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The Non-Singularity and Regularity of GP-V'-Rings

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Abstract In this paper, we introduce a non-trivial generalization of ZI-rings-quasi ZI-rings. A ring R is called a quasi ZI-ring, if for any non-zero elements $a, b \in R$, $ab = 0$ implies that there exists a positive integer n such that $a^n \neq 0$ and $a^n R b^n = 0$. The non-singularity and regularity of quasi ZI, GP-V'-rings are studied. Some new characterizations of strong regular rings are obtained. These effectively extend some known results.

Keywords GP-V'-rings; quasi ZI-rings; MELT(MERT) rings; weakly regular rings; generalized regular rings.

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1. Introduction

Throughout this paper, a ring R denotes an associative ring with identity and all modules are unitary. J, Z (resp., Y) will denote respectively the Jacobson radical, the left singular ideal (resp., right singular ideal) of R. $l(X)$ ($r(X)$) denotes the left (right) annihilator of X in R. If $X = \{a\}$, we will write $l(a)$, $r(a)$ for $l(X)$, $r(X)$.

Recall the following definitions and facts:

(1) R is said to be left (right) non-singular [1] if $Z = 0$ ($Y = 0$).

(2) A ring R is called (von Neumann) regular [2] if for any $a \in R$, there exists $b \in R$ such that $a = aba$. A ring R is called strong regular if for any $a \in R$, there exists $b \in R$ such that $a = a^2 b$.

(3) A ring R is called a zero insertive (briefly ZI) ring [3] if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$.

(4) A ring R is left (right) weakly regular [2] if $I^2 = I$ for each left (right) ideal I of R, or equivalently, if

$$RaR + l(a) = R \ (RaR + r(a) = R) \text{ for every } a \in R.$$

R is weakly regular if it is both right and left weakly regular.

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(5) A ring R is said to be reduced [4] if it contains no non-zero nilpotent element. If R is reduced, then $r(a) = l(a)$ for any $a \in R$.

(6) A ring R is said to be an MELT(MERT) [5] ring if every maximal essential left (right) ideal of R is an ideal. R is said to be generalized regular if every left ideal is generated by idempotents.

(7) A left R-module M is said to be YJ-injective [6], if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism from Ra^n into M extends to one from R into M. A ring R is called a left YJ-injective ring if ${}_RR$ is left YJ-injective. A ring R is called a left GP-V-(resp., GP-V'-)ring if every simple (resp., simple singular) left R-module is YJ-injective. The situation of the right can be defined similarly.

GP-V-(GP-V'-)rings have been extensively studied for several years. Many authors studied some connections between GP-V-(GP-V'-)rings and strongly von Neumann regular rings [7–9], [5–10], [11–13]. In [5], Xiao and Tong discussed the strong regularity and non-singularity of ZI, GP-V'-rings. In this paper, we extend the conception of ZI-rings to quasi ZI-rings, and study the non-singularity and strong regularity of quasi ZI, GP-V'-rings, which extend the main results in [5].

2. Quasi ZI-rings

Definition 2.1 *A ring R is called a quasi ZI-ring, if for any non-zero elements $a, b \in R$, $ab = 0$ implies that there exists a positive integer n such that $a^n \neq 0$ and $a^n R b^n = 0$.*

Example 2.2 Let $R' = \mathbb{Q}\langle \alpha, \beta, \gamma \rangle$ be a set of all polynomials in non-commuting indeterminates α, β, γ with coefficients in rational number field $\mathbb{Q}$. Let I be an ideal generated by

$$\langle \alpha^3, \beta^3, \gamma^3, \beta\alpha, \gamma\beta, \alpha\gamma, \gamma\alpha, \alpha^2\beta, \alpha\beta^2, \beta^2\gamma, \beta\gamma^2 \rangle$$

of R'. Put $R = R'/I$. Then R is a quasi ZI-ring but not a ZI-ring.

