\forall s \in S)$ is convergent. Set $f(s) = \lim_n f^n(s)$, $\forall s \in S$. Then $f \in B$.

In fact, for any arbitrary finite set $D \subset S$, we have

$$\|f_D\| =\| \sum_{s\in D} f(s)\chi_{\{s\}}\| = \| \sum_{s\in D} \lim_n f^n(s)\chi_{\{s\}}\|$$

$$= \lim_n \| \sum_{s \in D} f^n(s) \chi_{\{s\}} \| = \lim_n \| f^n_D \| \leq \overline{\lim_n} \| f^n \|$$
$$\leq \sup_n \| f^n \| < \infty.$$

Since B is bounded complete, $f \in B$.

For any $g \in B^*$, then $g(h) = \sum\limits_{s \in S} g(s)h(s)$ $(h \in B)$, which is absolutely convergent. Since each $h \in B$ can be finitely approximated, we may set

$$\bigcup_{k=1}^{\infty} \{s \in S : f^k(s) \neq 0, f(s) \neq 0\} = \{s_1, s_2, \ldots\}.$$

So

$$g(f^n) = \sum_{i=1}^{\infty} g(s_i) f^n(s_i), \quad g(f) = \sum_{i=1}^{\infty} g(s_i) f(s_i).$$

Write $c^n = (g(s_1)f^n(s_1), g(s_2)f^n(s_2), \ldots) \in l_1$. For $\forall c^* = (\alpha_1, \alpha_2, \ldots) \in l_\infty$, then

$$c^*(c^k - c^l) = \sum_{i=1}^{\infty} \alpha_i g(s_i)(f^k(s_i) - f^l(s_i))$$
$$= \sum_{i=1}^{\infty} \alpha_i g(s_i)(f^k - f^l)(s_i), \quad k, l \in \mathbb{N}.$$

Denote

$$\bar{g}(t) = \begin{cases} \alpha_i g(s_i), & t = s_i, i \in \mathbb{N}, \\ 0, & t \neq s_i, i \in \mathbb{N}. \end{cases}$$

Since $g \in B^*$ and $\{\alpha_i\}$ is bounded, $\bar{g} \in B^*$. Thus $c^*(c^k - c^l) = \bar{g}(f^k - f^l)$.

According to the hypothesis, we obtain that $\{c^n\}_{n \geq 1}$ is a w-Cauchy sequence in l_1. Since l_1 is weakly sequentially complete, $c^n \xrightarrow{w} c$ as $n \to \infty$ for some $c \in l_1$. Certainly, $c^n_i \longrightarrow c_i$, as $n \to \infty$, $\forall i \in \mathbb{N}$. So $c = (g(s_1)f(s_1), g(s_2)f(s_2), \ldots)$.

Take $c^* = (1, 1, \ldots) \in l_\infty$. Then

$$g(f^n) = \sum_{i=1}^{\infty} g(s_i) f^n(s_i) = c^*(c^n) \longrightarrow c^*(c) = \sum_{i=1}^{\infty} g(s_i) f(s_i) = g(f)$$

as $n \to \infty$. By the arbitrariness of g, $f^n \xrightarrow{w} f$ as $n \to \infty$, which shows that B is weakly sequentially complete. □

Now we give two main results in this paper.

Theorem 2.2 *Let B be a full function space and each function $f \in B$ can be finitely approximated. Then the following properties of B are equivalent:*

(1) B is reflexive;

(2) B is bounded complete and $\overline{\mathrm{span}}\{\delta_s : s \in S\} = B^*$;

(3) B is weakly sequentially complete and $\overline{\mathrm{span}}\{\delta_s : s \in S\} = B^*$.

Proof (1) $\Longrightarrow$ (2). Suppose B is reflexive. Then $\overline{\mathrm{span}}\{\delta_s : s \in S\} = B^*$. Indeed, if $\Phi \in B^* \backslash \overline{\mathrm{span}}\{\delta_s : s \in S\}$, then there exists $\hat{f} = J(f) \in B^{**} = J(B)$ such that $\Phi(f) = 1$, $\Psi(f) = 0$ for any $\Psi \in \overline{\mathrm{span}}\{\delta_s : s \in S\}$.

