Journal of Mathematical Research & Exposition
Nov., 2011, Vol. 31, No. 6, pp. 1047–1056
DOI:10.3770/j.issn:1000-341X.2011.06.012
Http://jmre.dlut.edu.cn

Positive Solutions for the Initial Value Problems of Impulsive Evolution Equations

He YANG
Department of Mathematics, Northwest Normal University, Gansu 730070*, P. R. China*

Abstract This paper deals with the existence of e-positive mild solutions (see Definition 1) for the initial value problem of impulsive evolution equation with noncompact semigroup

$$\begin{cases} u'(t)+Au(t)=f(t,\ u(t)),\ t\in[0,\ +\infty),\ t\neq t_k, \\ \triangle u|_{t=t_k}=I_k(u(t_k)),\ k=1,2,\ldots, \\ u(0)=x_0 \end{cases}$$

in an ordered Banach space E. By using operator semigroup theory and monotonic iterative technique, without any hypothesis on the impulsive functions, an existence result of e-positive mild solutions is obtained under weaker measure of noncompactness condition on nonlinearity of f. Particularly, an existence result without using measure of noncompaceness condition is presented in ordered and weakly sequentially complete Banach spaces, which is very convenient for application. An example is given to illustrate that our results are more valuable.

Keywords impulsive evolution equation; e-positive mild solution; equicontinuous semigroup; Measure of noncompactness.

Document code A
MR(2010) Subject Classification 34A37; 34B18; 34G25
Chinese Library Classification O175.15

1. Introduction and main results

During recent years, the impulsive differential equations have been an object of intensive investigation because of the wide possibilities for their applications in various fields of science and technology such as theoretical physics, population dynamics, economics, etc. [1, 2]. Correspondingly, the existence of solutions for impulsive differential equations in Banach spaces has also been studied by many authors [3–6]. But these results are for the case of ordinary differential equations. There are seldom the results on impulsive evolution equations [7, 8].

In this paper, we consider the initial value problem (IVP) of nonlinear impulsive evolution

Received January 3, 2010; Accepted January 12, 2011
Supported by the National Natural Science Foundation of China (Grant No. 10871160) and the Natural Science Foundation of Gansu Province (Grant No. 0710RJZA103).
E-mail address: yanghe256@163.com

equation

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \ t \in J_\infty, \ t \neq t_k, \\ \triangle u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \ldots, \\ u(0) = x_0, \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is a closed linear operator, $-A$ generates a C_0-semigroup $T(t)$ $(t \geq 0)$ in E. $f \in C(J_\infty \times E, E)$, $J_\infty = [0, +\infty)$. $0 < t_1 < t_2 < \cdots < t_m < \cdots$, $t_m \to +\infty (m \to +\infty)$, and $I_k : E \to E$, $k = 1, 2, \ldots$ are impulsive functions, $x_0 \in E$. $\triangle u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

Let $PC(J_\infty, E) := \{u : J_\infty \to E | u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots\}$. Let $J'_\infty = J_\infty \backslash \{t_1, t_2, \ldots, t_m, \ldots\}$, $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots$. Let λ_1 be the minimal positive real eigenvalue of the linear operator A, $e_1 \in D(A)$ be the positive eigenvector corresponding to λ_1.

In 1999, Liu [7] studied the existence and uniqueness of mild solutions for the problem

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \ t \in [0, T_0], \ t \neq t_k, \\ \triangle u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \ldots, p, \\ u(0) = x_0. \end{cases} \tag{2}$$

The existence theorem in [7] required that nonlinearity f and impulsive functions I_k's satisfy the following assumptions:

$$\|f(t, u) - f(t, v)\| \leq C^* \|u - v\|, \ t \in [0, T_0], \ u, v \in E, \tag{3}$$

$$\|I_k(u) - I_k(v)\| \leq h_k \|u - v\|, \ u, v \in E, \ k = 1, 2, \ldots, p, \tag{4}$$

where $C^* > 0$ and $h_k > 0$ satisfy

$$M^*(C^* T_0 + \sum_{k=1}^{p} h_k) < 1 \tag{5}$$

with $M^* = \max_{t \in [0, T_0]} \|T(t)\|$.