Proof One can easily check that $R = \{k_1\overline{1} + k_2\overline{\alpha} + k_3\overline{\beta} + k_4\overline{\gamma} + k_5\overline{\alpha^2} + k_6\overline{\beta^2} + k_7\overline{\gamma^2} + k_8\overline{\alpha\beta} + k_9\overline{\beta\gamma} + k_{10}\overline{\alpha\beta\gamma} \mid k_i \in \mathbb{Q}, i = 1, 2, \ldots, 10\}$, and the product of any four elements of $\{\alpha, \beta, \gamma\}$ is zero in R.

Note that $\overline{\alpha\gamma} = 0$ and $0 \neq \overline{\alpha\beta\gamma} \in \overline{\alpha}R\overline{\gamma}$, hence R is not a ZI-ring. But we claim that R is a quasi ZI-ring. We only consider monomial. Suppose that $\overline{ab} = 0$ for any $\overline{a} \neq 0$ and $\overline{b} \neq 0$ in R.

(1) If $\overline{a^2} = 0$, then $\overline{a} \notin \{\overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$. It is easy to see that $\overline{a}R\overline{b} = 0$.

(2) If $\overline{a^2} \neq 0$, then it is obvious that $\overline{a^2}R\overline{b^2} = 0$.

This completes the proof. □

From the above example, we know that quasi ZI-rings are non-trivial generalizations of ZI-rings.

Lemma 2.3 *If R is a quasi ZI-ring, then every idempotent is central in R.*

Proof Let $0, 1 \neq e^2 = e \in R$. Then $e(1 - e) = 0$. By the definition of quasi ZI-rings, we have

$eR(1-e)=0$. Similarly, $(1-e)Re=0$. Hence, e is a central idempotent.

3. The non-singularity of GP-V'-rings

Proposition 3.1 *If R is a quasi ZI, left GP-V'-ring, then*

(1) R is reduced;

(2) $I+l(a)=R$ for any non-zero ideal I of R and every $a\in I$.

Proof (1) Let $a\in R$ with $a^2=0$. If $a\neq 0$, then $l(a)\neq R$, and there exists a complement left ideal K of R such that $l(a)\oplus K$ is an essential left ideal of R. If $l(a)\oplus K=R$, then $l(a)=Re$, where $e^2=e\in R$. It follows from Lemma 2.3 that $a=ae=ea$. Note that $a\in r(l(a))=r(Re)=(1-e)R$, we have that $a=ea=0$. It is a contradiction. If $l(a)\oplus K\neq R$, then there exists a maximal left ideal M of R containing $l(a)\oplus K$. By assumption, the simple singular left R-module R/M is YJ-injective. Since $l(a)\subseteq M$, we can define a left R-homomorphism $f: Ra\to R/M$ by $f(ra)=r+M$ for all $r\in R$. Then there exists $b\in R$ such that $1-ab\in M$. Because R is a quasi ZI-ring, $aRa=0$. So $a=a-aba=ma$, where $m=1-ab\in M$. This implies that $1-m\in l(a)\subseteq M$ and $1\in M$, which is contradicting the maximality of M. So we have that $a=0$. This implies that R is reduced and (1) holds.

(2) Let I be a non-zero ideal of R. If there exists $b\in I$ such that $I+l(b)\neq R$, then there exists a maximal left ideal M of R containing $I+l(b)$. We claim that M is an essential left ideal of R. If not, then M is a direct summand of R. Thus $M=l(e)$ for some $e^2=e(\neq 0)\in R$. Note that $b\in l(e)$. It follows from Lemma 2.3 that $eb=be=0$. This implies that $e\in l(b)\subseteq M=l(e)$, and $e=e^2=0$, which is a contradiction. By assumption, the simple singular left R-module R/M is YJ-injective. So there exists a positive integer n such that $b^n\neq 0$ and any left R-homomorphism from Rb^n into R/M extends to one from R into R/M. It follows from (1) that $l(b^n)=l(b)\subseteq M$. So we may define a left R-homomorphism $f: Rb^n\to R/M$ by $f(rb^n)=r+M$ for all $r\in R$. Then there exists $c\in R$ such that $1+M=f(b^n)=b^nc+M$. Because $b^nc\subseteq I\subseteq M$, we have $1\in M$, which is contradicting the maximality of M. This implies that $I+l(a)=R$ for any non-zero ideal I of R and every $a\in I$, i.e., (2) holds.

