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On Ordered Ideals in Ordered Semirings

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Abstract An ordered semiring is a semiring S equipped with a partial order $\leq$ such that the operations are monotonic and constant 0 is the least element of S. In this paper, several notions, for example, ordered ideal, minimal ideal, and maximal ideal of an ordered semiring, simple ordered semirings, etc., are introduced. Some properties of them are given and characterizations for minimal ideals are established. Also, the matrix semiring over an ordered semiring is considered. Partial results obtained in this paper are analogous to the corresponding ones on ordered semigroups, and on the matrix semiring over a semiring.

Keywords ordered semiring; ordered ideal; simple semiring; minimal ideal; matrix.

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1. Introduction

A semiring is an algebra $S = (S, +, \cdot, 0)$ equipped with binary operations $+$ (sum or addition) and $\cdot$ (product or multiplication) and constant 0 such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot)$ is a semigroup and multiplication distributes over addition from both sides. Thus,

$$(a+b)c = ac+bc,$$

$$c(a+b) = ca+cb,$$

$$a0 = 0a = 0$$

hold for all $a,\ b,\ c \in S$.

A semiring S is called an antiring if it is zerosumfree, i.e., if the condition $a+b=0$ implies that $a=b=0$ for all $a,\ b \in S$.

A semiring S is called entire if $ab=0$ implies that $a=0$ or $b=0$.

An ordered semiring is a semiring S equipped with a partial order $\leq$ such that the operations are monotonic and the constant 0 is the least element of S.

If S is an ordered semiring and $a,\ b \in S$ such that $a+b=0$, then $a = a+0 \leq a+b = 0$ and so $a=0$. Similarly, $b=0$. Therefore, any ordered semiring is an antiring.

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A morphism of semirings is a function that preserves the operations and the constant 0. A morphism of ordered semirings also preserves the partial order.

A subsemiring of a semiring S is a nonempty subset of S which is closed under the operations and constant 0. A left (resp. right) ideal of a semiring S is a subsemiring I of S such that $s \in S$ and $x \in I$ implies $sx \in I$ (resp. $xs \in I$). I is an ideal if it is both a left and a right ideal.

Let H be a nonempty subset of S. Then the set $\{x \in S : x \leq h$ for some $h \in H\}$ is denoted by the notation $(H]$. For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$. It is clear that $H \subseteq (H]$, $((H]] = (H]$ and $A \subseteq B \Rightarrow (A] \subseteq (B]$ for any nonempty subset A, B of S.

We refer to [1–4] for all background information concerning semirings, semigroups, and universal algebra.

2. Ordered ideals of ordered semirings

In this section, we define our main concept, ordered ideals of ordered semirings, and establish some of their elementary properties.

From this section to the third section, S stands for an arbitrary ordered semiring.

Definition 1 *A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e., $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.*

Remark 1 It is clear that $\{0\}$ and S itself are ordered ideal of S. An ordered ideal I of S such that $I \neq \{0\}$ and $I \neq S$ is called a proper ordered ideal.

Example 1 Let $L = ([0,1], \vee, \cdot, 0)$, where $[0,1]$ is the unit interval, $a \vee b = \max\{a, b\}$ for $a,\ b \in [0,1]$ and $a \cdot b = (a + b - 1) \vee 0$ for $a,\ b \in [0,1]$. It is easy to verify that L equipped with the usual ordering $\leq$, is an ordered semiring. Let $I = [0, \frac{1}{2}]$. It is not difficult to verify that I is an ordered ideal of L.