The conditions (3)–(5) are all strongly restricted and are difficultly satisfied in applications. Recently, Cardinali and Rubbioni [8] extended and improved the result of Liu in [7]. They deleted the conditions (4) and (5) and improved the condition (3). They only required that nonlinearity f satisfies the following conditions:

$$\|f(t, x)\| \leq a(t)(1 + \|x\|), \ t \in [0, T_0], \ x \in E, \tag{6}$$

where $a \in L^1([0, T_0], \mathbb{R}^+)$ is a function, and for any bounded $D \subset E$

$$\alpha(f(t, D)) \leq k(t) \alpha(D), \ t \in [0, T_0], \tag{7}$$

where $k \in L^1([0, T_0], \mathbb{R}^+)$ is a function.

But the condition (7) is also difficult to verify in applications. In this paper, we will improve or delete the condition (7) by using order conditions in ordered Banach spaces. These order conditions are verified conveniently in applications. In addition, we obtain the existence of positive solutions for the initial value problem of impulsive evolution equations on J_∞.

Our main results are as follows:

Theorem 1 *Let E be an ordered Banach space with norm $\|\cdot\|$ and partial order "$\leq$", whose positive cone K is normal, and $-A$ generate a positive equicontinuous C_0-semigroup $T(t)$ $(t \geq 0)$. Let $x_0 \geq \sigma e_1$, $f(t, \sigma e_1) \geq \lambda_1 \sigma e_1$ for $\sigma > 0$ and $t \in J_\infty$. If the nonlinearity $f \in C(J_\infty \times K, E)$ satisfies the following conditions:*

(H_1) There exist $a, b \in C(J_\infty, J_\infty)$ such that

$$\|f(t,x)\| \leq a(t)\|x\| + b(t), \ t \in J_\infty, \ x \in K.$$

(H_2) For any $R > 0, T > 0$, there exists $C = C(R,T) > 0$ such that

$$f(t,x_2) - f(t,x_1) \geq -C(x_2 - x_1),$$

for any $t \in [0,T]$, $\theta \leq x_1 \leq x_2$, $\|x_1\|$, $\|x_2\| \leq R$.

(H_3) For any $R > 0, T > 0$, there exists $L = L(R,T) > 0$ such that

$$\alpha(f(t,D)) \leq L\alpha(D)$$

for any $t \in [0,T]$, and increasing monotonic sequence $D = \{x_n\} \subset K \cap \overline{B}(\theta, R)$. Then the IVP(1) has an e-positive mild solution on J_∞.

Remark 1 Analytic semigroup and differentiable semigroup are equicontinuous semigroups [11]. In applications of partial differential equations, such as strongly damped wave equation, parabolic type equation, etc., their solution semigroups are the analytic semigroups. Hence, it is convenient to apply Theorem 1 to these equations.

When E is an ordered and weakly sequentially complete Banach space, we delete the measure of noncompactness condition (H_3) of Theorem 1 and obtain the following result:

Corollary 1 *Let E be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, $-A$ generate a positive equicontinuous C_0-semigroup $T(t)$ $(t \geq 0)$. Let $x_0 \geq \sigma e_1$, $f(t, \sigma e_1) \geq \lambda_1 \sigma e_1$ for $\sigma > 0$ and $t \in J_\infty$. If the nonlinearity $f \in C(J_\infty \times K, E)$ satisfies the assumptions (H_1) and (H_2), then the IVP(1) has an e-positive mild solution on J_∞.*

Remark 2 In applications of some partial differential equations, we often choose $L^p(\Omega)$ as working space, which is weakly sequentially complete. Hence, it is very convenient in $L^p(\Omega)$ to apply the Corollary 1 to these equations.

