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A Note on the Exponential Diophantine Equation $(a^m-1)(b^n-1)=x^2$

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Abstract Let a and b be fixed positive integers. In this paper, using some elementary methods, we study the diophantine equation $(a^m-1)(b^n-1)=x^2$. For example, we prove that if $a\equiv 2 \pmod 6$, $b\equiv 3 \pmod{12}$, then $(a^n-1)(b^m-1)=x^2$ has no solutions in positive integers n,m and x.
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1. Introduction

Let a and b be fixed positive integers. There are many works concerning the diophantine equation $(a^m-1)(b^n-1)=x^2$. In [5], Szalay proved that the diophantine equation $(2^n-1)(3^n-1)=x^2$ has no solutions in positive integers n and x, $(2^n-1)(5^n-1)=x^2$ has the only solution $n=1,x=2$ in positive integers n and x, and $(2^n-1)((2^k)^n-1)=x^2$ has the only solution $k=2,n=3,x=21$ in positive integers $k\geq 2,n$ and x. In 2000, Hajdu and Szalay [1] proved the equation $(2^n-1)(6^n-1)=x^2$ has no solutions in positive integers (n,x), while the only solutions to the equation $(a^n-1)(a^{kn}-1)=x^2$, with $a>1,k>1,kn>2$ are $(a,n,k,x)=(2,3,2,21),(3,1,5,22)$, $(7,1,4,120)$. In 2000, Walsh [6] proved that $(2^n-1)(3^m-1)=x^2$ has no solutions in positive integers n,m and x.

Following these works, Luca and Walsh [4] showed that the diophantine equation $(a^k-1)(b^k-1)=x^n$ has finite solutions in positive integers (k,x,n) with $n>1$. Moreover, they showed how one can determine all integers $(k,x,2)$ of the equation above with $k\geq 1$, for almost all pairs (a,b) with $2\leq b<a\leq 100$. In 2009, Le [3] proved that if $3\mid b$, then $(2^n-1)(b^n-1)=x^2$ has no solutions in positive integers n and x. Recently, Li and Lzalay [2] proved that if $a\equiv 2 \pmod 6$ and $b\equiv 0 \pmod 3$, then the equation $(a^n-1)(b^n-1)=x^2$ has no positive integer solution (n,x).

In this paper, using some elementary methods, we obtain the following results:

Theorem 1 *If* $a\equiv 0 \pmod 2$, $b\equiv 15 \pmod{20}$, *then the equation*

$$(a^n-1)(b^n-1)=x^2 \tag{1}$$

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has no solutions in positive integers n *and* x.

Theorem 2 *If* $a \equiv 2 \pmod 6$, $b \equiv 3 \pmod{12}$, *then the equation*

$$(a^n-1)(b^m-1)=x^2 \tag{2}$$

has no solutions in positive integers n, m *and* x.

2. Proofs of Theorems

Let d be a positive integer which is not a square. It is well known that the Pell's equation $x^2-dy^2=1$ has infinitely many positive solutions. If (x_1,y_1) is the smallest positive integer solution, then for $n=1,2,3,\ldots$, define $x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^n$. The pairs (x_n,y_n) are all positive solutions of the Pell's equation. Moreover, the $x_n's$ and $y_n's$ satisfy the following recurrence relations

$$x_{2n}=2x_n^2-1, \quad x_{n+2}=2x_1x_{n+1}-x_n, \tag{3}$$

and

$$y_{2n}=2x_ny_n, \quad y_{n+2}=2x_1y_{n+1}-y_n. \tag{4}$$

Proof of Theorem 1 If Eq.(1) has a solution (n,x), then we have

$$a^n-1=dy^2, \tag{5}$$

and

$$b^n-1=dz^2, \tag{6}$$

where d, y and z are positive integers satisfying $dyz=x$, and d is square-free. Note that $a \equiv 0 \pmod 2$. By (5) we know that d is odd. Thus b^n-1 is properly divisible by an even power of 2. Hence $b^n-1 \equiv 3^n-1 \equiv 0 \pmod 4$, and we know that n must be even.

Let (x_1,y_1) denote the smallest positive integer solution, and $x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^n$ for $n \geq 1$. By (4), if n is even, then $y_n=2x_{n/2}y_{n/2}$ is even. Since $(x_n,y_n)=1$ $(n \geq 1)$, we have x_n is odd for all even values of n. Hence

$$a^{n/2}+y\sqrt{d}=x_r+y_r\sqrt{d} \tag{7}$$

holds for some odd positive integer r. By (3), we know that x_n is even for all odd positive integers n. Thus

$$b^{n/2}+y\sqrt{d}=x_s+y_s\sqrt{d} \tag{8}$$

holds for some positive even integers. Let $s=2t$. Then by (3) we have $b^{m/2}=x_{2t}=2{x_t}^2-1 \equiv 0 \pmod 5$. It follows that ${x_t}^2 \equiv 3 \pmod 5$, which is impossible.

This completes the proof of Theorem 1. □

Proof of Theorem 2 If Eq.(2) has a solution (n,m,x), then we have

$$a^n-1=dy^2, \tag{9}$$

and

$$b^m - 1 = dz^2, \tag{10}$$

where d, y and z are positive integers satisfying $dyz = x$, and d is square-free. Since $b \equiv 3 \pmod{12}$, by (10) we have $dz^2 \equiv 2 \pmod 3$, thus $3 \nmid d$, $3 \nmid z$, hence $z^2 \equiv 1 \pmod 3$, $d \equiv 2 \pmod 3$.

If $3 \nmid y$, then $y^2 \equiv 1 \mod 3$, $a^n = dy^2 + 1 \equiv 0 \pmod 3$, which is impossible. Thus $3 \mid y$, $a^n \equiv 2^n \equiv 1 \pmod 3$, which implies that n must be even.

By (9), we know that d is odd, thus $b^m - 1$ is properly divisible by an even power of 2. Hence $b^m - 1 \equiv 3^m - 1 \equiv 0 \pmod 4$, and we know that m must be even.

Let (x_1, y_1) denote the smallest positive integer solution, and $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ for $n \geq 1$. By (4), if n is even, then $y_n = 2x_{n/2}y_{n/2}$ is even. Noting that $(x_n, y_n) = 1$ $(n \geq 1)$, we have x_n is odd for all even values of n. Hence

$$a^{n/2} + y\sqrt{d} = x_r + y_r\sqrt{d} \tag{11}$$

holds for some odd positive integer r. By (3), we know that x_n is even for all odd positive integers n, thus

$$b^{m/2} + y\sqrt{d} = x_s + y_s\sqrt{d} \tag{12}$$

holds for some positive even integers. Let $s = 2t$. Then by (3) we have $b^{m/2} = x_{2t} = 2{x_t}^2 - 1 \equiv 0 \pmod 3$. It follows that ${x_t}^2 \equiv 2 \pmod 3$, which is impossible.

This completes the proof of Theorem 2. $\square$

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