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Boundedness for Multilinear Commutators of Bochner-Riesz Operator

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Abstract In this paper, the boundedness for the multilinear commutators of Bochner-Riesz operator is considered. We prove that the multilinear commutators generated by Bochner-Riesz operator and Lipschitz function are bounded from $L^p(\mathbb{R}^n)$ into $\dot{\Lambda}_{(\beta-\frac{n}{p})}(\mathbb{R}^n)$ and from $L^{\frac{n}{\beta}}(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$.

Keywords multilinear commutators; Bochner-Riesz operator; Lipschitz function; BMO space.

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1. Introduction and main results

In 2002, Pérez and Trujillo-González [1] introduced a kind of multilinear commutators of singular integral operators and obtained the sharp weighted estimates for this kind of multilinear commutators. From then, the properties of multilinear commutators have been widely studied in harmonic analysis. Hu, Meng and Yang [2, 3] proved the boundedness of multilinear commutators with non-doubling measures. Chen and Ma [4] established that multilinear commutators related to Calderón-Zygmund operator and fractional integral operator with Lipschitz function were bounded in the context of Triebel-Lizorkin space. Later, Mo and Lu [5] studied the boundedness for multilinear commutators of Marcinkiewicz integral operator on Triebel-Lizorkin space and Hardy space. Recently, the weighted estimates for multilinear commutators of Littlewood-Paley operator and Marcinkiewicz integral were established by Xue, Ding [6] and Zhang [7], respectively. The bounded properties for the multilinear commutators of θ-type Calderón-Zygmund operator were considered by authors in [8, 9]. In this paper, boundedness for the multilinear commutators generated by Bochner-Riesz operator and Lipschitz function is obtained. We prove that this kind of multilinear commutators is bounded from $L^p(\mathbb{R}^n)$ into $\dot{\Lambda}_{(\beta-\frac{n}{p})}(\mathbb{R}^n)$ and from $L^{\frac{n}{\beta}}(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$.

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Let us firstly introduce some notations and definitions. Throughout this paper, C always denotes a constant independent of the main parameters involved, but it may be different from line to line. Q will denote a cube of $\mathbb{R}^n$ with side parallel to the axes. For $0 < \beta < 1$, the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h\in\mathbb{R}^n;h\neq 0} |\triangle_h^{[\beta]+1} f(x)|/|h|^\beta < \infty,$$

where $\triangle_h^k$ denotes the k-th difference operator [5, 8].

Given any positive integer m, for $1 \le i \le m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ of $\{1, 2, \ldots, m\}$ with i different elements. For any $\sigma \in C_i^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \ldots, m\}\backslash\sigma$. Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a finite family of locally integrable functions. For all $1 \le i \le m$ and $\sigma = \{\sigma(1), \ldots, \sigma(i)\} \in C_i^m$, we denote $\vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(i)})$ and the product $b_\sigma = b_{\sigma(1)}, \ldots, b_{\sigma(i)}$. With this notation, if $\beta_{\sigma(1)} + \cdots + \beta_{\sigma(i)} = \beta_\sigma$, we write

$$\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}} = \|b_{\sigma(1)}\|_{\dot{\Lambda}_{\beta_{\sigma(1)}}} \cdots \|b_{\sigma(i)}\|_{\dot{\Lambda}_{\beta_{\sigma(i)}}}.$$

For the product of all the functions, we simply write

$$\|\vec{b}\|_{\dot{\Lambda}_\beta} = \prod_{i=1}^m \|b_i\|_{\dot{\Lambda}_{\beta_i}},$$

where $\sum_{i=1}^m \beta_i = \beta$.

Definition 1 *Let $B_t^\delta(f)^\wedge(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a finite family of locally integrable functions with $b_i \in \dot{\Lambda}_{\beta_i}(\mathbb{R}^n)$, $1 \le i \le m$. Let*

$$B_{\vec{b},t}^\delta(f)(x) = \int_{\mathbb{R}^n} \prod_{i=1}^m (b_i(x) - b_i(y)) B_t^\delta(x - y) f(y) \mathrm{d}y,$$

where $B_t^\delta(x) = t^{-n} B^\delta(x/t)$ for $t > 0$. The multilinear commutators of Bochner-Riesz operator is defined by

$$B_{\vec{b},*}^\delta(f)(x) = \sup_{t>0} |B_{\vec{b},t}^\delta(f)(x)|. \tag{1}$$

We also define that

$$B_*^\delta(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the Bochner-Riesz operator [10].

Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$. Then it is clear that

$$B_*^\delta(f)(x) = \|B_t^\delta(f)(x)\| \text{ and } B_{\vec{b},*}^\delta(f)(x) = \|B_{\vec{b},t}^\delta(f)(x)\|.$$

The main results in this paper are as follows.

Theorem 1 *Suppose that $B_{\vec{b},*}^\delta$ is defined by (1), where $\vec{b}$ is as above such that $b_i \in \dot{\Lambda}_{\beta_i}(\mathbb{R}^n)$, $1 \le i \le m$, $0 < \beta_i < 1$, $0 < \sum_{i=1}^m \beta_i = \beta < 1$, $\delta > \frac{n-1}{2}$ and $\max\{\beta - \min_{1\le i\le m} \beta_i, \frac{n-1}{2} - \delta + \beta\} < \frac{n}{p} < \beta$. Then there exists a positive constant C such that*

$$\|B_{\vec{b},*}^\delta(f)\|_{\dot{\Lambda}_{(\beta - \frac{n}{p})}(\mathbb{R}^n)} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Theorem 2 *Suppose that $B^{\delta}_{\vec{b},*}$ is defined by (1), where $\vec{b}$ is as above such that $b_i \in \dot{\Lambda}_{\beta_i}(\mathbb{R}^n)$, $1 \le i \le m$, $0 < \beta_i < 1$, $0 < \sum_{i=1}^m \beta_i = \beta < 1$ and $\delta > \frac{n-1}{2}$. Then there exists a positive constant C such that*

$$\|B^{\delta}_{\vec{b},*}(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}.$$

2. Proofs of Theorems

Proof of Theorem 1 Assume that $p < \infty$. Consider $x \neq y$ and let Q be the cube with center x and length $r = 3|x-y|$. Then

$$\begin{aligned}
&|B^{\delta}_{\vec{b},*}(f)(x) - B^{\delta}_{\vec{b},*}(f)(y)| = \big|\|B^{\delta}_{\vec{b},t}(f)(x)\| - \|B^{\delta}_{\vec{b},t}(f)(y)\|\big| \le \|B^{\delta}_{\vec{b},t}(f)(x) - B^{\delta}_{\vec{b},t}(f)(y)\| \\
&= \Big\| \int_{\mathbb{R}^n} \prod_{i=1}^m [b_i(x) - b_i(z)] B^{\delta}_t(x-z) f(z)\mathrm{d}z - \int_{\mathbb{R}^n} \prod_{i=1}^m [b_i(y) - b_i(z)] B^{\delta}_t(y-z) f(z)\mathrm{d}z \Big\| \\
&\le \Big\| \int_{2Q} \prod_{i=1}^m [b_i(x) - b_i(z)] B^{\delta}_t(x-z) f(z)\mathrm{d}z \Big\| + \Big\| \int_{2Q} \prod_{i=1}^m [b_i(y) - b_i(z)] B^{\delta}_t(y-z) f(z)\mathrm{d}z \Big\| + \\
&\quad \Big\| \int_{(2Q)^c} \prod_{i=1}^m [b_i(x) - b_i(z)] B^{\delta}_t(x-z) f(z)\mathrm{d}z - \int_{(2Q)^c} \prod_{i=1}^m [b_i(y) - b_i(z)] B^{\delta}_t(y-z) f(z)\mathrm{d}z \Big\| \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Firstly, we estimate I. By the property of B^{δ}_t (see [11]), we obtain

$$\begin{aligned}
|B^{\delta}_t(x-y)| &\le Ct^{-n}(1 + \frac{|x-y|}{t})^{-(\delta+(n+1)/2)} \\
&\le C(\frac{t}{t+|x-y|})^{\delta-(n-1)/2} \frac{1}{(t+|x-y|)^n} \le C|x-y|^{-n}. \qquad (2)
\end{aligned}$$

Note that $(n-\beta)p' = n - (\beta - \frac{n}{p})p'$, by the generalized Minkowski inequality and Hölder's inequality, we get

$$\begin{aligned}
\mathrm{I} &\le \int_{2Q} \Big\| \prod_{i=1}^m [b_i(x) - b_i(z)] B^{\delta}_t(x-z) f(z) \Big\| \mathrm{d}z \\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta} \int_{2Q} |x-z|^{\beta} |f(z)| |x-z|^{-n}\mathrm{d}z \\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} \Big(\int_{2Q} \frac{1}{|x-z|^{(n-\beta)p'}}\mathrm{d}z \Big)^{\frac{1}{p'}} \\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} r^{\beta - \frac{n}{p}} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} |x-y|^{\beta - \frac{n}{p}}.
\end{aligned}$$