The proof of Theorem 1 will be introduced in next section. In Section 3, an example will be given to illustrate that our results are more valuable.

2. Proof of main result

Let $(E, \|\cdot\|)$ be a Banach space, $A : D(A) \subset E \to E$ be a closed linear operator, $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ in E. Then there exist $M > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\delta t}, \ t \geq 0.$$

Let $I = [t_0, T]$, $C(I, E)$ denote the Banach space of all continuous E-value functions on interval I with norm $\|u\|_C = \max_{t \in I} \|u(t)\|$. We consider the initial value problem (IVP) of linear evolution equation without impulse in E

$$\begin{cases} u'(t) + Au(t) = \varphi(t), \ t \in I, \\ u(t_0) = x_0. \end{cases} \tag{8}$$

It is well-known [1, Chapter 4, Theorem 2.9] that when $x_0 \in D(A)$ and $\varphi \in C^1(I, E)$, the IVP(8) has unique classical solution $u \in C^1(I, E) \cap C(I, E_1)$ (where E_1 is a Banach space generated by $D(A)$ with norm $\|x\|_1 = \|x\| + \|Ax\|$) expressed by

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)\varphi(s)\mathrm{d}s, \ t \in I. \tag{9}$$

Generally, when $x_0 \in E$ and $\varphi \in C(I, E)$, the function u given by (9) belongs to $C(I, E)$ and it is called a mild solution of the IVP(8).

Similarly, for the initial value problem (IVP) of the linear impulsive evolution equation in E

$$\begin{cases} u'(t) + Au(t) = \varphi(t), \ t \in J_\infty, \ t \neq t_k, \\ \triangle u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \ldots, \\ u(0) = x_0. \end{cases} \tag{10}$$

Definition 1 *If an abstract function $u \in PC(J_\infty, E)$ satisfies the following integral equation*

$$u(t) = T(t)x_0 + \int_0^t T(t - s)\varphi(s)\mathrm{d}s + \sum_{0 < t_k < t} T(t - t_k)I_k(u(t_k)),$$

then we call it a mild solution of the IVP(10). Furthermore, if there exist $e \geq 0$ and $\sigma > 0$ such that $u(t) \geq \sigma e$ for $t \in J_\infty$, then we call it an e-positive mild solution of the IVP(10).

Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set in E and $C(I, E)$. We refer to [12] for the details of the definition and the properties of the measure of noncompactness. For any $B \subset C(I, E)$ and $t \in I$, set $B(t) = \{u(t) | u \in B\} \subset E$. If B is bounded in $C(I, E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$. The following lemmas will be used in the proof of Theorem 1.

Lemma 1 *Let $B \subset C(I, E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on I, and*

$$\alpha(B) = \max_{t \in I} \alpha(B(t)).$$

This lemma can be found in [13, Theorem 1.1.2].

Lemma 2 *Let $B = \{u_n\} \subset C(I, E)$ be countable. If there exists $\psi \in L^1(I)$ such that $\|u_n(t)\| \leq$*

$\psi(t)$, *a. e.* $t \in I$, $n = 1, 2, \ldots$, *then* $\alpha(B(t))$ *is Lebesgue integral on* I, *and*

$$\alpha(\{\int_I u_n(t)\mathrm{d}t | u_n \in B\}) \le 2\int_I \alpha(B(t))\mathrm{d}t.$$

This lemma can be found in [14, Corollary 3.1(b)].

Proof of Theorem 1 (I) We prove the global existence of e-positive mild solutions for the IVP(1) on $J_0 = [0, t_1]$.

In this case, the IVP(1) is equivalent to the initial value problem(IVP) of evolution equation without impulse in E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)),\ t \in J_0, \\ u(0) = x_0. \end{cases} \tag{11}$$

Next the proof will be divided into two steps.

(i) The local existence of e-positive mild solutions.