Proposition 1 *Let $f : R \to S$ be a morphism of ordered semirings. Then $K = \{x \in R : f(x) = 0\}$ is an ordered ideal of R.*

Proof Clearly K is nonempty since $0 \in K$. Suppose that $r \in R$ and $x, y \in K$. Then we have

$$f(rx) = f(r)f(x) = f(r)0 = 0,\ f(xr) = f(x)f(r) = 0f(r) = 0,$$

and

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0.$$

Hence $rx,\ xr,\ x + y \in K$. In addition, if $a \leq x$ for some $x \in K$, then $f(a) \leq f(x) = 0$ since f is a morphism. Thus $f(a) = 0$ as 0 is the least element of S. It follows that $a \in K$. This completes the proof. □

Lemma 1 *Let $\{I_\lambda, \lambda \in \wedge\}$ be a family of ordered ideals of S. Then $\bigcap_{\lambda \in \wedge} I_\lambda$ is also an ordered ideal of S.*

Proof Clearly $\bigcap_{\lambda \in \wedge} I_\lambda$ is an ideal of S (see [2]). Suppose that $a \leq x$ for some $x \in \bigcap_{\lambda \in \wedge} I_\lambda$.

Since $x \in I_\lambda$ and I_λ is an ordered ideal of S, we have $a \in I_\lambda$ for each $\lambda \in \wedge$. Thus $a \in \bigcap_{\lambda\in\wedge} I_\lambda$. This completes the proof. □

The intersection of all ordered ideals of S containing a nonempty subset A of S is the ordered ideal of S generated by A, denoted by $L(A)$. For $A = \{a\}$, we denote by $L(a)$ the ordered ideal of S generated by a ($a \in S$).

Proposition 2 *Let I be an ideal of S. Then $(I]$ is an ordered ideal of S generated by I.*

Proof Clearly $(I]$ is nonempty since $I \subseteq (I]$. Suppose that $r \in S$ and $a, b \in (I]$. Then there exist $x, y \in I$ such that $a \le x$ and $b \le y$ by the definition of $(I]$. Since S is an ordered semiring and I is an ideal of S, we have $ra \le rx \in I, ar \le xr \in I$ and $a + b \le x + y \in I$. It follows that $ra \in (I], ar \in (I]$ and $a + b \in (I]$. As for $((I]] \subseteq (I]$, it is clear. Hence $(I]$ is an ordered ideal of S containing I. Moreover, if J is an arbitrary ordered ideal of S containing I, then $(I] \subseteq (J] \subseteq J$. Thus $(I]$ is the least ordered ideal of S containing I. That is to say, $(I]$ is an ordered ideal of S generated by I, as required. □

Corollary 1 *For any $a \in S$, $L(a) = (A]$, where $A = \{\sum_{fin}(n_i a + x_i a + a y_i + u_i a v_i : n_i$ are nonnegative integers, $x_i, y_i, u_i, v_i \in S\}$.*

Proof It is easy to verify that A is an ideal of S generated by a (see [2]). So by Proposition 2, we have $(A]$ is an ordered ideal of S containing a. Thus $L(a) \subseteq (A]$. On the other hand, $L(a)$ is also an ideal of S containing a, so we have $A \subseteq L(a)$. Thus $(A] \subseteq L(a)$ since $(A]$ is an ordered ideal of S generated by A. Therefore $L(a) = (A]$, as required. □

Corollary 2 *Let A, B be ideals of S. Then $L(A \cup B) = (A + B]$, where $A + B = \{a + b : a \in A, b \in B\}$.*

Proof It is easy to verify that $A+B$ is an ideal of S generated by $A\cup B$ (see [2]). So by Proposition 2, we have $(A + B]$ is an ordered ideal of S containing $A \cup B$. Thus $L(A \cup B) \subseteq (A + B]$. On the other hand, $L(A \cup B)$ is also an ideal of S containing $A \cup B$, so we have $A + B \subseteq L(A \cup B)$. It follows that $(A + B] \subseteq L(A \cup B)$ since $(A + B]$ is an ordered ideal of S generated by $A + B$. Therefore $L(A \cup B) = (A + B]$. □

Corollary 3 *Let A, B, C be ordered ideals of S. Then $(C\cap A+C\cap B] \subseteq C\cap(A+B]$. Moreover, if $x \le y+z$ implies $x \le y$ or $x \le z$ for all x, y, z in the poset $(S, \le)$, then $(C\cap A+C\cap B] = C\cap(A+B]$.*

Proof It is easy to verify that $C \cap (A + B]$ is an ordered ideal of S containing $C \cap A$ and $C \cap B$ by Lemma 1 and Proposition 2. While by Proposition 2, we have $(C \cap A + C \cap B] = L((C \cap A) \cup (C \cap B))$. So $(C \cap A + C \cap B] \subseteq C \cap (A + B]$.

