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On the Mean Value of General k-th Gauss Sums

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Abstract The main purpose of this paper is using residue system and character sums methods to investigate the mean value properties of general k-th Gauss sums, and two exact calculating formulas are given.

Keywords general Gauss sums; character sums; residue system; mean value.

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1. Introduction

Let $q \geq 3$ be a positive integer. For any integer n and positive integer k, the general k-th Gauss sums $G(n,k,\chi;q)$ are defined as follows

$$G(n,k,\chi;q) = \sum_{b=1}^{q}{}' \chi(b) e\left(\frac{nb^k}{q}\right), \tag{1}$$

where $\sum_b'$ denotes the summation of all b such that $(b,q)=1$, χ denotes a Dirichlet character modulo q, and $e(y)=e^{2\pi i y}$.

This summation is very important, because it can be regarded as a generalization of the classical Gauss sums $G(n,1,\chi;q)$. The various properties of $G(n,1,\chi;q)$ were investigated by many authors. Perhaps the most famous properties of $G(n,1,\chi;q)$ are the identities in [1, 3]

$$G(n,1,\chi;q) = \chi(\overline{n}) G(1,1,\chi;q) \quad \text{and} \quad |G(1,1,\chi;q)| = \sqrt{q}, \tag{2}$$

where q is an odd prime, $\overline{n}$ is the inverse of n such that $n\overline{n} \equiv 1 \pmod q$, and χ is a non-principal character modulo q.

Interest in general k-th Gauss sums stems from its good mean value properties. In recent years, interest has been aroused to deal with the $2l$-th mean value of general k-th Gauss sums,

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which is defined by

$$\sum_{\chi \bmod q} |G(n,k,\chi;q)|^{2l}, \tag{3}$$

where l is a positive integer. For example, let $q=p$ be an odd prime, Zhang [5] showed that

$$\sum_{\chi \bmod p} |G(n,2,\chi;p)|^4 = \begin{cases} (p-1)\left[3p^2-6p-1+4\sqrt{p}\left(\dfrac{n}{p}\right)\right], & p \equiv 1 \pmod 4, \\ (p-1)(3p^2-6p-1), & p \equiv 3 \pmod 4. \end{cases} \tag{4}$$

And if $p \equiv 3 \pmod 4$, then

$$\sum_{\chi \bmod p} |G(n,2,\chi;p)|^6 = (p-1)(10p^3-25p^2-4p-1),$$

where n is any integer with $(n,p)=1$, and $(\frac{n}{p})$ is the Legendre symbol. Very recently, the authors [2] gave the higher mean value of general quadratic Gauss sums. The idea of the present paper stems from the work of Zhang and Liu [4, 6]. We show the following mean value distribution of general k-th Gauss sums by using residue system and character sums methods.

Theorem 1.1 *Let $q \geq 3$ be a positive integer, and let k be any positive integer with $(k,q)=1$. If $d=(k,p-1)$, then*

$$\sum_{\chi \bmod q} \sum_{n=1}^{q} |G(n,k,\chi;q)|^4 = q\phi^2(q) \prod_{p^a \| q} \left[\phi(p^a)\big((a-1)d^2+2d\big) - d^2 p^{a-1}\right],$$

where $\prod_{p^a \| q}$ denotes the product over all p such that $p^a \mid q$ but $p^{a+1} \nmid q$, and $\phi(n)$ is the Euler function.

Theorem 1.2 *Let p be an odd prime, and let n be any integer with $(n,p)=1$. If k is a positive integer with $(k,p-1)=1$, then*

$$\sum_{\chi \bmod p} |G(n,k,\chi;p)|^{2l} = p^{l+1} - 2p^l + 1.$$

2. Several lemmas

In order to complete the proofs of the above theorems, we need the following several lemmas. For convenience, we always denote by $\phi(n)$ the Euler function in this section.