For any $t_0 \ge 0$ and $x_0 \in E$, we prove that the initial value problem (IVP) of evolution equation

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)),\ t > t_0, \\ u(t_0) = x_0 \end{cases} \tag{12}$$

has an e-positive mild solution on $I = [t_0, t_0 + h_{t_0}]$, where $h_{t_0} \in (0, 1)$ is pending. Let

$$M_{t_0} = \sup\{\|T(t)\| | 0 \le t \le t_0 + 1\},\ R_{t_0} = 2M_{t_0}(\|x_0\| + 1) + \sigma e_1,$$
$$a_{t_0} = \max_{t \in [0, t_0+1]} a(t), b_{t_0} = \max_{t \in [0, t_0+1]} b(t),\ L_{t_0} = L(t_0 + 1, R_{t_0}).$$

Let $C = C(t_0 + 1, R_{t_0})$ be the constant in condition (H_2). The IVP(12) can be rewritten as the form

$$\begin{cases} u'(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t),\ t > t_0, \\ u(t_0) = x_0. \end{cases} \tag{13}$$

We define the mapping Q by

$$(Qu)(t) = S(t - t_0)x_0 + \int_{t_0}^{t} S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s,\ t \in I, \tag{14}$$

where $S(t) = e^{-Ct}T(t)$ $(t \ge 0)$ is the C_0-semigroup generated by $-(A + CI)$. Then $S(t)$ $(t \ge 0)$ is a positive equicontinuous C_0-semigroup. From condition (H_2) and the continuity of f, $Q : C(I, K) \to C(I, E)$ is continuous and increasing, and a solution of the IVP(13) on I is equivalent to a fixed point of Q.

Denote $\Omega := \{u \in C(I, K) | \|u(t)\| \le R_{t_0}, u(t) \ge \sigma e_1, t \in I\}$. Then $\Omega \subset C(I, K)$ is a nonempty bounded convex closed set. Let $h_{t_0} \le \min\{1, \frac{\|x_0\|+1}{(a_{t_0}+C)R_{t_0}+b_{t_0}}\}$. Then for every $u \in \Omega$ and $t \in I$, by assumption (H_1) and (14), we have

$$\|(Qu)(t)\| \le \|S(t - t_0)x_0\| + \|\int_{t_0}^{t} S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s\|$$

$$\leq M_{t_0}\|x_0\| + M_{t_0}\int_{t_0}^{t} a(s)\|u(s)\| + b(s) + C\|u(s)\|\mathrm{d}s$$
$$\leq M_{t_0}\|x_0\| + M_{t_0}[(a_{t_0}+C)R_{t_0} + b_{t_0}]h_{t_0} \leq R_{t_0}.$$

Let $v_0 = \sigma e_1$. Then

$$\varphi(t) \triangleq v_0'(t) + (A+CI)v_0(t) = \lambda_1\sigma e_1 + C\sigma e_1 \leq f(t,\sigma e_1) + C\sigma e_1.$$

Since $S(t)$ is a positive C_0-semigroup and Q is an increasing operator, from (14), we have

$$\begin{aligned}\sigma e_1 = v_0(t) &= S(t-t_0)v_0(t_0) + \int_{t_0}^{t} S(t-s)\varphi(s)\mathrm{d}s\\ &\leq S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)[f(t,\sigma e_1) + C\sigma e_1]\mathrm{d}s\\ &= Q(\sigma e_1)(t) \leq (Qu)(t), \quad t \in I.\end{aligned}$$

Thus, $Q : \Omega \to \Omega$ is continuous and increasing. By the similar method of Li [10], we can prove that $Q(\Omega)$ is a family of equicontinuous functions in $C(I,K)$.

Let $v_0 = \sigma e_1 \in \Omega$. Define interval sequence $\{v_n\}$ by

$$v_n = Qv_{n-1}, \quad n = 1, 2, \ldots. \tag{15}$$

Since Q is an increasing operator and $v_1 = Qv_0 \geq v_0$, we have

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots. \tag{16}$$

Hence, $\{v_n\} = \{Qv_{n-1}\} \subset Q(\Omega) \subset \Omega$ is bounded and equicontinuous.