Let $\{s \in S : f(s) \neq 0\} = \{s_1, s_2, \ldots\}$. By Lemma 1.1, there exists $g \in Y$ such that $\Phi(f) = \sum_{i=1}^{\infty} g(s_i) f(s_i)$. But

$$\sum_{i=1}^{\infty} g(s_i) f(s_i) = \sum_{i=1}^{\infty} \left(g(s_i)\delta_{s_i}\right)(f) = 0,$$

i.e., $\Phi(f) = 0$, a contradiction. This means $\overline{\text{span}}\{\delta_s : s \in S\} = B^*$.

Let f be a function on S with $\sup\{\|f_D\| : D \subset S,\ D \text{ is finite set}\} < \infty$. Define $D_1 \preceq D_2 \Leftrightarrow D_1 \subset D_2$. Then $\Gamma = \{D : D \subset S,\ D \text{ is finite set}\}$ with partial order $\preceq$ is a directed set and $\{f_D : D \in \Gamma\}$ is a net of B.

Since B is reflexive, there exists a subnet $\{f_{D'} : D' \in \Gamma'\}$ and $g \in B$ such that $f_{D'} \xrightarrow{w} g$, where Γ' is a cofinal subset of Γ. Then $f_{D'}(s) \longrightarrow g(s)$ $(s \in S)$. Given $s \in S$, denote $D = \{s\}$. Then there is $D_\alpha \in \Gamma'$ satisfying $D_\alpha \succeq D$, i.e., $s \in D_\alpha$. Whenever $D' \succeq D_\alpha$, $f_{D'}(s) = f(s)$. Therefore $f(s) = g(s)$. That is $f = g \in B$. Thus B is bounded complete.

(2)$\Longrightarrow$ (3). It is obtained by Theorem 2.1.

(3)$\Longrightarrow$ (1). Assume $\{f_k\}_{k\geq 1} \subset U(B)$. Let

$$\bigcup_{k=1}^{\infty} \{s \in S : f_k(s) \neq 0\} = \{s_1, s_2, \ldots\}.$$

By the diagonal method, we can choose a subsequence $\{f_{k_j}\}_{j\geq 1} \subset \{f_k\}_{k\geq 1}$ such that

$$\lim_{j\to\infty} f_{k_j}(s_i) = \alpha_i, \quad \forall i \in \mathbb{N}.$$

According to the condition $\overline{\text{span}}\{\delta_s : s \in S\} = B^*$, $\{f_{k_j}\}$ is a weak-Cauchy sequence by Lemma 2.2. So, there is $f \in B$ such that $f_{k_j} \xrightarrow{w} f$ as $j \to \infty$, and $\|f\| \leq 1$, which shows B is reflexive. □

Theorem 2.3 *Let B be a full function space and each function $f \in B$ can be finitely approximated. If B is weakly sequentially complete and $\overline{\text{span}}\{\delta_s : s \in S\} = B^*$. Then $K \subset P_B B_s$ is relatively weakly sequentially campact if and only if*

(1) K is bounded;

(2) $T_s(K)$ is relatively weakly sequentially compact in B_s for any $s \in S$, where $T_s : P_B B_s \to B_s$, $T_s(y) = y(s)$, $\forall y \in P_B B_s$.

Proof We only prove the sufficiency.

Let $K \subset P_B B_s$ be bounded and $\{y^k\}_{k\geq 1} \subset K$. Set

$$\bigcup_{k=1}^{\infty} \{s \in S,\ y^k(s) \neq 0\} = \{s_1, s_2, \ldots\},$$

then $\{y^k(s_j)\}_{k\geq 1} \subset B_{s_j}$ $(j \in \mathbb{N})$ is relatively weakly sequentially campact. We may assume that $y^k(s_j) \xrightarrow{w} y_{s_j} \in B_{s_j}$ as $k \to \infty$ for any $i \in \mathbb{N}$ (otherwise, we can choose a subsequence of $\{y^k\}_{k\geq 1}$ by the diagonal method). Then

$$\|y_{s_j}\| \leq \varliminf_{k} \|y^k(s_j)\|, \quad j \in \mathbb{N}.$$

Since $\{\|y^k\|\}_{k\geq 1}$ is bounded, there exist $\{y^{k_i}\}_{i\geq 1} \subset \{y^k\}_{k\geq 1}$ and sequence of numbers