As for the second claim, suppose that $x \in C \cap (A + B]$. Then $x \in C$ and $x \le a + b$ for some $a \in A$ and $b \in B$. By hypothesis, we have $x \le a$ or $x \le b$. Thus $x \in C \cap A$ or $x \in C \cap B$. It follows that $x \in (C \cap A) \cup (C \cap B) \subseteq ((C \cap A) + (C \cap B)]$. Thus $C \cap (A + B] \subseteq (C \cap A + C \cap B]$ and so $(C \cap A + C \cap B] = C \cap (A + B]$, as required. □

Remark 2 In general, the equation $(C \cap A + C \cap B] = C \cap (A + B]$ does not hold in an arbitrary ordered semiring.

Example 2 Let $(N_5 = \{0, 1, a, b, c\}, \ \leq)$ be the pentagon lattice depicted in Figure 1. We define two binary operations $+$(addition) and $\cdot$(multiplication) on N_5 as follows, $x + y = x \vee y$ and $x \cdot y = 0$ for all x, y in N_5. It is not difficult to verify that $(N_5, +, \cdot, 0, \leq)$ is an ordered semiring. Let $A = \{0, a\}$, $B = \{0, b\}$, $C = \{0, b, c\}$. Then it is easy to verify that A, B, C are ordered ideals of N_5, and $(A + B] = N_5$. So $(C \cap A + C \cap B] = (C \cap B] = B \neq C = C \cap (A + B]$.

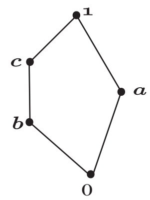

Figure 1 The 'pentagon' lattice N_5

Proposition 3 *Let $I(S) = \{J : J$ is an ordered ideal of $S\}$. Then $I(S)$, with $\subseteq$ as the partial ordering, is a complete lattice.*

Proof It is obvious that $(I(S), \subseteq)$ is a poset. In addition, for any I and J in $I(S)$, by Lemma 1 and Corollary 2, we have $I \wedge J = I \cap J$ and $I \vee J = (I + J]$. Thus $(I(S), \subseteq)$ is a lattice. Moreover, it is clear that $\{0\}$ is the smallest element and S is the largest element in the poset $(I(S), \subseteq)$. Again by Lemma 1, $I(S)$ is closed under arbitrary intersection. Thus $(I(S), \subseteq)$ is a complete lattice. $\Box$

3. Minimal ideals and simple ordered semirings

An ordered semiring S is called simple if it does not contain proper ordered ideals. A proper ordered ideal I of S is called minimal if there is no proper ordered ideal J of S such that $J \subseteq I$. Equivalently, if for any ordered ideal J of S such that $J \subseteq I$, then we have $J = I$ or $J = \{0\}$.

Definition 2 *A subsemiring K of S is called simple, if the ordered semiring $(K, +, \cdot, 0, \leq)$ is simple.*

Remark 3 That an ordered semiring S is simple does not imply that a subsemiring of S is also simple.

Example 3 Let $L = ([0, +\infty), \vee, \cdot, 0)$, where $\vee = \max$ and $\cdot$ is the usual multiplication of real numbers. Let $R = [0, 1]$. It is easy to verify that L equipped with the usual ordering $\leq$, is an ordered semiring and simple; and R is a subsemiring of S but not an ideal. Also R is not simple since $T = [0, 1)$ is a proper ordered ideal of R.