Let $B = \{v_n | n \in \mathbb{N}\}$, $B_0 = \{v_{n-1} | n \in \mathbb{N}\}$. Since $B_0 = B \cup \{v_0\}$, from the property of measure of noncompactness, we have $\alpha(B(t)) = \alpha(B_0(t))$ for $t \in I$. From condition (H_3), Lemma 2 and (14), for any $t \in I$, we have

$$\begin{aligned}\alpha(B(t)) &= \alpha(Q(B_0)(t)) = \alpha(\{S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)[f(s,v_{n-1}(s)) + Cv_{n-1}(s)]\mathrm{d}s | n \in \mathbb{N}\})\\ &= \alpha(\{\int_{t_0}^{t} S(t-s)[f(s,v_{n-1}(s)) + Cv_{n-1}(s)]\mathrm{d}s | n \in \mathbb{N}\})\\ &\leq 2\int_{t_0}^{t} \|S(t-s)\| \cdot \alpha(\{f(s,v_{n-1}(s)) + Cv_{n-1}(s) | n \in \mathbb{N}\})\mathrm{d}s\\ &\leq 2M_{t_0}\int_{t_0}^{t} \alpha(f(s,B_0(s)) + C\alpha(B_0(s))))\mathrm{d}s \leq 2M_{t_0}(L_{t_0}+C)\int_{t_0}^{t} \alpha(B_0(s))\})\mathrm{d}s\\ &= 2M_{t_0}(L_{t_0}+C)\int_{t_0}^{t} \alpha(B(s))\})\mathrm{d}s.\end{aligned}$$

By Bellman inequality, we have $\alpha(B(t)) \equiv 0$ for $t \in I$. By Lemma 1, we deduce that $\alpha(B) = \max_{t\in I}\alpha(B(t)) = 0$, i.e., $\{v_n\}$ is relatively compact in $C(I,K)$. Hence, there exists subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \to u^* \in \Omega$ as $k \to \infty$. Combining this with (16) and the normality of cone K, it is easy to prove that $v_n \to u^*$ as $n \to \infty$. Let $n \to \infty$ in (15). From the continuity of operator Q, we have $u^* = Qu^*$. Therefore, $u^* \in \Omega \subset C(I,K)$ is an e-positive mild solution of the IVP(13).

(ii) The global existence of e-positive mild solutions for the IVP(11) on J_0.

From (i), we easily see that the IVP(11) has an e-positive mild solution $u_0 \in C([0,h_0],K)$ expressed by

$$u_0(t)=S(t)x_0+\int_0^t S(t-s)[f(s,u_0(s))+Cu_0(s)]\mathrm{d}s.$$

By the extension theorem [11], u_0 can be extended to a saturated solution of the IVP(11), which is also denoted by $u_0 \in C([0,T),K)$, whose existence interval is $[0,T)$.

Next, we show that $T>t_1$. Denote

$$\overline{a}=\max_{t\in[0,T+1]}a(t),\ \overline{b}=\max_{t\in[0,T+1]}b(t),\ M_1=\sup_{t\in[0,T+1]}\|T(t)\|.$$

If $T\le t_1$, then by assumption (H_1), we have

$$\begin{aligned}\|u_0(t)\| &\le \|S(t)x_0\|+\int_0^t\|S(t-s)[f(s,u_0(s))+Cu_0(s)]\|\mathrm{d}s\\ &\le M_1\|x_0\|+M_1\overline{b}T+M_1(\overline{a}+C)\int_0^t\|u_0(s)\|\mathrm{d}s.\end{aligned}$$

By Bellman inequality, we have

$$\|u_0(t)\|\le M_1(\|x_0\|+\overline{b}T)e^{M_1(\overline{a}+C)t}\le M_1(\|x_0\|+\overline{b}T)e^{M_1(\overline{a}+C)T}\triangleq M_2.$$