$(\alpha_{s_j})_{j\geq 1}$ such that $\lim_{i\to\infty}\|y^{k_i}(s_j)\| = \alpha_{s_j}$ for any $j\in\mathbb{N}$. Denoting

$$g(t) = \begin{cases} \alpha_{s_j}, & t = s_j, j\in\mathbb{N}, \\ 0, & t\neq s_j, j\in\mathbb{N}, \end{cases}$$

and $f^{k_i}(s) = \|y^{k_i}(s)\|$, we have

$$f^{k_i}(s) \longrightarrow g(s) \text{ as } i\to\infty, \quad \forall s\in S.$$

According to the hypothesis that $\overline{\text{span}}\{\delta_s : s\in S\} = B^*$ and the weak sequential completeness of B, we obtain that $f^{k_i} \xrightarrow{w} g\in B$ as $i\to\infty$. Let

$$y(t) = \begin{cases} y_{s_j}, & t = s_j, j\in\mathbb{N}, \\ 0, & t\neq s_j, j\in\mathbb{N}, \end{cases}$$

and $f(s) = \|y(s)\|$. Then $|f(s)|\leq |g(s)|$ by (1). So $f\in B$ and $y\in P_B B_s$. In virtue of Lemma 2.1, $y^k \xrightarrow{w} y$ as $k\to\infty$. □

By Theorems 2.1 and 2.3, we have

Corollary 2.1 ([3]) *Let B be a full function space and each function $f\in B$ can be finitely approximated. If B is bounded complete and $\overline{\text{span}}\{\delta_s : s\in S\} = B^*$. Then $K\subset P_B B_s$ is relatively weakly sequentially compact if and only if*

(1) K is bounded;

(2) $T_s(K)$ is relatively weakly sequentially compact in B_s for any $s\in S$.

By Theorem 2.3, we can also obtain the following result.

Theorem 2.4 *Let B be a reflexive full function space and each function $f\in B$ can be finitely approximated. Then $K\subset P_B B_s$ is relatively weakly sequentially compact if and only if*

(1) K is bounded;

(2) $T_s(K)$ is relatively weakly sequentially compact in B_s for any $s\in S$, where $T_s : P_B B_s\to B_s$, $T_s(y) = y(s)$, $\forall y\in P_B B_S$.

By Theorem 2.4, we can obtain the following three corollaries.

Corollary 2.2 ([4]) *Let X be a reflexive Banach space with the hyperorthogonal basis and $\{X_n\}$ be a sequence of Banach space. $P_X X_n$ is the substitution space of X_n in X. Then $K\subset P_X X_n$ is relatively weakly sequentially compact if and only if*

(1) K is bounded;

(2) $T_n(K)$ is relatively weakly sequentially compact in X_n for any $n\in\mathbb{N}$, where $T_n : P_X X_n\to X_n$, $T_n(x) = x_n$, $\forall x = (x_1, x_2, \ldots, x_n, \ldots)\in P_X X_n$.

Corollary 2.3 *Let X_n be a Banach space, $n = 1, 2, \ldots, 1\leq p<\infty$. Then a subset $K\subset l^p(X_n)$ is relatively weakly sequentially compact if and only if*

(1) K is bounded;

(2) $T_n(K)$ is relatively weakly sequentially compact in X_n for any $n\in\mathbb{N}$, where $T_n : l^p(X_n)\to X_n$, $T_n(x) = x_n$, $\forall x = (x_1, x_2, \ldots, x_n, \ldots)\in l^p(X_n)$.

In particular, the subset $K\subset l^p(X)$ is relatively weakly sequentially compact if and only if

(1) K is bounded;

(2) $T_n(K)$ is relatively weakly sequentially compact in X, where $T_n : l^p(X) \to X$, $T_n(x) = x_n$, $\forall x = (x_1, x_2, \dots, x_n, \dots) \in l^p(X)$ (see [1]).