With the same estimate for II, we immediately obtain

$$\mathrm{II} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} |x-y|^{\beta - \frac{n}{p}}.$$

For III, we write

$$\int_{(2Q)^c} \prod_{i=1}^m [b_i(x) - b_i(y) + b_i(y) - b_i(z)] B^{\delta}_t(x-z) f(z)\mathrm{d}z$$

$$= \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} [b_i(x) - b_i(y)]_\sigma \int_{(2Q)^c} [b_i(y) - b_i(z)]_{\sigma'} B_t^\delta(x-z) f(z) \mathrm{d}z$$

$$= \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} [b_i(x) - b_i(y)]_\sigma \int_{(2Q)^c} [b_i(y) - b_i(z)]_{\sigma'} B_t^\delta(x-z) f(z) \mathrm{d}z +$$

$$\prod_{i=1}^{m} [b_i(x) - b_i(y)] \int_{(2Q)^c} B_t^\delta(x-z) f(z) \mathrm{d}z + \int_{(2Q)^c} \prod_{i=1}^{m} [b_i(y) - b_i(z)] B_t^\delta(x-z) f(z) \mathrm{d}z.$$

Thus

$$\begin{aligned} \mathrm{III} \le & \Big\| \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} [b_i(x) - b_i(y)]_\sigma \int_{(2Q)^c} [b_i(y) - b_i(z)]_{\sigma'} B_t^\delta(x-z) f(z) \mathrm{d}z \Big\| + \\ & \Big\| \prod_{i=1}^{m} [b_i(x) - b_i(y)] \int_{(2Q)^c} B_t^\delta(x-z) f(z) \mathrm{d}z \Big\| + \\ & \Big\| \int_{(2Q)^c} \prod_{i=1}^{m} [b_i(y) - b_i(z)] [B_t^\delta(x-z) - B_t^\delta(y-z)] f(z) \mathrm{d}z \Big\| \\ := & \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3. \end{aligned}$$

Since $\beta - \min_{1 \le i \le m} \beta_i < \frac{n}{p}$, $\beta_{\sigma'} - \frac{n}{p} < 0$. By the generalized Minkowski inequality, the inequality (2) and Hölder's inequality, we have

$$\begin{aligned} \mathrm{III}_1 \le & \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} |[b_i(x) - b_i(y)]_\sigma| \int_{(2Q)^c} \|[b_i(y) - b_i(z)]_{\sigma'} B_t^\delta(x-z) f(z)\| \mathrm{d}z \\ \le & C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_\sigma}} |x-y|^{\beta_\sigma} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_{\sigma'}}} \int_{(2Q)^c} \frac{|y-z|^{\beta_{\sigma'}}}{|x-z|^n} |f(z)| \mathrm{d}z \\ \le & C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_\sigma}} |x-y|^{\beta_\sigma} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_{\sigma'}}} \|f\|_{L^p} \sum_{k=1}^{\infty} \Big(\int_{2^{k+1}Q \setminus 2^k Q} \frac{|y-z|^{\beta_{\sigma'} p'}}{|x-z|^{np'}} \mathrm{d}z \Big)^{\frac{1}{p'}} \\ \le & C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_\sigma}} |x-y|^{\beta_\sigma} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_{\sigma'}}} \|f\|_{L^p} \sum_{k=1}^{\infty} \frac{(2^{k+1}r)^{\beta_{\sigma'}}}{|2^k r|^n} |2^{k+1}Q|^{\frac{1}{p'}} \\ \le & C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_\sigma}} |x-y|^{\beta_\sigma} \|\vec{b_\sigma}\|_{\dot{\Lambda}_{\beta_{\sigma'}}} \|f\|_{L^p} |x-y|^{\beta_{\sigma'} - \frac{n}{p}} \sum_{k=1}^{\infty} 2^{k(\beta_{\sigma'} - \frac{n}{p})} \\ \le & C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} |x-y|^{\beta - \frac{n}{p}}. \end{aligned}$$

By the generalized Minkowski inequality, the inequality (2) and Hölder's inequality, we get

$$\begin{aligned} \mathrm{III}_2 \le & \prod_{i=1}^{m} [b_i(x) - b_i(y)] \int_{(2Q)^c} \|B_t^\delta(x-z) f(z)\| \mathrm{d}z \\ \le & C \|\vec{b}\|_{\dot{\Lambda}_\beta} |x-y|^\beta \int_{(2Q)^c} \frac{|f(z)|}{|x-z|^n} \mathrm{d}z \\ \le & C \|\vec{b}\|_{\dot{\Lambda}_\beta} |x-y|^\beta \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \frac{|f(z)|}{|x-z|^n} \mathrm{d}z \end{aligned}$$

$$\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}|x-y|^\beta\|f\|_{L^p}\sum_{k=1}^\infty\frac{1}{|2^kQ|}|2^{k+1}Q|^{\frac{1}{p'}}$$
$$\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}|x-y|^{\beta-\frac{n}{p}}.$$