Lemma 2.1 *Let p be an odd prime and a be a positive integer. If k is a positive integer with $(k,p)=1$, then*

$$\sum_{\substack{b=1 \\ p^a \mid (b^k-1)}}^{p^a}{}' \sum_{\substack{c=1 \\ (c^k-1)}}^{p^a}{}' 1 = \phi(p^a)\big((a-1)d^2+2d\big) - d^2 p^{a-1}, \tag{6}$$

where $\sum_b'$ denotes the summation of all b such that $(b,p^a)=1$, $\sum_c'$ in the same way, and $d=(k,p-1)$.

Proof Let m, n be positive integers with $(m,n)=1$. Suppose that g is a given primitive

root modulo m, then the k-th congruent equation $x^k \equiv n \pmod m$ is solvable if and only if $(k, \phi(m)) \mid \mathrm{ind}_g n$, where $\mathrm{ind}_g n$ is the index of n modulo m such that $n \equiv g^{\mathrm{ind}_g n} \pmod m$. Moreover, the number of solutions is $(k, \phi(m))$ when it is solvable. Without loss of generality, since there exists a primitive root g modulo p^a and $\mathrm{ind}_g 1 = 0$, the k-th congruent equation $x^k \equiv 1 \pmod{p^a}$ has only $(k, \phi(p^a))$ solutions. Now the key idea is to split the left of the identity (2.1) into $\mathrm{M}_1 + \mathrm{M}_2 + \mathrm{M}_3$, where

$$\mathrm{M}_1 = \sideset{}{'}\sum_{\substack{b=1\\ p^a \mid (b^k-1)\\ p \nmid (b^k-1)}}^{p^a} \sideset{}{'}\sum_{\substack{c=1\\ (c^k-1)}}^{p^a} 1, \quad \mathrm{M}_2 = \sum_{i=1}^{a-1} \sideset{}{'}\sum_{\substack{b=1\\ p^a \mid (b^k-1)\\ p^{a-i} \| (b^k-1)}}^{p^a} \sideset{}{'}\sum_{\substack{c=1\\ (c^k-1)}}^{p^a} 1, \quad \mathrm{M}_3 = \sideset{}{'}\sum_{\substack{b=1\\ p^a \mid (b^k-1)}}^{p^a} \sideset{}{'}\sum_{\substack{c=1\\ (c^k-1)}}^{p^a} 1. \tag{7}$$

It is trivial to obtain M_3. We have

$$\mathrm{M}_3 = \phi(p^a)(k, \phi(p^a)) = d\phi(p^a). \tag{8}$$

Thus, it remains to get M_1 and M_2. Let S be a reduced residues system modulo k, and let d be a positive divisor of k. By the factorization of reduced residue system [1, Theorem 5.33], it is known that S is constituted of the union of $\phi(k)/\phi(d)$ disjoint sets, each of which is a reduced residue system modulo d. Thus

$$\mathrm{M}_1 = (k, \phi(p^a))\left(\phi(p^a) - (k, \phi(p))\frac{\phi(p^a)}{\phi(p)}\right) = d\phi(p^a) - d^2 p^{a-1}, \tag{9}$$

and

$$\begin{aligned}\mathrm{M}_2 &= \sum_{i=1}^{a-1} \frac{\phi(p^a)}{\phi(p^i)}(k, \phi(p^i))\left((k, \phi(p^{a-i}))\frac{\phi(p^a)}{\phi(p^{a-i})} - (k, \phi(p^{a-i+1}))\frac{\phi(p^a)}{\phi(p^{a-i+1})}\right)\\ &= \sum_{i=1}^{a-1} p^{a-i} d(dp^i - dp^{i-1}) = (a-1)d^2\phi(p^a).\end{aligned} \tag{10}$$

Combining (7)–(10), Lemma 2.1 follows immediately.