Hence let $N_0=N(0,M_2):=\sup_{t\in[0,T+1],\|x\|\le M_2}\|f(t,x)\|$. Since $S(t)$ is continuous in operator norm for $t>0$, for any $0<\tau_1<\tau_2<T$, we have

$$\begin{aligned}&\|u_0(\tau_2)-u_0(\tau_1)\|\\ &\le \|S(\tau_2)x_0-S(\tau_1)x_0\|+\int_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|\cdot\|f(s,u_0(s))+Cu_0(s)\|\mathrm{d}s+\\ &\quad\int_{\tau_1}^{\tau_2}\|S(\tau_2-s)\|\cdot\|f(s,u_0(s))+Cu_0(s)\|\mathrm{d}s\\ &\le \|S(\tau_2)x_0-S(\tau_1)x_0\|+(N_0+CM_2)\int_0^T\|S(\tau_2-\tau_1+s)-S(s)\|\mathrm{d}s+\\ &\quad M_1(N_0+CM_2)(\tau_2-\tau_1)\to 0,(\tau_1,\tau_2\to T^-).\end{aligned}$$

Hence by Cauchy criterion, there exists $\overline{x}\in K$ such that $\lim_{t\to T^-}u_0(t)=\overline{x}$. We consider the initial value problem (IVP) of evolution equation without impulsive in E

$$\begin{cases} u'(t)+(A+CI)u(t)=f(t,u(t))+Cu(t),\ t>T,\\ u(T)=\overline{x}.\end{cases}\tag{17}$$

From (i), the IVP(17) has an e-positive mild solution v on $[T,T+h_T]$. Let

$$\widetilde{u}(t)=\begin{cases} u_0(t),\ t\in[0,T),\\ v(t),\ t\in[T,T+h_T].\end{cases}$$

It is easy to see that $\widetilde{u}(t)$ is an e-positive mild solution of the IVP(11) on $[0,T+h_T]$. Therefore, $\widetilde{u}(t)$ is an extension of $u_0(t)$, this is a contradiction. Hence, $T>t_1$, i.e., the global e-positive mild solution $u_0(t)$ of the IVP(11) exists on J_0, which is also an e-positive mild solution of the IVP(1) on J_0.

(II) We show that the IVP(1) has global e-positive mild solution on J_∞.

At first, we prove that the IVP(1) has global e-positive mild solution on $J_1 = (t_1, t_2]$. We consider the initial value problem (IVP) of evolution equation without impulse on J_1

$$\begin{cases} u'(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t), \ t \in J_1, \\ u(t_1^+) = u_0(t_1) + I_1(u_0(t_1)). \end{cases} \tag{18}$$

Clearly, a global e-positive mild solution of the IVP(18) on J_1 is also an e-positive mild solution of the IVP(1) on J_1. By the argument similar to the proof of (I), the IVP(18) has an e-positive mild solution $u_1 \in C(J_1, K)$ expressed by

$$\begin{aligned} u_1(t) &= S(t - t_1)(u_0(t_1) + I_1(u_0(t_1))) + \int_{t_1}^{t} S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s \\ &= S(t)x_0 + \int_0^t S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s + S(t - t_1)I_1(u_0(t_1)). \end{aligned}$$

Assume that, for $t \in J_{k-1}$ $(k = 3, 4, \ldots)$, the IVP(1) has an e-positive mild solution $u_{k-1} \in C(J_{k-1}, K)$ $(k = 3, 4, \ldots)$. Then, for $t \in J_k$ $(k = 2, 3, \ldots)$, the initial value problem (IVP) of evolution equation without impulse in E

$$\begin{cases} u'(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t), \ t \in J_k, \ k = 2, 3, \ldots, \\ u(t_k^+) = u_{k-1}(t_k) + I_k(u_{k-1}(t_k)) \end{cases}$$

