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Distance of Edges in the Curve Complex of a Surface

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Abstract Let S be a closed orientable surface of genus $g \geq 2$, and $C(S)$ the curve complex of S. In the paper, we introduce the concepts of 2-path between edges in $C(S)$, which can be regarded as an analogue to the edge path between vertices in $C(S)$. We show that $C(S)$ is $2P$-connected, and the 2-diameter of $C(S)$ is infinite.

Keywords curve complex; distance of vertices; 2-path between edges; 2-diameter of $C(S)$.

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1. Introduction

In the late 1970s, Harvey [4] associated to a surface S a finite-dimensional simplicial complex $C(S)$, called the complex of curves, which was intended to capture some properties of combinatorial topology of S. The vertices of Harvey's complex are homotopy classes of simple closed curves in S, and the simplices are collections of curves that can be realized disjointly. Harer [2, 3] considered the complex from a cohomological point of view, Ivanov [6–9] considered its applications to the structure of the mapping class group of S, Masur-Minsky [10] started a study of the intrinsic geometry of $C(S)$, and Hempel [5] applied it to study the topology of 3-manifolds. It has proved to be of fundamental importance in the study of many problems related to surfaces in topology, geometry and complex analysis. See [11] for a survey that gives a good account of the history of the mathematics of the curve complex, continuing up to the recent advance.

An important fact is that the complex $C(S)$, and also its 1-skeleton, can be given the structure of a metric space by assigning length 1 to every edge and making each simplex an Euclidean simplex with edges of length 1, see [10].

In the present paper, we introduce the concepts of 2-path between edges in $C(S)$, which can be regarded as an analogue to the edge path between vertices in $C(S)$. We mainly show that for the closed orientable surface $S = S_{g,0}$ with $g \geq 2$, $C(S)$ is $2P$-connected, and the 2-diameter of $C(S)$ is infinite. The arguments are based on the corresponding known results in "lower" case by Ivanov [7] and Masur-Minsky [10].

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The paper is organized as follows. In Section 2, we review some necessary definitions and notions and collect some known results which will be used later. The main results and their proofs are included in Section 3. The terminologies not defined in the paper are all standard, see, for example, [7].

2. Preliminaries

A simplicial complex consists of a family of vertices and a family of simplices. Simplices are non-empty finite sets of vertices, subject only to the following two conditions: a non-empty subset of a simplex σ is a simplex (which is called a face of σ); every vertex belongs to some simplex. Let $\sigma = \{v_0, v_1, \ldots, v_p\}$ be a simplex. p is called the dimension of σ, and is denoted by $\dim \sigma$, i.e. $\dim \sigma =$the number of the vertices in it minus 1. A 1-dimensional simplex is also called an edge.

We use $S_{g,b}$ to denote the compact, connected, orientable surface of genus g which has b boundary components. A simple closed curve on $S_{g,b}$ is called a circle. A circle on $S_{g,b}$ is non-trivial if it does not bound a disk in $S_{g,b}$ and it is not ∂-parallel. The isotopy class of a circle C is denoted by $\langle C \rangle$.

Definition 2.1 *Let $S = S_{g,b}$. The curve complex $C(S)$ is defined as follows: The vertices of $C(S)$ are the isotopy classes of non-trivial circles on S. A simplex of $C(S)$ is a set of vertices $\{\gamma_0, \gamma_1, \ldots, \gamma_p\}$ such that $\gamma_0 = \langle C_0 \rangle, \gamma_1 = \langle C_1 \rangle, \ldots, \gamma_p = \langle C_p \rangle$ for a collection of pairwise disjoint circles $C_0, C_1, \ldots, C_p$.*

Clearly, $C(S_{g,b}) = \emptyset$ if $g = 0$ and $b = 0, 1, 2$ or 3; $\dim S_{1,0} = 0$; for the other cases of g and b, $\dim C(S_{g,b}) = 3g - 4 + b$.

Definition 2.2 *Let $S = S_{g,b}$, and α, γ be two vertices in $C(S)$. An edge-path in $C(S)$ from α to γ is a finite sequence $\beta_0, \beta_1, \ldots, \beta_n$ of vertices in $C(S)$ such that $\beta_0 = \alpha, \gamma = \beta_n$, and the adjacent β_{j-1}, β_j are vertices of an edge in $C(S)$, for $1 \leq j \leq n$. n is called the length of the edge-path. We usually say that an edge-path between two vertices in $C(S)$ is a 1-path. $C(S)$ is P-connected (or simply, connected) if for any two vertices α, γ in $C(S)$, there is an edge path in $C(S)$ from α to γ.*

The following is a well-known theorem due to N.V. Ivanov (see [6] or [7]).