As for III_3, we consider the following two cases:

Case 1 $0<t\leq r$. In this case, notice that [11]

$$|B^\delta(z)|\leq C(1+|z|)^{-(\delta+(n+1)/2)}.$$

By the generalized Minkowski inequality and Hölder's inequality, we obtain

$$\begin{aligned}
\mathrm{III}_3 &\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\int_{(2Q)^c}\|[B_t^\delta(x-z)-B_t^\delta(y-z)]|y-z|^\beta f(z)\|\mathrm{d}z\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}t^{-n}(1+\frac{|y-z|}{t})^{-(\delta+\frac{n+1}{2})}|y-z|^\beta|f(z)|\mathrm{d}z\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}(\frac{t}{r})^{\delta-\frac{n-1}{2}}\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}\frac{|x-y|^{\delta-\frac{n-1}{2}}}{|y-z|^{\delta+\frac{n+1}{2}-\beta}}|f(z)|\mathrm{d}z\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}\sum_{k=1}^\infty\Big\{\int_{2^{k+1}Q\backslash 2^kQ}(\frac{|x-y|^{\delta-\frac{n-1}{2}}}{|y-z|^{\delta+\frac{n+1}{2}-\beta}})^{p'}\mathrm{d}z\Big\}^{\frac{1}{p'}}\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}\sum_{k=1}^\infty\frac{r^{\delta-\frac{n-1}{2}}}{(2^kr)^{\delta+\frac{n+1}{2}-\beta}}|2^{k+1}Q|^{\frac{1}{p'}}\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}r^{\beta-\frac{n}{p}}\sum_{k=1}^\infty 2^{-k(\frac{n}{p}+\delta-\frac{n-1}{2}-\beta)}\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}|x-y|^{\beta-\frac{n}{p}}.
\end{aligned}$$

Case 2 $t>r$. In this case, we choose δ_0 such that $\beta+(n-1)/2<\delta_0<\min\{\delta,(n+1)/2\}$. Notice that [11]

$$|\bigtriangledown^\alpha B^\delta(z)|\leq C(1+|z|)^{-(\delta+(n+1)/2)}$$

for any $\alpha=(\alpha_1,\ldots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$, where $\bigtriangledown^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. By the generalized Minkowski inequality and Hölder's inequality, we have

$$\begin{aligned}
\mathrm{III}_3 &\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\int_{(2Q)^c}\|[B_t^\delta(x-z)-B_t^\delta(y-z)]|y-z|^\beta f(z)\|\mathrm{d}z\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}t^{-n-1}|x-y|(1+\frac{|y-z|}{t})^{-(\delta+\frac{n+1}{2})}|y-z|^\beta|f(z)|\mathrm{d}z\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}t^{-n-1}|x-y|(1+\frac{|y-z|}{t})^{-(\delta_0+\frac{n+1}{2})}|y-z|^\beta|f(z)|\mathrm{d}z\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}t^{\delta_0-\frac{n+1}{2}}\|f\|_{L^p}\sum_{k=1}^\infty\Big\{\int_{2^{k+1}Q\backslash 2^kQ}(\frac{|x-y|}{|y-z|^{\delta_0+\frac{n+1}{2}-\beta}})^{p'}\mathrm{d}z\Big\}^{\frac{1}{p'}}\\
&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}r^{\delta_0-\frac{n+1}{2}}\|f\|_{L^p}\sum_{k=1}^\infty\Big\{\int_{2^{k+1}Q\backslash 2^kQ}(\frac{|x-y|}{|y-z|^{\delta_0+\frac{n+1}{2}-\beta}})^{p'}\mathrm{d}z\Big\}^{\frac{1}{p'}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta} r^{\delta_0-\frac{n+1}{2}}\|f\|_{L^p}\sum_{k=1}^{\infty}\frac{r(2^{k+1}r)^{\frac{n}{p'}}}{(2^k r)^{\delta_0+\frac{n+1}{2}-\beta}}\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p} r^{\beta-\frac{n}{p}}\sum_{k=1}^{\infty}2^{-k(\frac{n}{p}+\delta_0-\frac{n-1}{2}-\beta)}\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}|x-y|^{\beta-\frac{n}{p}}.
\end{aligned}
$$