Corollary 2.4 *Let $X_1, X_2, \dots, X_n$ be Banach spaces, and $\psi \in \Psi_n$. Then $K \subset (X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$, where $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is a ψ-direct sum, is a relatively weakly sequentially compact set if and only if*

(1) K is bounded;

(2) $T_i(K)$ is relatively weakly sequentially compact in X_i, $i = 1, 2, \dots, n$, where $T_i : (X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi \to X_i$, $T_i(x) = x_i$, $\forall x \in (X_1 \oplus X_2 \oplus \cdots \oplus X_n)$.

Theorem 2.5 *Let B be a full function space on S and each function $f \in B$ can be finitely approximated. If $P_B B_s$ has property (u), then B_s also has property (u) for every $s \in S$.*

Proof Let $s \in S$, $\{x_s^n\}_{n \geq 1}$ be weak Cauchy sequence in B_s. Set

$$x^n(t) = \begin{cases} x_s^n, & t = s, \\ 0, & t \neq s, \end{cases}$$

then $x^n \in P_B B_s (\forall n \in \mathbb{N})$ and $\{x^n\}$ is weak Cauchy sequence.

Since $P_B B_s$ has property (u), there exists wuC series $\sum_{i=1}^{\infty} y^i$ in $P_B B_s$ such that $x^n - \sum_{i=1}^{n} y^i \xrightarrow{w} 0 (n \to \infty)$. Therefore $x_s^n - \sum_{i=1}^{n} y^i(s) \xrightarrow{w} 0$ as $n \to \infty$. Also

$$\sup_{\Delta \in F} \| \sum_{i \in \Delta} y^i(s) \| = \sup_{\Delta \in F} \frac{\| \sum\limits_{i \in \Delta} y^i(s) \| \| \chi_{\{s\}} \|}{\| \chi_{\{s\}} \|} \leq \frac{1}{\| \chi_{\{s\}} \|} \sup_{\Delta \in F} \| \sum_{i \in \Delta} y^i \| < \infty,$$

so $\sum_{i=1}^{\infty} y^i(s)$ is wuC series in B_s.

Under certain conditions, if every B_s has property (u), then $P_B B_s$ also has property (u). □

Theorem 2.6 *Let B be a bounded complete full function space and each function $f \in B$ can be finitely approximated, and $\overline{\mathrm{span}}\{\delta_s : s \in S\} = B^*$. If B_s has property (u) for each $s \in S$, and $c = \sup_{s \in S} u_s < \infty$, then $P_B B_s$ has property (u), where u_s is (u)-model of B_s.*

Proof Let $\{x^n\}_{n=1}^{\infty} \subset P_B B_s$ be a weak Cauchy sequence. Then $\{x^n\}$ is bounded. Set

$$\bigcup_{k=1}^{\infty} \{s \in S,\ x^n(s) \neq 0\} = \{s_1, s_2, \dots\},$$

then $\{x^n(s_j)\}_{n \geq 1} \subset B_{s_j}$ $(\forall j \in \mathbb{N})$ is bounded. Take a subsequence $\{x^{n_k}\} \subset \{x^n\}$ by the diagonal method such that $\|x^{n_k}(s_i)\|$ is convergent for any $i \in \mathbb{N}$. For any $i \in \mathbb{N}$, since $\{x^{n_k}(s_i)\}_{k=1}^{\infty}$ is weak Cauchy sequence in B_{s_i} and B_{s_i} has property (u), by Lemma 1.3 there exists wuC series $\sum_{j=1}^{\infty} y_{s_i}^j$ in B_{s_i} satisfying

(a) $x^{n_k}(s_i) - \sum_{j=1}^{k} y_{s_i}^j \xrightarrow{w} 0$ as $k \to \infty$,

(b) $\sup_{\Delta \in F} \| \sum_{j \in \Delta} y_{s_i}^j \| \leq u_{s_i} \overline{\lim}_{k \to \infty} \|x^{n_k}(s_i)\| + \frac{1}{i^2 \|\chi_{\{s_i\}}\|} \leq c \lim_{k \to \infty} \|x^{n_k}(s_i)\| + \frac{1}{i^2 \|\chi_{\{s_i\}}\|}$.