Lemma 2 *S is simple if and only if $L(a) = S$ for all $a \in S$ and $a \neq 0$.*

Lemma 3 *Let I be an ordered ideal of S and K be a simple subsemiring. If $K \cap I \neq \{0\}$, then $K \subseteq I$.*

Proof Let $0 \neq a \in K \cap I$. Since K is simple and $0 \neq a \in K$, we have $L_K(a) = K$ by Lemma 2, where $L_K(a)$ denotes the ordered ideal of K generated by a, that is, $L_K(a) = (A]$, where $A = \{\sum_{fin}(n_i a + x_i a + a y_i + u_i a v_i : n_i$ are nonnegative integers, $x_i, y_i, u_i, v_i \in K\}$. Therefore $K = L_K(a) \subseteq L(a) \subseteq I$ since $a \in I$, as required. □

Theorem 1 *Let S be an ordered semiring and entire. Then an ordered ideal of S is minimal if and only if it is simple.*

Proof $\Rightarrow$. Let I be a minimal ordered ideal of S, and $J \neq \{0\}$ be an ordered ideal of I. Let $H = \{h \in J : h \leq \sum_{\text{fin}} k_i a_i l_i$ for some $k_i, l_i \in I$ and $a_i \in J\}$. Then $H \subseteq J \subseteq I$.

In the following, we will prove that H is an ordered ideal of S and $H \neq \{0\}$. We shall prove it in three steps.

(1) Since $I \neq \{0\}$ and $J \neq \{0\}$, we can choose $0 \neq a \in I$ and $0 \neq b \in J$. Thus $0 \neq aba \in H$ since S is entire. It follows that $H \neq \{0\}$.

(2) H is an ideal of S. In fact, suppose that $s \in S$ and $x,\ y \in H$. We have $x \leq \sum_{\text{fin}} k_i a_i l_i$ and $y \leq \sum_{\text{fin}} u_j b_j v_j$ for some $k_i, l_i, u_j, v_j \in I$ and $a_i, b_j \in J$. Then

$$x + y \leq \sum_{\text{fin}}(k_i a_i l_i + u_j b_j v_i),$$

and

$$sx \leq s\sum_{\text{fin}} k_i a_i l_i = \sum_{\text{fin}}(sk_i)a_i l_i,\ xs \leq (\sum_{\text{fin}} k_i a_i l_i)s = \sum_{\text{fin}} k_i a_i (l_i s).$$

We immediately have $x + y \in H$. At the same time, since $sx \in SH \subseteq SI \subseteq I$, $\sum_{\text{fin}}(sk_i)a_i l_i \in IJI \subseteq J$ and J is an ordered ideal of I, we have $sx \in J$. Moreover, since $sk_i \in SI \subseteq I$, $a_i \in J$ and $l_i \in I$, we get $sx \in H$. A similar argument shows that $xs \in H$. Thus H is an ideal of S.

(3) $(H] \subseteq H$. Indeed, suppose that $x \leq h$ for some $h \in H$. Since $h \in H$, we have $h \in J$ and $h \leq \sum_{\text{fin}} k_i a_i l_i$ for some $k_i, l_i \in I$ and $a_i \in J$. Thus $x \leq \sum_{\text{fin}} k_i a_i l_i \in SJS \subseteq SIS \subseteq I$. It follows that $x \in I$ since I is an ordered ideal of S. As $x \leq \sum_{\text{fin}} k_i a_i l_i \in IJI \subseteq J$, $x \in I$ and J is an ordered ideal of I, we have $x \in J$. Then since $x \in J$, $x \leq \sum_{\text{fin}} k_i a_i l_i$, $k_i, l_i \in I$ and $a_i \in J$, we have $x \in H$.

Thus H is an ordered ideal of S and $H \neq \{0\}$. Since I is a minimal ideal of S, we have $H = I$ and so $J = I$. Therefore I is simple.