Lemma 2.2 *Let p be an odd prime and a be a positive integer. If $d = (k, p-1)$, then for positive integer k with $(k, p) = 1$,*

$$\sum_{\chi \bmod p^a} \sum_{n=1}^{p^a} |G(n, k, \chi; p^a)|^4 = p^a \phi^2(p^a)\big[\phi(p^a)\big((a-1)d^2 + 2d\big) - d^2 p^{a-1}\big].$$

Proof Let $(c, p^a) = 1$. It is clear that bc runs over a reduced residue system modulo p^a as b does. Thus

$$|G(n, k, \chi; p^a)|^2 = \sideset{}{'}\sum_{b=1}^{p^a} \chi(b) e\left(\frac{nb^k}{p^a}\right) \sideset{}{'}\sum_{c=1}^{p^a} \overline{\chi}(c) e\left(\frac{-nc^k}{p^a}\right) = \sideset{}{'}\sum_{b=1}^{p^a} \chi(b) \sideset{}{'}\sum_{c=1}^{p^a} e\left(\frac{nc^k(b^k-1)}{p^a}\right). \tag{11}$$

So from (11) and the orthogonality relation for Dirichlet characters we have

$$\sum_{\chi \bmod p^a} |G(n, k, \chi; p^a)|^4$$

$$
\begin{aligned}
&= \sum_{r=1}^{\phi(p^a)} \sideset{}{'}\sum_{b=1}^{p^a} \chi_r(b) \sideset{}{'}\sum_{c=1}^{p^a} e\Big(\frac{nc^k(b^k-1)}{p^a}\Big) \sideset{}{'}\sum_{x=1}^{p^a} \overline{\chi}_r(x) \sideset{}{'}\sum_{y=1}^{p^a} e\Big(\frac{-ny^k(x^k-1)}{p^a}\Big) \\
&= \sideset{}{'}\sum_{b=1}^{p^a} \sideset{}{'}\sum_{x=1}^{p^a} \sideset{}{'}\sum_{c=1}^{p^a} \sideset{}{'}\sum_{y=1}^{p^a} e\Big(\frac{nc^k(b^k-1)}{p^a}\Big) e\Big(\frac{-ny^k(x^k-1)}{p^a}\Big) \sum_{r=1}^{\phi(p^a)} \chi_r(b)\overline{\chi}_r(x) \\
&= \phi(p^a) \sideset{}{'}\sum_{b=1}^{p^a} \Big| \sideset{}{'}\sum_{c=1}^{p^a} e\Big(\frac{nc^k(b^k-1)}{p^a}\Big)\Big|^2. \qquad (12)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\Big| \sideset{}{'}\sum_{c=1}^{p^a} e\Big(\frac{nc^k(b^k-1)}{p^a}\Big)\Big|^2 &= \sideset{}{'}\sum_{c_1=1}^{p^a} e\Big(\frac{nc_1^k(b^k-1)}{p^a}\Big) \sideset{}{'}\sum_{c_2=1}^{p^a} e\Big(\frac{-nc_2^k(b^k-1)}{p^a}\Big) \\
&= \sideset{}{'}\sum_{c_1=1}^{p^a} \sideset{}{'}\sum_{c_2=1}^{p^a} e\Big(\frac{nc_2^k(c_1^k-1)(b^k-1)}{p^a}\Big).
\end{aligned}
$$

It follows from (12) and (13) that

$$
\sum_{\chi \bmod p^a} |G(n,k,\chi;p^a)|^4 = \phi(p^a) \sideset{}{'}\sum_{b=1}^{p^a} \sideset{}{'}\sum_{c_1=1}^{p^a} \sideset{}{'}\sum_{c_2=1}^{p^a} e\Big(\frac{nc_2^k(c_1^k-1)(b^k-1)}{p^a}\Big). \qquad (14)
$$

Thus, by Lemma 2.1 and the famous trigonometric sum formula:

$$
\sum_{m=0}^{k-1} e^{\frac{2\pi i m n}{k}} = \begin{cases} 0, & \text{if } k \nmid n, \\ k, & \text{if } k \mid n, \end{cases} \qquad (15)
$$

we obtain

$$
\begin{aligned}
\sum_{\chi \bmod p^a} \sum_{n=1}^{p^a} |G(n,k,\chi;p^a)|^4 &= \phi(p^a) \sideset{}{'}\sum_{b=1}^{p^a} \sideset{}{'}\sum_{c_1=1}^{p^a} \sideset{}{'}\sum_{c_2=1}^{p^a} \sideset{}{'}\sum_{n=1}^{p^a} e\Big(\frac{nc_2^k(c_1^k-1)(b^k-1)}{p^a}\Big) \\
&= p^a\phi^2(p^a)\big[\phi(p^a)\big((a-1)d^2+2d\big) - d^2p^{a-1}\big].
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

