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Uniqueness of Entire Function Related to Shared Set

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Abstract In this paper, uniqueness of entire function related to shared set is studied. Let f be a non-constant entire function and k be a positive integer, d be a finite complex number. There exists a set S with 3 elements such that if f and its derivative $f^{(k)}$ satisfy $E(S,f)=E(S,f^{(k)})$, and the zeros of $f(z)-d$ are of multiplicity $\geq k+1$, then $f=f^{(k)}$.

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1. Introduction and main results

In this paper, we use the symbols as given in Nevanlinna theory of meromorphic functions [1–3].

Let f and g be two non-constant meromorphic functions, and $a\in\overline{\mathbb{C}}=\mathbb{C}\bigcup\{\infty\}$. We say that f and g share the value a IM (ignoring multiplicities) if $f-a$ and $g-a$ have the same zeros, and they share the value a CM (counting multiplicities) if $f-a$ and $f-b$ have the same zeros with the same multiplicities. When $a=\infty$ the zeros of $f-a$ means the poles of f (see [3]).

Let f be a non-constant meromorphic function in the complex plane and let S be a set of distinct complex numbers. Put

$$E(s,f)=\bigcup_{a\in S}\{z:f(z)-a=0,\mathrm{CM}\},\quad \overline{E}(S,f)=\bigcup_{a\in S}\{z:f(z)-a=0,\mathrm{IM}\}.$$

If $E(S,f)=E(S,g)$, we say that f and g share the set S CM. If $E(S,f)=E(S,g)$, we say that f and g share the set S IM. Especially, when $S=\{a\}$, $a\in\overline{\mathbb{C}}$, $E(a,f)=E(a,g)$ or $\overline{E}(a,f)=\overline{E}(a,g)$ means f and g share the value a CM or IM respectively.

In 2003, Fang and Zalcman [4] proved the following result.

Theorem A *There exists a set S with 3 elements such that if a non-constant entire function f and its derivative f' satisfy $E(S,f)=E(S,g)$, then $f=f'$.*

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It is natural to ask whether Theorem A remains valid for $f^{(k)}$. In this paper, we use the theory of normal families to prove

Theorem 1 *Let f be a non-constant entire function and k be a positive integer, d be a finite complex number. There exists a set S with 3 elements such that if f and its derivative $f^{(k)}$ satisfy $E(S,f) = E(S,f^{(k)})$, and the zeros of $f(z)-d$ are of multiplicity $\geq k+1$, then $f = f^{(k)}$.*

2. Some lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 ([5]) *Let $\mathscr{F}$ be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least k. Suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathscr{F}$ is not normal, there exist, for each $\alpha(0 \leq \alpha \leq k)$,*

(a) points z_n with $|z_n| < r < 1$,

(b) functions $f_n \in \mathscr{F}$, and

(c) positive numbers $\rho_n \to 0^+$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a non-constant entire function, all of whose zeros have multiplicity at least k, such that $g^{\#}(\xi) \leq g^{\#}(0) = kA+1$. Here $g^{\#}(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative of g.

Lemma 2 ([3]) *Let f be a non-constant meromorphic function and k be a positive integer. Then*

$$N(r, \frac{1}{f^{(k)}}) < N(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f).$$

Lemma 3 ([6]) *Let g be a meromorphic function on $\mathbb{C}$. If its spherical derivative is uniformly bounded on $\overline{\mathbb{C}}$. Then the order of g is at most 2. If g is an entire function, then the order of g is at most 1.*

Lemma 4 *Let $\mathscr{F}$ be a family of holomorphic functions in D and k be a positive integer, a be a finite complex number. There exists a set with 3 elements such that if any $f \in \mathscr{F}$ satisfies $\overline{E}(S,f) = \overline{E}(S,f^{(k)})$, and the zeros of $f(z)-a$ are of multiplicity $\geq k+1$, then $\mathscr{F}$ is normal in D.*

Proof Without loss of generality, we may assume $D = \Delta$, $S = \{a_1, a_2, a_3\}$. $\mathscr{F}$ is not normal in Δ. We consider two cases.

Case 1 $a \in S$. We need only consider that $a = a_1$. Set $A = \max\limits_{a \in S}|s| + 1$, then by $\overline{E}(S,f) = \overline{E}(S,f^{(k)})$ and Lemma 1, there exist points z_n with $|z_n| < r < 1$, functions $f_n \in \mathscr{F}$, and positive numbers $\rho_n \to 0^+$, such that $\rho_n^{-k} f_n(z_n + \rho_n \zeta) - \rho_n^{-k} a = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a non-constant holomorphic function, all of whose zeros have multiplicity at least $k+1$, and $g^{\#}(\zeta) \leq g^{\#}(0) = kA+1$.