$\Leftarrow$. Let K be an ordered ideal of S and simple. Let J be an ordered ideal of S and $J \neq \{0\}$ such that $J \subseteq K$. Then J is an ordered ideal of K. Since K is simple, we have $J = K$. Therefore, K is a minimal ordered ideal of S. This completes the proof. □

Theorem 2 *Let S be an ordered semiring and entire. Assume that S has proper ordered ideals and the equation $(C \cap A + C \cap B] = C \cap (A + B]$ holds for any ordered ideals $A,\ B,\ C$ of S. Then every proper ordered ideal of S is minimal (simple) if and only if S contains exactly one proper ordered ideal or S contains exactly two proper ordered ideals $J,\ K$ such that $S = (J + K]$.*

Proof $\Rightarrow$. Let J be a proper ordered ideal of S. By hypothesis, J is a minimal ordered ideal of S. Then we have the following two cases:

(1) $S = L(a),\ \forall a \in S \backslash J$ ($S \backslash J$ is the complement of J in S).

Suppose that K is also a proper ordered ideal of S and $K \neq J$. If $K\backslash J = \emptyset$, then $K \subseteq J$ and so $K \subset J$, which is impossible (since J is minimal). If $K\backslash J \neq \emptyset$, then $S = L(a) \subseteq K$, for some $a \in K\backslash J \subseteq S\backslash J$ and so $S = K$. Contradiction. Thus, in this case, J is the unique proper ordered ideal of S.

(2) $S \neq L(a)$, for some $a \in S\backslash J$.

Then $L(a)$ is a proper ideal of S. By hypothesis, $L(a)$ is minimal. By Corollary 2, $(L(a)+J]$ is an ordered ideal of S. Assume that $(L(a)+J] \neq S$. Then $(L(a)+J]$ is a proper ordered ideal of S. By hypothesis $(L(a)+J]$ is a minimal ordered ideal of S. On the other hand $J \subset (L(a)+J]$. We get a contradiction. Thus $(L(a)+J] = S$.

Let K be an arbitrary ordered ideal of S. By hypothesis, K is minimal. Since $K = K \cap S = K \cap (L(a)+J] = (K \cap L(a) + K \cap J]$, we have the following two cases:

(a) $K \cap J \neq \{0\}$. Then $K \cap J$ is an ordered ideal of S by Lemma 1. Since $K \cap J \subseteq J$, J is minimal, we have $K \cap J = J$, i.e., $J \subseteq K$. Since K is minimal, we have $J = K$.

(b) $K \cap L(a) \neq \{0\}$. A similar argument shows that $K = L(a)$.

Therefore, in this case, S contains exactly two proper ideals J and $L(a)$ such that $S = (L(a)+J]$.

$\Leftarrow$. If S contains exactly one proper ordered ideal J, it is obvious that J is minimal. Suppose that S contains exactly two proper ordered ideals J and K such that $S = (J+K]$. Then $J \nsubseteq K$ and $K \nsubseteq J$. Otherwise $S \neq (J+K]$. Let $I \neq \{0\}$ be an ordered ideal of S such that $I \subseteq J$. Then $I \subseteq J \subset S$, and so I is a proper ordered ideal of S. Since $I \subseteq J$ and $K \nsubseteq J$, we have $I \neq K$. Since S contains exactly two proper ordered ideals J, K, we have $I = J$. Thus J is minimal. A similar argument shows that K is minimal. $\square$

4. Matrices

Let S be a semiring with identity element 1, i.e., $(S, \cdot, 1)$ is a monoid. We denote by $M_{m\times n}(S)$ the set of all $m \times n$ matrices over S. Especially, we denote by $M_n(S)$ the set of all square matrices of order n over S.

For $A \in M_{m\times n}(S)$, we denote by a_{ij} or A_{ij} the element of S corresponding to the (i,j)th entry of A. For convenience, we use $\underline{n}$ to denote the set $\{1, 2, \ldots, n\}$ for any positive integer n. If $m = n$ and $a_{ij} = 0$ for all $i, j \in \underline{n}$, then A is called the zero matrix and denoted by 0_n. If $a_{ij} = 0$ for all i and j provided that $i \neq j$ and $a_{ii} = 1$ for all $i \in \underline{n}$, then A is called the identity matrix and denoted by I_n.