Corollary 3.2 ([5]) *If R is a ZI, left GP-V'-ring, then*

(1) R is reduced;

(2) $I+l(a)=R$ for any non-zero ideal I of R and every $a\in I$.

Lemma 3.3 *If Z (resp., Y) is reduced, then $Z=0$ (resp., $Y=0$).*

Proof Suppose that Z is reduced. For any $a\in Z$, if $a\neq 0$, then $l(a)\cap Ra\neq 0$. Thus there exists $r\in R$ such that $ra\neq 0$ and $raa=0$. Since Z is reduced, we have $ara=0$. This implies that $ra=0$, which is a contradiction. So $Z=0$. Similarly we have $Y=0$ if Y is reduced.

By Proposition 3.1(1) and Lemma 3.3, we can easily obtain the following theorem.

Theorem 3.4 *If R is a quasi ZI, left GP-V'-ring, then R is left and right non-singular.*

Corollary 3.5 ([5]) *If R is a ZI, left GP-V-ring, then R is left and right non-singular.*

Corollary 3.6 *Let R be a quasi ZI, left YJ-injective ring. If R is a left GP-V'-ring, then R is right non-singular.*

Corollary 3.7 ([5]) Let R be a ZI, left YJ-injective ring. If R is a left GP-V'-ring, then R is right non-singular.

Corollary 3.8 *If R is a quasi ZI, left GP-V'-ring and satisfies the descending chain conditions of special left annihilators in Z, then R is left non-singular.*

Corollary 3.9 ([5]) *If R is a ZI, left GP-V'-ring and satisfies the descending chain conditions of special left annihilators in Z, then R is left non-singular.*

4. The regularity of GP-V'-rings

By Proposition 3.1, we have immediately the following.

Proposition 4.1 *If R is a quasi ZI, left GP-V'-ring, then R is a weakly regular ring.*

Corollary 4.2 ([5]) *If R is a ZI, left GP-V'-ring, then R is a weakly regular ring.*

From [14, Theorem 7], R is a strongly regular ring if and only if R is an Abelian, right quasi-duo ring whose simple singular R-modules are GP-injective. Now, we give some new characterizations of strongly regular rings as follows.

Theorem 4.3 *Let R be a ring. Then the following statements are equivalent:*

(1) R is a strongly regular ring;

(2) R is a quasi ZI, $MELT$, left GP-V-ring;

(3) R is a quasi ZI, $MERT$, right GP-V-ring;

(4) R is a quasi ZI, $MELT$, left GP-V'-ring;

(5) R is a quasi ZI, $MERT$, right GP-V'-ring.

Proof (1) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). It suffices to prove that $Ra + l(a) = R$ for any $a \in R$. If not, then there exists $a \in R$ such that $Ra + l(a) \neq R$. Thus there exists a maximal left ideal L of R containing $Ra + l(a)$. With the similar discussion to the proof of Proposition 3.1(2), we get that L is an essential left ideal of R. By assumption, R/L is left YJ-injective. Thus there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism from Ra^n into R/L extends to one from R into R/L. Since R is reduced, we can define a left R-homomorphism $f : Ra^n \to R/L$ by $f(ra^n) = r + L$ for all $r \in R$. Then there exists $c \in R$ such that $1 - a^n c \in L$. Since R is $MELT$, L is an ideal of R and $RaR \subseteq L$, which implies that $1 \in L$. This is a contradiction. Therefore there exist $b \in R, d \in l(a)$ such that $1 = ba + d$, and $a = ba^2$. So R is of strong regularity.