First, we claim: $\overline{E}(S, g^{(k)}) = \overline{E}(0, g)$. Suppose that $g(\zeta_0) = 0$. Then by Hurwitz's theorem, there exists a sequence $\{\zeta_n\}$ with $\zeta_n \to \zeta_0$, such that (for n sufficiently large) $g_n(\zeta_n) = 0$. Thus $f_n(z_n + \rho_n\zeta_n) = a_1$. Since $\overline{E}(S, f) = \overline{E}(S, f^{(k)})$, we have $f_n^{(k)}(z_n + \rho_n\zeta_n) \in S$, thus $g_n^{(k)}(\zeta_n) \in S$, so $g^{(k)}(\zeta_0) \in S$. Therefore $\overline{E}(0, g) \subseteq \overline{E}(S, g^{(k)})$.

Now suppose that $g^{(k)}(\zeta_0) = s$, $s \in S$. We claim that $g^{(k)}(\zeta) \not\equiv s$. If $g^{(k)}(\zeta) \equiv s$, then $g(\zeta)$ is a polynomial of the degree at most k, which contradicts the fact that the zeros of $g(\zeta)$ are of multiplicity $\geq k + 1$.

By Hurwitz's theorem, there exists a sequence ζ_n with $\zeta_n \to \zeta_0$, such that $g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) = s$, and since $\overline{E}(S, f) = \overline{E}(S, f^{(k)})$, we have $f_n(z_n + \rho_n\zeta_n) \in S$. Hence, there exists a subsequence of $\{f_n\}$, still denoted by $\{f_n\}$, such that $f_n(z_n + \rho_n\zeta_n) = s'$, $s' \in S$.

If $s' \neq a_1$, then $g(\zeta_0) = \lim\limits_{n\to\infty} g_n(\zeta_n) = \lim\limits_{n\to\infty} \frac{s'-a_1}{\rho_n^k} = \infty$, which contradicts $g^{(k)}(\zeta_0) = s$.

If $s' = a_1$, then $g(\zeta_0) = \lim\limits_{n\to\infty} g_n(\zeta_n) = \lim\limits_{n\to\infty} \frac{f_n(z_n+\rho_n\zeta_n)-a_1}{\rho_n^k} = 0$.
Namely, $\overline{E}(S, g^{(k)}) \subseteq \overline{E}(0, g)$. Hence $\overline{E}(S, g^{(k)}) = \overline{E}(0, g)$.

Now we consider the following two subcases:

Subcase 1.1 $g(\zeta)$ is transcendental entire function. By the second fundamental theorem to $g^{(k)}(\zeta)$, we have

$$2T(r, g^{(k)}) \leq \overline{N}(r, g) + \sum_{i=1}^{3} \overline{N}(r, \frac{1}{g^{(k)} - a_i}) + S(r, g^{(k)}) \leq \overline{N}(r, \frac{1}{g}) + S(r, g^{(k)}). \tag{2.1}$$

Since the zeros of $g(\zeta)$ are of multiplicity $\geq k + 1$, we get

$$\overline{N}(r, \frac{1}{g}) \leq \overline{N}(r, \frac{1}{g^{(k)}}) \leq T(r, g^{(k)}). \tag{2.2}$$

Thus (2.1) and (2.2) yield $T(r, g^{(k)}) = S(r, g^{(k)})$, which is a contradiction.

Subcase 1.2 $g(\zeta)$ is a polynomial. Set

$$g(\zeta) = c_0\zeta^m + c_1\zeta^{m-1} + \cdots + c_m, \quad m \geq k + 1,$$

where c_j $(j = 0, 1, \ldots, m)$ are finite complex numbers, and $c_0 \neq 0$.

$$T(r, g^{(k)}) = (m - k)\log r + O(1), \quad \text{as} \quad r \to \infty.$$

$$\overline{N}(r, \frac{1}{g}) \leq \frac{m}{k+1}\log r + O(1) \leq \frac{m}{2} + O(1), \; S(r, g^{(k)}) = O(1).$$

From (2.1), we obtain

$$2(m - k)\log r \leq \frac{m}{2}\log r + O(1),$$

thus $m \leq \frac{4}{3}k$. Since the zeros of $g(\zeta)$ are of multiplicity $\geq k + 1$, $g(\zeta)$ has only one zero ζ_0. Then $g(\zeta) = c_0(\zeta - \zeta_0)^m$, and $g^{(k)}(\zeta_0) = m(m - 1)\cdots(m - k + 1)|c_0|(\zeta - \zeta_0)^{m-k}$. Obviously, $g^{(k)}(\zeta) = a_j (j = 1, 2, 3)$ have $3(m - k)(\geq 3)$ zeros, which contradicts $\overline{E}(0, g) = \overline{E}(S, g^{(k)})$.