Given $A, B \in M_{m\times n}(S)$ and $C \in M_{n\times l}(S)$, we define:

$$A + B = (a_{ij} + b_{ij})_{m\times n}; \quad AC = (\sum_{k\in\underline{n}} a_{ik}c_{kj})_{m\times l}.$$

It is easy to verify that $(M_n(S), +, \cdot, 0_n)$ is a semiring with the identity element I_n.

Moreover, if S is an ordered semiring, we define:

$$A \leq B \Leftrightarrow \quad (\forall i \in \underline{m})\ (\forall j \in \underline{n})\ \ a_{ij} \leq b_{ij}.$$

It is easy to verify that $(M_n(S), +, \cdot, 0_n, I_n, \leq)$ is also an ordered semiring.

From now on, S stands for an arbitrary ordered semiring with identity element 1.

Proposition 4 *Let I be an ordered ideal of S. Then $M_n(I)$ is an ordered ideal of $M_n(S)$.*

Proof Since I is an ideal of S, we have $M_n(I)$ is an ideal of $M_n(S)$ (see [7], Proposition 2). In the following, we will prove that $(M_n(I)] \subseteq M_n(I)$.

Suppose that $A \leq B$ for some $B \in (M_n(I)]$. Then $a_{ij} \leq b_{ij}$ and $b_{ij} \in I$ for all $i, j \in \underline{n}$. Since I is an ordered ideal of S, we have $a_{ij} \in I$. Thus $A \in M_n(I)$. Therefore, $(M_n(I)] \subseteq M_n(I)$ and so $M_n(I)$ is an ordered ideal of $M_n(S)$. □

Proposition 5 *Let K be an ordered ideal of $M_n(S)$. Then there exists a unique ordered ideal I of S such that $K = M_n(I)$.*

Proof Let $I = \{a \in S : A_{ij} = a \text{ for some } A \in K\}$. We denote by e_{ij} the matrix in $M_n(S)$ with 1 as the (i, j)th entry, 0 otherwise; p_{ij} the elementary matrix in $M_n(S)$ with the ith row and the jth row of the identity matrix I_n permuted. Since $A_{ij}e_{ij} = e_{ii}Ae_{jj}$, $ae_{11} = p_{1i}(ae_{ij})p_{1j}$ for all $i, j \in \underline{n}$, it is easy to verify that $I = \{a \in S : A_{ij} = a \text{ for some } A \in K\} = \{a \in S : ae_{ij} \in K \text{ for some } i, j \in \underline{n}\} = \{a \in S : ae_{11} \in K\}$.

In the following, we will prove that I is the unique ordered ideal of S such that $K = M_n(I)$. We shall prove it in four steps.

(1) I is an ideal of S. Suppose that $a,\ b \in I$ and $s \in S$. Then $ae_{11},\ be_{11} \in K$ and so $(a+b)e_{11} = ae_{11} + be_{11} \in K$, $(sa)e_{11} = (se_{11})(ae_{11}) \in K$, $(as)e_{11} = (ae_{11})(se_{11}) \in K$. It follows that $a + b,\ sa,\ as \in I$. Thus I is an ideal of S.

(2) $(I] \subseteq I$. Assume that $x \leq a$ for some $a \in I$. We have $xe_{11} \leq ae_{11}$ and $ae_{11} \in K$. Since K is an ordered ideal of $M_n(S)$, we have $xe_{11} \in K$. Consequently, $x \in I$.

(3) $K = M_n(I)$. Suppose that $A = (a_{ij})_{n\times n} \in K$. Then $a_{ij} \in I$ for each $i, j \in \underline{n}$. Thus $A \in M_n(I)$. It follows that $K \subseteq M_n(I)$.

On the other hand, if $B = (b_{ij})_{n\times n} \in M_n(I)$, then $b_{ij} \in I$ and so $b_{ij}e_{ij} \in K$ for each $i, j \in \underline{n}$. Consequently, $B = (b_{ij})_{n\times n} = \sum_{i,j\in\underline{n}} b_{ij}e_{ij} \in K$. It follows that $M_n(I) \subseteq K$. Thus $K = M_n(I)$.