Similarly, we can prove (5) $\Rightarrow$ (1). This completes the proof. □

Corollary 4.4 ([5]) *Let R be a ring. Then the following statements are equivalent:*

(1) R is a strongly regular ring;

(2) R is a ZI, $MELT$, left GP-V-ring;

(3) R is a ZI, $MERT$, right GP-V-ring;

(4) R is a ZI, $MELT$, left GP-V'-ring;

(5) R is a ZI, $MERT$, right GP-V'-ring.

We know that a regular ring is generalized regular. But there exists a generalized regular ring which is not regular [5].

An additive subgroup L of a ring is called a weakly right ideal of R if for every $x \in L$ and $r \in R$ there exists a natural number n such that $(xr)^n \in L$. There exists a weakly right ideal K of R which is not a right ideal of R (see [15]).

Theorem 4.5 *Let R be a ring. Then the following statements are equivalent:*

(1) R is a strongly regular ring;

(2) R is a quasi ZI, generalized regular ring;

(3) R is a quasi ZI, left GP-V'-ring whose every maximal essential left ideal is a weakly right ideal;

(4) R is a quasi ZI, left GP-V-ring whose every maximal essential left ideal is a weakly right ideal.

Proof (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Suppose that R is a generalized regular ring. For any $a \in R$, we have that $Ra = \Sigma Re_i$, where e_i is an idempotent in R. We may assume that $a = r_1e_1 + \cdots + r_ne_n$. It follows that $Ra = Re_1 + \cdots + Re_n$. Put $f_1 = e_1 + e_2 - e_1e_2$. Note that all e_i are central idempotents by Lemma 2.3. So $f_1^2 = f_1$, $e_1f_1 = e_1$, $e_2f_1 = e_2$ and $Re_1 + Re_2 = Rf_1$. Putting $f_2 = f_1 + e_3 - f_1e_3, \ldots, f_{n-2} = f_{n-3} + e_{n-1} - f_{n-3}e_{n-1}$ and $e = f_{n-2} + e_n - f_{n-2}e_n$, and repeating the above process, one can see that

$$Ra = Rf_1 + Re_3 + \cdots + Re_n = Rf_2 + \cdots + Re_n = \cdots = Re = eR.$$

So there exist $r, b \in R$ such that $a = re$ and $e = ba$, which implies that $a = ae = ea = ba^2$. Therefore R is of strong regularity.

(3) $\Rightarrow$ (1). For any $a \in R$, we prove that $Ra + l(a) = R$. If not, there would exist a maximal left ideal L of R containing $Ra + l(a)$. With the same discussions as the proof of Theorem 4.3, we have $b \in R$ such that $1 - a^nb \in L$. By the hypothesis, L is a weakly right ideal of R. Note that $a^n \in L$. Therefore there exists a natural number m such that $(a^nb)^m \in L$. Since L is also a left ideal of R and $1 - a^nb \in L$, we have that

$$a^nb - (a^nb)^2 = a^nb(1 - a^nb) \in L$$

$$(a^nb)^2 - (a^nb)^3 = (a^nb)^2(1 - a^nb) \in L$$

$$\cdots$$

$$(a^nb)^{m-1} - (a^nb)^m = (a^nb)^{m-1}(1 - a^nb) \in L.$$

This implies that $a^n b \in L$. Then $1 = 1 - a^n b + a^n b \in L$, which contradicts $L \neq R$. Thus $Ra + l(a) = R$ for any $a \in R$. So R is a strongly regular ring. This completes the proof. □

Corollary 4.6 ([5]) *Let R be a ring. Then the following statements are equivalent:*

(1) R is a strongly regular ring;

(2) R is a ZI, generalized regular ring;

(3) R is a ZI, left GP-V'-ring whose every maximal essential left ideal is a weakly right ideal;

(4) R is a ZI, left GP-V-ring whose every maximal essential left ideal is a weakly right ideal.

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