has an e-positive mild solution $u_k \in C(J_k, K)$ expressed by

$$\begin{aligned} u_k(t) =& S(t - t_k)(u_{k-1}(t_k) + I_k(u_{k-1}(t_k))) + \int_{t_k}^{t} S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s \\ =& S(t - t_k)[I_k(u_{k-1}(t_k)) + S(t_k - t_{k-1})(u_{k-2}(t_{k-1}) + I_{k-1}(u_{k-2}(t_{k-1}))) + \\ & \int_{t_{k-1}}^{t_k} S(t_k - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s + \int_{t_k}^{t} S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s \\ =& \cdots \\ =& S(t)x_0 + \int_0^t S(t - s)[f(s, u(s)) + Cu(s)]\mathrm{d}s + \sum_{0 < t_j < t} S(t - t_j)I_j(u_{j-1}(t_j)). \end{aligned}$$

Now, we define a function u by:

$$u(t) = \begin{cases} u_0(t), & t \in J_0, \\ u_1(t), & t \in J_1, \\ \cdots \\ u_k(t), & t \in J_k \ (k = 2, 3, \ldots), \\ \cdots . \end{cases} \tag{19}$$

It is clear that $u(t) \in PC(J_\infty, K)$ is an e-positive mild solution of the IVP(1), which satisfies

$$u(t) = S(t)x_0 + \int_0^t S(t-s)[f(s, u(s)) + Cu(s)]\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k)I_k(u(t_k)).$$

By the global existence property of $u_i(t)$ on J_i, $i = 0, 1, 2, \ldots$, the $u(t)$ defined by (19) is a global e-positive mild solution of the IVP(1) on J_∞. □

3. Applications

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. We consider the initial boundary value problem of parabolic type

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla^2 u = g(x, t, u), \; x \in \Omega, \; t \in J_\infty, \; t \neq t_k, \\ \triangle u|_{t=t_k} = I_k(u(x, t_k)), \; x \in \Omega, \; k = 1, 2, \ldots, \\ u|_{\partial\Omega} = 0, \\ u(x, 0) = \varphi(x), \; x \in \Omega, \end{cases} \tag{20}$$

where ∇^2 denotes a Laplace operator, $g(x, t, u) : \overline{\Omega} \times J_\infty \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. For the problem (20), we have the following existence result:

Theorem 2 *Let λ_1 be the minimal positive real eigenvalue of Laplace operator $-\nabla^2$ with Dirichlet boundary value condition $u|_{\partial\Omega} = 0$, $e_1 \in L^2(\Omega)$ be the positive eigenvector corresponding to λ_1. Let $\varphi(x) \geq e_1(x), g(x, t, \sigma e_1(x)) \geq \lambda_1 \sigma e_1(x)$ for $x \in \overline{\Omega}, t \in J_\infty$ and $\sigma > 0$. If the nonlinearity $g(x, t, u) : \overline{\Omega} \times J_\infty \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies the following conditions:*

(P_1) There exist $a, b \in C(\overline{\Omega} \times J_\infty, \mathbb{R}^+)$ such that

$$|g(x, t, \eta)| \leq a(x, t)|\eta| + b(x, t),$$

for any $x \in \overline{\Omega}$, $t \in J_\infty$, $\eta \in \mathbb{R}^+$.

(P_2) For any $R > 0$, $T > 0$, there exists $M = M(R, T) > 0$ such that

$$g(x, t, \eta_2) - g(x, t, \eta_1) \geq -M(\eta_2 - \eta_1),$$

for any $x \in \overline{\Omega}$, $t \in J_\infty$, and $0 \leq \eta_1 \leq \eta_2, |\eta_1|, |\eta_2| \leq R$, then the problem (20) has an e-positive mild solution.