3. The proofs of Theorems 1.1 and 1.2

The Proof of Theorem 1.1 Suppose that χ_i is a Dirichlet character modulo $p_i^{a_i}$, then $\chi = \chi_1 \cdots \chi_r$ is a Dirichlet character modulo $q = p_1^{a_1} \cdots p_r^{a_r}$. So if χ_i runs over $\phi(p_i^{a_i})$ Dirichlet characters modulo $p_i^{a_i}$, then we can get $\phi(q)$ Dirichlet characters modulo q. Now, let $m_1, m_2, \ldots, m_k$ be k positive integers with $(m_i, m_j) = 1$, where $i \neq j$ and $i, j = 1, \ldots, k$. If $m = m_1 m_2 \cdots m_k = m_j M_j$ $(1 \le j \le k)$ and $x = M_1 x^{(1)} + \cdots + M_k x^{(k)}$, then x runs over a reduced (complete) residue system modulo m if and only if $x^{(1)}, \ldots, x^{(k)}$ respectively runs over a reduced (complete) residue system modulo m_1, …, modulo m_k. Thus

$$
|G(n,k,\chi;q)| = \Big| \sideset{}{'}\sum_{b=1}^{p_1^{a_1}\cdots p_r^{a_r}} \chi(b) e\Big(\frac{nb^k}{p_1^{a_1}\cdots p_r^{a_r}}\Big)\Big|
$$

$$
\begin{aligned}
&= \Bigg|\sideset{}{'}\sum_{b_1=1}^{p_1^{a_1}} \cdots \sideset{}{'}\sum_{b_r=1}^{p_r^{a_r}} \chi_1\left(\frac{q}{p_1^{a_1}}b_1 + \cdots + \frac{q}{p_r^{a_r}}b_r\right)\cdots \\
&\quad \chi_r\left(\frac{q}{p_1^{a_1}}b_1 + \cdots + \frac{q}{p_r^{a_r}}b_r\right) e\left(\frac{n(\frac{q}{p_1^{a_1}}b_1 + \cdots + \frac{q}{p_r^{a_r}}b_r)^k}{p_1^{a_1}\cdots p_r^{a_r}}\right)\Bigg| \\
&= \Bigg|\sideset{}{'}\sum_{b_1=1}^{p_1^{a_1}} \chi_1\left(\frac{q}{p_1^{a_1}}b_1\right) e\left(\frac{n(\frac{q}{p_1^{a_1}}b_1)^k}{p_1^{a_1}\cdots p_r^{a_r}}\right)\Bigg| \cdots \Bigg|\sideset{}{'}\sum_{b_r=1}^{p_r^{a_r}} \chi_r\left(\frac{q}{p_r^{a_r}}b_r\right) e\left(\frac{n(\frac{q}{p_r^{a_r}}b_r)^k}{p_1^{a_1}\cdots p_r^{a_r}}\right)\Bigg| \\
&= \Bigg|\sideset{}{'}\sum_{b_1=1}^{p_1^{a_1}} \chi_1(b_1) e\left(\frac{nq^{k-1}b_1^k}{(p_1^{a_1})^k}\right)\Bigg| \cdots \Bigg|\sideset{}{'}\sum_{b_r=1}^{p_r^{a_r}} \chi_r(b_r) e\left(\frac{nq^{k-1}b_r^k}{(p_r^{a_r})^k}\right)\Bigg| \\
&= \left|G\left(n\left(\frac{q}{p_1^{a_1}}\right)^{k-1}, k, \chi_1; p_1^{a_1}\right)\right| \cdots \left|G\left(n\left(\frac{q}{p_r^{a_r}}\right)^{k-1}, k, \chi_r; p_r^{a_r}\right)\right|. \qquad (16)
\end{aligned}
$$