(4) I is unique. Suppose that J is also an ordered ideal of S such that $K = M_n(J)$. Then $K = M_n(I) = M_n(J)$. In the following, we will prove $I = J$. Indeed, if $a \in I$, then $ae_{11} \in M_n(I)$ and so $ae_{11} \in M_n(J)$. Thus $a \in J$. It follows that $I \subseteq J$. A similar argument shows that $J \subseteq I$. Therefore $I = J$. This completes the proof. □

Theorem 3 *Let $A = \{I : I \text{ is an ordered ideal of } S\}$ and $B = \{K : K \text{ is an ordered ideal of } M_n(S)\}$. Then the two lattices $(A, \subseteq)$ and $(B, \subseteq)$ are isomorphic.*

Proof Let $f : I \mapsto M_n(I)$ be a function from A to B.

In the following, we will prove that f is an isomorphism. We shall prove it in three steps.

(1) It follows from Proposition 5 that f is onto one-to-one.

(2) f is order-preserving. Assume that $I \subseteq J$ holds in A. It is obvious that $M_n(I) \subseteq M_n(J)$. That is to say $f(I) \subseteq f(J)$.

(3) f^{-1} is order-preserving. Suppose that $K_1 \subseteq K_2$ holds in B. By Proposition 5, there

exist I, J in A such that $K_1 = M_n(I)$ and $K_2 = M_n(J)$. It follows that $M_n(I) \subseteq M_n(J)$. If $a \in I$, then $ae_{11} \in M_n(I)$ and so $ae_{11} \in M_n(J)$. Thus $a \in J$. It follows that $I \subseteq J$, i.e., $f^{-1}(K_1) \subseteq f^{-1}(K_2)$. This completes the proof. □

Definition 3 *An ordered ideal $I \neq S$ of S is called maximal if, for any ordered ideal J of S, $I \subset J$ implies $J = S$.*

Corollary 4 *Let I be an ordered ideal of S. Then I is maximal if and only if $M_n(I)$ is a maximal ordered ideal of $M_n(S)$.*

Proof Assume that I is maximal and K is an ordered ideal of $M_n(S)$ such that $M_n(I) \subset K$. By Proposition 5, there exists a unique ordered ideal J of S such that $K = M_n(J)$. Thus $M_n(I) \subset M_n(J) \subseteq M_n(S)$. By Theorem 3, we have $I \subset J \subseteq S$. Since I is maximal, we get $J = S$. Thus $K = M_n(J) = M_n(S)$. Consequently, $M_n(I)$ is a maximal ordered ideal of $M_n(S)$.

Conversely, if $M_n(I)$ is a maximal ordered ideal of $M_n(S)$ and T is an ordered ideal of S such that $I \subset T$. By Theorem 3, we have $M_n(I) \subset M_n(T) \subseteq M_n(S)$. Since $M_n(I)$ is maximal, we get $M_n(T) = M_n(S)$. Thus $T = S$. Consequently, I is a maximal ordered ideal of S. This completes the proof. □

Corollary 5 *S is simple if and only if $M_n(S)$ is simple.*

Proof Assume that S is simple and $K \neq \{0\}$ is an ordered ideal of $M_n(S)$. By Proposition 5, there exists a unique ordered ideal J of S such that $K = M_n(J)$. Thus $M_n(0) \subset M_n(J) \subseteq M_n(S)$. By Theorem 3, we have $\{0\} \subset J \subseteq S$. Since S is simple, we get $J = S$. Thus $K = M_n(J) = M_n(S)$. Consequently, $M_n(S)$ is simple.

Conversely, if $M_n(S)$ is simple and $T \neq \{0\}$ is an ordered ideal of S, then $M_n(T)$ is an ordered ideal of $M_n(S)$ and $M_n(T) \neq \{0\}$. Since $M_n(S)$ is simple, we get $M_n(T) = M_n(S)$. Thus $T = S$. Consequently, S is simple. This completes the proof. □

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