Theorem 2.3 *$C(S_{g,b})$ is connected for $g = 0$ and $b \geq 5$, $g = 1$ and $b \geq 2$, all $g \geq 2$ and $b \geq 0$.*

Definition 2.4 *Let $S = S_{g,b}$. Assume that $C(S_{g,b})$ is connected. For any two vertices α, γ in $C(S)$, the distance of α and γ is defined to be the minimal length of all edge-paths in $C(S)$ from α to γ, and is denoted by $d(\alpha, \gamma)$. The diameter of $C(S)$ is defined to be the supremum over the distances between vertices in $C(S)$, and is denoted by* $\operatorname{diam} C(S)$.

As to the diameter of $C(S)$, we have the following remarkable and completely unexpected theorem, which is due to Masur and Minsky [10].

Theorem 2.5 *Let $S = S_{g,b}$. If $C(S)$ is connected, then* $\operatorname{diam} C(S) = \infty$.

Theorem 2.5 asserts that for any natural number N, there exists two vertices α, γ in $C(S)$ so that $d(\alpha, \gamma) \geq N$, i.e., α and γ cannot be connected by an edge-path in $C(S)$ of length less than N.

3. 2-paths between two 1-simplices in $C(S)$

We now introduce the concept of a 2-path between two 1-simplices in $C(S)$, which is analogous to the edge path between vertices in $C(S)$.

Definition 3.1 *Let $S = S_{g,0}$ with $g \geq 2$. Let α, γ be two 1-simplices in $C(S)$. A 2-path in $C(S)$ from α to γ is a finite sequence $\beta_0, \beta_1, \ldots, \beta_n$ of 1-simplices in $C(S)$ such that $\beta_0 = \alpha$, $\gamma = \beta_n$, and the adjacent β_{j-1}, β_j are 1-faces of a 2-simplex in $C(S)$, for $1 \leq j \leq n$. It is also called a 2-path between α and γ. n is called the length of the 2-path. $C(S)$ is 2P-connected if for any two edges α, γ in $C(S)$, there is a 2-path in $C(S)$ from α to γ.*

Figure 1 below shows a 2-path between the two edges α and γ in $C(S)$.

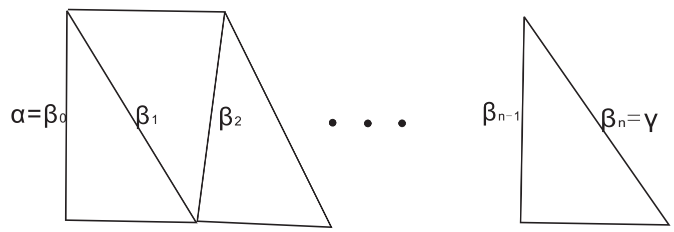

Figure 1 A 2-path between the two edges α and γ

For a collection $\mathcal{C}$ of pairwise disjoint circles on a surface S, we use $S \setminus \mathcal{C}$ to denote the surface obtained by cutting S open along $\mathcal{C}$.

First we have

Lemma 3.2 *Let $S = S_{g,b}$, where when $g = 0$, $b \geq 6$; when $g = 1$, $b \geq 3$; and when $g \geq 2$, $b \geq 0$. Let $\sigma = \{u_0, u_1\}$ and $\tau = \{u_0', u_1,\}$ be two 1-simplices in $C(S)$. Then there exists a 1-path $u_0 = v_0, v_1, \ldots, v_k = u_0'$ in $C(S \setminus C_1)$ from u_0 to u_0', where $u_1 = \langle C_1 \rangle$, and for each i, $0 \leq i \leq k$, $\varsigma_i = \{v_i, u_1\}$ is a 1-simplex in $C(S)$. Moreover, set $\xi_i = \{v_{i-1}, v_i, u_1\}$, then ξ_i is a 2-simplex in $C(S)$, having ς_{i-1} and ς_i as 1-faces, $1 \leq i \leq k$.*

Proof If u_0 and u_0' have representatives which are disjoint, then u_0, u_0' is a 1-path from u_0 to u_0'. In the following we assume $J \cap J' \neq \emptyset$ for any $J \in u_0$ and $J' \in u_0'$. For $u_1 = \langle C_1 \rangle$, cut S open along C_1 to get a surface $S' = S \setminus C_1$. Choose $J \in u_0$ and $J' \in u_0'$ so that $J \cap C_1 = \emptyset$ and $J' \cap C_1 = \emptyset$. Since $J \cap J' \neq \emptyset$, so J and J' are lying in the same component, say S^*, of S'.