It follows from (16) that

$$
\sum_{\chi \bmod q} |G(n,k,\chi;q)|^4 = \prod_{i=1}^{r} \sum_{\chi_i \bmod p_i^{a_i}} \left|G\left(n\left(\frac{q}{p_i^{a_i}}\right)^{k-1}, k, \chi_i; p_i^{a_i}\right)\right|^4. \qquad (17)
$$

So from Lemma 2.2 and (17), we derive

$$
\begin{aligned}
\sum_{n=1}^{q}\sum_{\chi \bmod q} |G(n,k,\chi;q)|^4 &= \prod_{i=1}^{r} \sum_{\chi_i \bmod p_i^{a_i}} \sum_{n_i=1}^{p_i^{a_i}} \left|G\left(n_i\left(\frac{q}{p_i^{a_i}}\right)^{k-1}, k, \chi_i; p_i^{a_i}\right)\right|^4 \\
&= \prod_{i=1}^{r} \sum_{\chi_i \bmod p_i^{a_i}} \sum_{n_i=1}^{p_i^{a_i}} |G(n_i,k,\chi_i;p_i^{a_i})|^4 \\
&= \prod_{i=1}^{r} p_i^{a_i}\phi^2(p_i^{a_i})\big[\phi(p_i^{a_i})\big((a_i-1)(k,p_i-1)^2 + 2(k,p_i-1)\big) - (k,p_i-1)^2 p^{a_i-1}\big] \\
&= q\phi^2(q) \prod_{p^a \| q} \big[\phi(p^a)\big((a-1)(k,p-1)^2 + 2(k,p-1)\big) - (k,p-1)^2 p^{a-1}\big],
\end{aligned}
$$

as desired.

The Proof of Theorem 1.2 It is easy to see that if a runs over a reduced residue system modulo p, then a^m also runs over such a system, where m is a positive integer such that $km \equiv 1 \pmod{(p-1)}$. Thus

$$
\begin{aligned}
G(n,k,\chi;p) &= \sum_{b=1}^{p-1} \chi(b^m) e\left(\frac{nb^{km}}{p}\right) = \sum_{b=1}^{p-1} \chi^m(b) e\left(\frac{nb^{km}}{p}\right) \\
&= \sum_{b=1}^{p-1} \chi^m(b) e\left(\frac{nb}{p}\right). \qquad (18)
\end{aligned}
$$

Since $(n,p)=1$, $|\chi(n)|^2 = \chi(n)\overline{\chi}(n) = 1$. It follows from (18) that

$$
G(n,k,\chi;p) = \sum_{b=1}^{p-1} \chi^m(b)\chi^m(n)\overline{\chi^m}(n) e\left(\frac{nb}{p}\right) = \overline{\chi^m}(n) \sum_{b=1}^{p-1} \chi^m(nb) e\left(\frac{nb}{p}\right)
$$

$$= \overline{\chi^m}(n) \sum_{b=1}^{p-1} \chi^m(b) e\left(\frac{b}{p}\right). \tag{19}$$

Assume that χ is a non-principal character modulo p, then χ^m is a non-principal character modulo p. In fact, if χ^m is the principal character modulo p, then by the properties of character group, m must be a divisor, or a multiple of $p-1$. Since $km \equiv 1 \pmod{(p-1)}$, it is impossible that m is a multiple of $p-1$. Hence, m is a positive divisor of $p-1$. In this case, we can get that m divides $km-1$, which results in $m=1$, a contradiction. So if χ_o is the principal character modulo p, then by (2) and (15) we have

$$\left| \sum_{a=1}^{p-1} \chi^m(a) e\left(\frac{a}{p}\right) \right| = \begin{cases} \sqrt{p}, & \chi \neq \chi_o; \\ 1, & \chi = \chi_o. \end{cases} \tag{20}$$

It follows from (19) and (20) that

$$\begin{aligned} \sum_{\chi \bmod p} |G(n,k,\chi;p)|^{2l} &= \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_o}} |G(n,k,\chi;p)|^{2l} + |G(n,k,\chi_o;p)|^{2l} \\ &= (p-2)p^l + 1 = p^{l+1} - 2p^l + 1. \end{aligned}$$

This completes the proof of Theorem 1.2. □

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