Proof Let $E = L^2(\Omega)$. Then E is an ordered and weakly sequentially complete Banach space. Let $K = \{u \in L^2(\Omega) | u(x) \geq 0$, a. e. $x \in \Omega\}$. Then K is normal in E. We define an operator A by

$$D(A) = H^2(\Omega) \cup H_0^1(\Omega), \; Au = -\nabla^2 u. \tag{21}$$

The operator $A : D(A) \subset E \to E$ defined by (21) is a positive definite selfadjoint operator with compact resolvent [15]. Hence, the spectrum $\sigma(A)$ of A consists of all positive real eigenvalues and it can be arrayed in sequence as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \; \lambda_n \to \infty, \quad n \to \infty.$$

It is well-known [11, 16] that $-A$ generates an analytic semigroup $T(t)$ $(t \geq 0)$ in L^2 (Ω). By the maximum principle of parabolic equation, it is easy to see that $T(t)$ $(t \geq 0)$ is a positive operator semigroup. For $v \in L^2$ (Ω), let $f(t, v) := g(\cdot, t, v(\cdot))$, $I_k(v(t_k)) = I(v(\cdot, t_k))$. Then the impulsive parabolic equation (20) is rewritten into the form of the impulsive evolution equation (1) in $E = L^2(\Omega)$. Obviously, all the conditions of Corollary 1 are satisfied. From Corollary 1, the impulsive evolution equation (1) has an e-positive mild solution $u^* \in PC(J_\infty, E)$, which is also an e-positive mild solution of the problem (20). $\square$

References

[1] LAKSHMIKANTHAM V, BAINOV D D, SIMEONOV P S. *Theory of Impulsive Differential Equations* [M]. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[2] BAINOV D D, SIMEONOV P S. *Impulsive Differential Equations: Asymptotic Properties of the Solutions* [M]. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[3] LIU Xinzhi, GUO Dajun. *Initial value problems for first order impulsive integro-differential equations in Banach spaces* [J]. Comm. Appl. Nonlinear Anal., 1995, **2**(1): 65–83.

[4] SUN Jinli, MA Yihai. *Initial value problems for the second order mixed monotone type of impulsive differential equations in Banach spaces* [J]. J. Math. Anal. Appl., 2000, **247**(2): 506–516.

[5] LI Yongxiang, LIU Zhe. *Monotone iterative technique for addressing impulsive integro-differential equations in Banach spaces* [J]. Nonlinear Anal., 2007, **66**(1): 83–92.

[6] LIU Yuji. *Positive solutions of periodic boundary value problems for nonlinear first-order impulsive differential equations* [J]. Nonlinear Anal., 2009, **70**(5): 2106–2122.

[7] LIU J H. *Nonlinear impulsive evolution equations* [J]. Dynam. Contin. Discrete Impuls. Systems, 1999, **6**(1): 77–85.

[8] CARDINALI T, RUBBIONI P. *Mild solutions for impulsive semilinear evolution differential inclusions* [J]. J. Appl. Funct. Anal., 2006, **1**(3): 303–325.

[9] SUN Jingxian. *Some new criteria for compactness in Banach spaces and their applications* [J]. Chinese Ann. Math. Ser. A, 1990, **11**(4): 407–412. (in Chinese)

[10] LI Yongxiang. *Existence of solutions to initial value problems for abstract semilinear evolution equations* [J]. Acta Math. Sinica (Chin. Ser.), 2005, **48**(6): 1089–1094. (in Chinese)

[11] PAZY A. *Semigroups of Linear Operators and Applications to Partial Differential Equations* [M]. Springer-Verlag, New York, 1983.

[12] DEIMLING K. *Nonlinear Functional Analysis* [M]. Springer-Verlag, Berlin, 1985.

[13] GUO Dajun, SUN Jingxian. *Oreinary Differential Equations in Abstract Spaces* [M]. Shangdong Science and Technology, Ji'nan, 1989. (in Chinese)

[14] HEINZ H R. *On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions* [J]. Nonlinear Anal., 1983, **7**: 1351–1371.

[15] LI Yongxiang. *Existence and uniqueness of periodic solution for a class of semilinear evolution equations* [J]. J. Math. Anal. Appl., 2009, **349**(1): 226–234.

[16] AMANN H. *Periodic Solutions of Semilinear Parabolic Equations* [M]. Academic Press, New York, 1978.