If C_1 is non-separating in S, $S^* = S'$ is either a $(b+2)$-punctured 2-sphere with $b + 2 \geq 5$, or a $(b+2)$-punctured surface of positive genus, the conclusion follows from Theorem 2.3.

Next assume that C_1 is separating in S. Then C_1 cuts S into two pieces, one of which is S^*, the other one is denoted by S^{**}. Since C_1 is non-trivial on S, S^{**} is neither a disk nor an annulus. If S^{**} is a 3-punctured 2-sphere, then S^* is either a $(b-1)$-punctured 2-sphere with $b - 1 \geq 5$, or a $(b-1)$-punctured torus with $b - 1 \geq 2$, or a $(b-1)$-punctured surface with genus $g \geq 2$. In the last case, $S = S_{g,b}$ with $g \geq 2$ and $b \geq 2$. The conclusion again follows from Theorem 2.3. If S^{**} is either a p-punctured 2-sphere with $p \geq 4$, or a punctured surface of positive genus, then there exists a non-trivial circle K on S^{**}. Let $v = \langle K \rangle$. Thus u_0, v, u_0' is a 1-path in $C(S)$ from u_0 to u_0', and $\{u_0, u_1, v\}$ and $\{u_0', u_1, v\}$ are 2-simplices in $C(S)$. The conclusion holds.

This completes the proof. □

Theorem 3.3 *Let $S = S_{g,0}$ with $g \geq 2$. Then $C(S)$ is $2P$-connected.*

Proof It suffices to show that for any two 1-simplices σ, τ in $C(S)$, there always exists an 2-path in $C(S)$ from σ to τ.

Assume $\sigma = \{u_0, u_1\}$, $\tau = \{w_0, w_1\}$. By Theorem 2.3, there exists a 1-path $u_1 = v_0, v_1, \ldots, v_p = w_1$ in $C(S)$ from u_1 to w_1, and $\sigma_j = \{v_{j-1}, v_j\}$ is a 1-simplex in $C(S)$, $1 \leq j \leq p$. Now $\sigma = \{u_0, u_1\}$ and $\sigma_1 = \{v_1, u_1\}$ have one vertex in common. By applying Lemma 3.2 to σ and σ_1, there exists a 1-path $u_0 = x_0, x_1, \ldots, x_s = v_1$ in $C(S \setminus C_1)$, where $u_1 = \langle C_1 \rangle$, and for each i, $0 \leq i \leq s$, $l_i = \{x_i, u_1\}$ is a 1-simplex in $C(S)$. Moreover, set $\xi_i = \{x_{i-1}, x_i, u_1\}$, ξ_i is a 2-simplex in $C(S)$ which has l_{i-1} and l_i as 1-faces, $1 \leq i \leq s$. Thus $\sigma = l_0, l_1, \ldots, l_s = \sigma_1$ is a 2-path from σ to σ_1. See Figure 2 below. Similarly, there exists a 2-path from σ_j to σ_{j+1}, $1 \leq j \leq p-1$. Thus there exists a 2-path in $C(S)$ from σ to τ.

This completes the proof. □

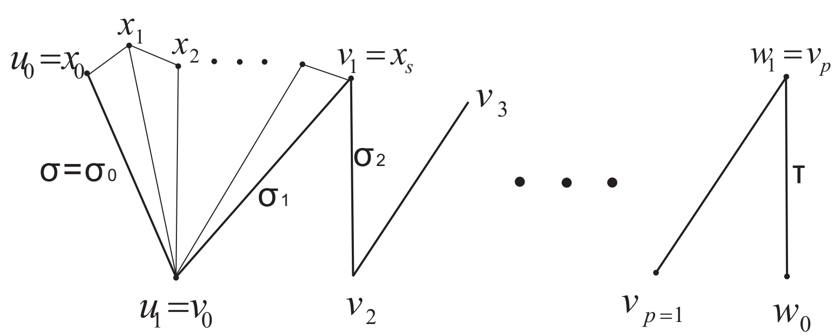

Figure 2 A 2-path from σ to τ

From Theorem 3.3, we know that for any two edges in $C(S_{g,0})$ with $g \geq 2$ there exists a 2-path between them. It is natural to introduce the following definitions.