Set

$$y^k(t) = \begin{cases} y_{s_i}^k, & t = s_i, i \in \mathbb{N}, \\ 0, & t \neq s_i, i \in \mathbb{N}, \end{cases}$$

then $y^k \in P_B B_s$ $(\forall k \in \mathbb{N})$ and $\sum_{j=1}^{\infty} y^j$ is a wuC series. In fact, denote $f^{n_k}(t) = \|x^{n_k}(t)\|$ $(\forall t \in S)$, then $f^{n_k} \in B$ and $\|f^{n_k}\| = \|x^{n_k}\| \le \sup_n \|x^n\| < \infty$. Let D be any finite subset in S. Then

$$\sup_{\Delta\in F}\Big\|\sum_{s_i\in D\cap\{s_1,s_2,\ldots\}}\|\sum_{j\in\Delta} y^j(s_i)\|\chi_{\{s_i\}}\Big\| \le \Big\|\sum_{s_i\in D\cap\{s_1,s_2,\ldots\}}\big(c\lim_{k\to\infty}\|x^{n_k}(s_i)\| + \frac{1}{i^2\|\chi_{\{s_i\}}\|}\big)\chi_{\{s_i\}}\Big\|$$

$$\le c\lim_{k\to\infty}\Big\|\sum_{s_i\in D\cap\{s_1,s_2,\ldots\}}\|x^{n_k}(s_i)\|\chi_{\{s_i\}}\Big\| + \sum_{i=1}^{\infty}\frac{1}{i^2} = c\lim_{k\to\infty}\|f_D^{n_k}\| + \sum_{i=1}^{\infty}\frac{1}{i^2}$$

$$\le c\sup_n\|f^n\| + \sum_{i=1}^{\infty}\frac{1}{i^2} < \infty.$$

Specially, for any $j \in \mathbb{N}$, taking $g^j(s) = \|y^j(s)\|$ $(\forall s \in S)$, we have

$$\|g_D^j\| = \Big\|\sum_{s_i\in D\cap\{s_1,s_2,\ldots\}}\|y^j(s_i)\|\chi_{\{s_i\}}\Big\| \le c\sup_n\|f^n\| + \sum_{i=1}^{\infty}\frac{1}{i^2} < \infty.$$

By the hypothesis that B is bounded complete and the arbitrariness of D, $g^j \in B$ and $y^j \in P_B B_s$.

Notice that for any $n \in \mathbb{N}$,

$$\sup_{\Delta\in F}\Big\|\sum_{i=1}^{n}\|\sum_{j\in\Delta} y^j(s_i)\|\chi_{\{s_i\}}\Big\| < c\sup_n\|f^n\| + \sum_{i=1}^{n}\frac{1}{i^2} < \infty,$$

we obtain that

$$\sup_{\Delta\in F}\|\sum_{j\in\Delta} y^j\| = \sup_{\Delta\in F}\Big\|\sum_{i=1}^{\infty}\|\sum_{j\in\Delta} y^j(s_i)\|\chi_{\{s_i\}}\Big\| < \infty.$$

Thus $\sum_j y^j$ is a wuC series in $P_B B_s$.

In order to complete the proof of the theorem, we only need to show that $x^{n_k} - \sum_{j=1}^{k} y^j \xrightarrow{w} 0$ as $k \to \infty$. Since $x^{n_k}(s) - \sum_{j=1}^{k} y^j(s) \xrightarrow{w} 0$ as $k \to \infty$ for any $s \in S$, $x^{n_k} - \sum_{j=1}^{k} y^j \xrightarrow{w} 0$ as $k \to \infty$ by Lemma 2.1. □

Remark 2.1 By Theorems 2.5 and 2.6, we can also obtain some corresponding corollaries in the mentioned three special types of substitution spaces in Section 1.

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