Case 2 $a \notin S$. By Lemma 1, there exist points z_n with $|z_n| < r < 1$, functions $f_n \in \mathscr{F}$, and positive numbers $\rho_n \to 0^+$, such that $g_n = f_n(z_n + \rho_n\zeta) - a \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a non-constant holomorphic function, all of whose zeros have multiplicity at least $k+1$.

Using the same argument as Case 1, we have $\overline{E}(a_i - a, g) \subseteq \overline{E}(0, g^{(k)})$, $i = 1, 2, 3$. By the second fundamental theorem and Lemma 2, we have

$$\begin{aligned}3T(r,g) &\le \overline{N}(r,\frac{1}{g}) + \sum_{i=1}^{3}\overline{N}(r,\frac{1}{g-(a_i-a)}) + S(r,g)\\ &\le \frac{1}{k+1}N(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g^{(k)}}) + S(r,g)\\ &\le (1+\frac{1}{k+1})N(r,\frac{1}{g}) + S(r,g)\\ &\le (1+\frac{1}{k+1})T(r,g) + S(r,g).\end{aligned}$$

Since $k \ge 1$, $T(r,g) = S(r,g)$, which is a contradiction. Lemma 4 is proved. □

3. Proof of Theorem 1

Proof Set $S = \{0, a, b\}$, where a, b are two non-zero distinct finite complex numbers satisfying

$$a^2 \ne b^2,\ a \ne 2b,\ a^2 - ab + b^2 \ne 0,\ 3d^2 - 2(a+b)d + ab \ne 0.$$

First, we prove $\rho_f \le 1$. Set $\mathscr{F} = \{f(z+\omega)\}$, $z \in \{z : |z| < 1\}$. Then $\mathscr{F}$ is a family of holomorphic functions in D. Obviously $\forall g(z) = f(z+\omega) \in \mathscr{F}$, we have $E(S,g) = E(S, g^{(k)})$, and the zeros of $g - d$ are of multiplicity $\ge k+1$. By Lemma 4, $\mathscr{F}$ is normal in D. Thus by Marty's criteria, there exists $M(>0)$ satisfying

$$f^{\#}(\omega) = \frac{|f'(\omega)|}{1+|f(\omega)|^2} = \frac{|g'(0)|}{1+|g(0)|^2} = g^{\#}(0) \le M$$

for ω all in $\mathbb{C}$. By Lemma 3, $\rho_f \le 1$. Set

$$\varphi(z) = \frac{f^{(k)}(z)[f^{(k)}(z)-a][f^{(k)}(z)-b]}{f(z)[f(z)-a][f(z)-b]}. \tag{3.1}$$

Then by $E(S,f) = E(S,f^{(k)})$, there exists an entire function $\alpha(z)$ satisfying

$$\varphi(z) = e^{\alpha(z)}. \tag{3.2}$$

Standard computations involving the lemma on the logarithmic derivative show that

$$m(r,\varphi) = S(r,f), \tag{3.3}$$

and hence

$$T(r,\varphi) = m(r,\varphi) + N(r,\varphi) = S(r,f). \tag{3.4}$$

By $\rho_f \le 1$, $T(r,f) = O(r)$, $S(r,f) = O(logr)$. It then follows from (3.4) that φ is a polynomial, so by (3.2) φ must be a non-zero constant c. Hence

$$\frac{f^{(k)}(z)[f^{(k)}(z)-a][f^{(k)}(z)-b]}{f(z)[f(z)-a][f(z)-b]} = c,$$

that is,

$$f^{(k)}(z)[f^{(k)}(z)-a][f^{(k)}(z)-b]=cf(z)[f(z)-a][f(z)-b]. \tag{3.5}$$

Differentiating the two sides of (3.5), we obtain

$$[3(f^{(k)})^2-2(a+b)f^{(k)}+ab]f^{(k+1)}=c[3f^2-2(a+b)f+ab]f'. \tag{3.6}$$

We claim $f(z)-d\neq 0$. Indeed, suppose that z_0 is $p(\geq k+1)$ zero of $f(z)-d$. Then the left-hand side of (3.6) vanishes at z_0 to order $p-k-1$,while the right-hand side vanishes to the order at least $p-1$, a contradiction. Hence

$$f(z)=d+Be^{Az} \tag{3.7}$$

and

$$f^{(k)}(z)=BA^ke^{Az}, \tag{3.8}$$

where $A\neq 0, B\neq 0$, and d are constants.