Definition 3.4 *Let $S = S_{g,0}$ with $g \geq 2$. For any two 1-simplices σ, τ in $C(S)$, there always exists a 2-path in $C(S)$ from σ to τ. Define the 2-distance of σ and τ, denoted by $d_2(\sigma, \tau)$, to be the minimal length of all 2-paths in $C(S)$ between σ and τ. The 2-diameter of $C(S)$ is defined to be the supremum over the 2-distances between edges in $C(S)$, and is denoted by $\mathrm{diam}_2 C(S)$.*

Theorem 3.5 *Let $S = S_{g,0}$ with $g \geq 2$. Let σ and τ be two distinct edges in $C(S)$. Let u be a vertex of σ, and w a vertex of τ. Then $d(u, w) \leq d_2(\sigma, \tau)$.*

Proof Let $n = d_2(\sigma, \tau)$. It suffices to show that there exists a 1-path in $C(S)$ from u to w of length $\leq 2n$. Let $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \tau$ be a 2-path in $C(S)$ of the minimal length n. Then for each i, $1 \leq i \leq n$, σ_{i-1} and σ_i are edges of a 2-simplex $\triangle_i$ in $C(S)$. We induct on n to show that for any vertex u in σ_0 and any vertex w in σ_n, there exists a 1-path of length $\leq n$ in $\triangle_1 \cup \cdots \cup \triangle_n$ from u to w. When $n = 1$, it is obviously true. Assume the conclusion holds for $n \leq k$. Consider the case $n = k + 1$. By induction, for any vertex u in σ_0 and any vertex v in σ_k, which is also a vertex in $\triangle_n$, there exists a 1-path of length $\leq k$ in $\triangle_1 \cup \cdots \cup \triangle_k$ from u to v. For any vertex w in $\triangle_n$, $d(v, w) \leq 1$. So for any vertex u in σ_0 and any vertex w in σ_n, there exists a 1-path of length $\leq k + 1 = n$ in $\triangle_1 \cup \cdots \cup \triangle_k \cup \triangle_n$ from u to w. See Figure 3.

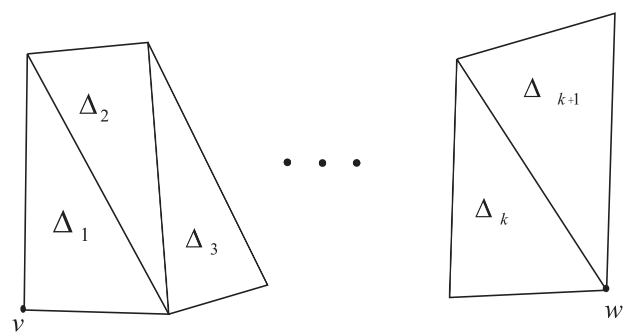

Figure 3 A 1-path of length from v to w

Theorem 3.6 *Let $S = S_{g,0}$ with $g \geq 2$. Then* $\mathrm{diam}_2 C(S) = \infty$.

Proof Suppose the conclusion is not true. Then there is a positive constant M_S so that $\mathrm{diam}_2 C(S) \leq M_S$. By Theorem 2.3, $C(S)$ is connected. By Theorem 2.5, $\mathrm{diam} C(S) = \infty$. So there exist two vertices u and v in $C(S)$ with $d(u, v) \geq M_S + 1$. Let σ and τ be two edges in $C(S)$ such that u is a vertex of σ and v is a vertex of τ. Then $d_2(\sigma, \tau) \leq \mathrm{diam}_2 C(S) \leq M_S$. By Theorem 3.5, $d(u, v) \leq d_2(\sigma, \tau) \leq \mathrm{diam}_2 C(S) \leq M_S$, which contradicts $d(u, v) \geq M_S + 1$.

The proof is completed. □

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