The case $p=\infty$ is similar, even easier. We omit the details. Thus the proof of Theorem 1 is completed. $\square$

Proof of Theorem 2 Let

$$
h_Q=\Big\|\int_{(2Q)^c}\prod_{i=1}^{m}[b_i(x_0)-b_i(y)]B_t^\delta(x_0-y)f(y)\mathrm{d}y\Big\|,
$$

where x_0 is the center of the cube Q. In order to prove Theorem 2, it suffices to show that

$$
\int_Q|B_{\vec{b},*}^\delta(f)(x)-h_Q|\mathrm{d}x\le C|Q|\|\vec{b}\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}.
$$

Write

$$
\begin{aligned}
&|B_{\vec{b},*}^\delta(f)(x)-h_Q|\\
&\le\Big|\Big\|\int_{\mathbb{R}^n}\prod_{i=1}^{m}[b_i(x)-b_i(y)]B_t^\delta(x-y)f(y)\mathrm{d}y\Big\|-\Big\|\int_{(2Q)^c}\prod_{i=1}^{m}[b_i(x_0)-b_i(y)]B_t^\delta(x_0-y)f(y)\mathrm{d}y\Big\|\Big|\\
&\le\Big\|\int_{\mathbb{R}^n}\prod_{i=1}^{m}[b_i(x)-b_i(y)]B_t^\delta(x-y)f(y)\mathrm{d}y-\int_{(2Q)^c}\prod_{i=1}^{m}[b_i(x_0)-b_i(y)]B_t^\delta(x_0-y)f(y)\mathrm{d}y\Big\|\\
&\le\Big\|\int_{2Q}\prod_{i=1}^{m}[b_i(x)-b_i(y)]B_t^\delta(x-y)f(y)\mathrm{d}y\Big\|+\\
&\quad\Big\|\int_{(2Q)^c}\prod_{i=1}^{m}[b_i(x)-b_i(y)]B_t^\delta(x-y)f(y)\mathrm{d}y-\int_{(2Q)^c}\prod_{i=1}^{m}[b_i(x_0)-b_i(y)]B_t^\delta(x_0-y)f(y)\mathrm{d}y\Big\|\\
&:=\mathrm{J}+\mathrm{JJ}.
\end{aligned}
$$

By the generalized Minkowski inequality and the inequality (2), we have

$$
\begin{aligned}
\mathrm{J}&\le\int_{2Q}\Big\|\prod_{i=1}^{m}[b_i(x)-b_i(y)]B_t^\delta(x-y)f(y)\Big\|\mathrm{d}y\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\int_{2Q}\frac{1}{|x-y|^{n-\beta}}|f(y)|\mathrm{d}y\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\mathrm{I}_\beta(|f\chi_{2Q}|)(x).
\end{aligned}
$$

By $(L^{\frac{n}{\beta}},L^\infty)$-boundedness of the fractional integral operator I_β, we get

$$
\int_Q\mathrm{J}\,\mathrm{d}x\le C|Q|\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^{\frac{n}{\beta}}}.
$$

For JJ, writing III similarly to that in the proof of Theorem 1, we have

$$\begin{aligned}\mathrm{JJ} \le & \Big\| \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} [b_i(x) - b_i(x_0)]_\sigma \int_{(2Q)^c} [b_i(x_0) - b_i(y)]_{\sigma'} B_t^\delta(x-y) f(y) \mathrm{d}y \Big\| + \\ & \Big\| \prod_{i=1}^{m} [b_i(x) - b_i(x_0)] \int_{(2Q)^c} B_t^\delta(x-y) f(y) \mathrm{d}y \Big\| + \\ & \Big\| \int_{(2Q)^c} \prod_{i=1}^{m} [b_i(x_0) - b_i(y)][B_t^\delta(x-y) - B_t^\delta(x_0-y)] f(y) \mathrm{d}y \Big\| \\ := & \mathrm{JJ}_1 + \mathrm{JJ}_2 + \mathrm{JJ}_3.\end{aligned}$$

With the same method as used to estimate III_1, III_2 and III_3 in the proof of Theorem 1, and noting that $p = \frac{n}{\beta}$, one easily obtains that

$$\mathrm{JJ} \le \mathrm{JJ}_1 + \mathrm{JJ}_2 + \mathrm{JJ}_3 \le C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^{\frac{n}{\beta}}}.$$

Thus

$$\int_Q \mathrm{JJ}\, \mathrm{d}x \le C|Q| \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^{\frac{n}{\beta}}}.$$

Therefore the proof of Theorem 2 is completed. □

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