$$T(r,f)\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-d})+S(r,f)=\overline{N}(r,\frac{1}{f})+S(r,f),$$

that is

$$\overline{N}(r,\frac{1}{f})=T(r,f)+S(r,f). \tag{3.9}$$

Similarly, we have

$$\overline{N}(r,\frac{1}{f-a})=T(r,f)+S(r,f),\overline{N}(r,\frac{1}{f-b})=T(r,f)+S(r,f). \tag{3.10}$$

By (3.9), (3.10), $\overline{E}(S,f)=\overline{E}(S,f^{(k)})$, and the second fundamental theorem, we have

$$\begin{aligned}
3T(r,f)&\leq \overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-a})+\overline{N}(r,\frac{1}{f-b})+S(r,f)\\
&\leq \overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f)\\
&\leq 2T(r,f^{(k)})+S(r,f)\\
&\leq 2m(r,\frac{f^{(k)}}{f})+2m(r,f)+S(r,f)\\
&\leq 2T(r,f)+S(r,f).
\end{aligned}$$

Hence we obtain $T(r,f)=S(r,f)$, which contradicts (3.6). Thus $d\in S$. Now we consider the following three cases.

Case 1 $d=0$. By (3.6) and (3.7), we have

$$f(z)=Be^{Az}(A\neq 0,B\neq 0),f^{(k)}(z)=BA^ke^{Az}\neq 0. \tag{3.11}$$

Suppose $f(z_1)=a$. Then since $E(S,f)=E(S,f^{(k)})$, we have either $f^{(k)}(z_1)=a$ or $f^{(k)}(z_1)=b$. If $f^{(k)}(z_1)=a$, then by (3.11), $A^k=1$, so $f\equiv f^{(k)}$. If $f^{(k)}(z_1)=b$, then by (3.11),

$$A^k=\frac{a}{b}. \tag{3.12}$$

Similarly, if $f(z_2)=b$, then either $f^{(k)}(z_2)=a$ or $f^{(k)}(z_2)=b$. If $f^{(k)}(z_2)=a$, then by (3.11),

$$A^k=\frac{a}{b}. \tag{3.13}$$

If $f^{(k)}(z_2)=b$, then $f\equiv f^{(k)}$. Thus either $f\equiv f^{(k)}$ or, by (3.12) and (3.13), $a^2=b^2$. However, this contradicts $a^2\neq b^2$. It follows that if $d=0$, then $f\equiv f^{(k)}$.

Case 2 $d=a$. By (3.6) and (3.7), we have

$$f(z)=a+Be^{Az}, f^{(k)}(z)=BA^ke^{Az}\neq 0. \tag{3.14}$$

Let $f(z_3)=0$. Then since $E(S,f)=E(S,f^{(k)})$, either $f^{(k)}(z_3)=a$ or $f^{(k)}(z_3)=b$. Assume first that $f^{(k)}(z_3)=a$. Then by (3.14), $A^k=-1$. Thus

$$f(z)=a+Be^{Az},\ f^{(k)}(z)=-Be^{Az}. \tag{3.15}$$

Let $f(z_4)=b$. Then since $E(S,f)=E(S,f^{(k)})$, either $f^{(k)}(z_4)=a$ or $f^{(k)}(z_4)=b$. If $f^{(k)}(z_4)=a$, (3.15) gives $b=0$, which contradicts $b\neq 0$. If $f^{(k)}(z_4)=b$, we obtain $a=2b$, which also contradicts $a\neq 2b$. A similar argument applies in case $f^{(k)}(z_4)=b$. In that case, $A^k=-\frac{b}{a}$ and

$$f(z)=a+Be^{Az},\ f^{(k)}(z)=-\frac{b}{a}\cdot Be^{Az}. \tag{3.16}$$

Choosing z_5 so that $f(z_5)=b$, we have either $f^{(k)}(z_5)=a$ or $f^{(k)}(z_5)=b$. If $f^{(k)}(z_5)=a$, (3.16) yields $a^2-ab+b^2=0$, which contradicts $a^2-ab+b^2\neq 0$. Similarly $f^{(k)}(z_5)=b$ leads to $b=0$, which is also ruled out. It follows that Case 2 cannot occur.

Case 3 $d=b$. This case is symmetric to Case 2 and can be eliminated by the same arguments.

In the above discussion we have shown that $f\equiv f^{(k)}$. This completes the proof of Theorem 1. □

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