Journal of Mathematical Research & Exposition
Nov., 2011, Vol. 31, No. 6, pp. 1097–1107
DOI:10.3770/j.issn:1000-341X.2011.06.019
Http://jmre.dlut.edu.cn

Weighted Composition Followed and Proceeded by Differentiation Operators from $Q_k(p,q)$ Space to Weighted α-Bloch Space

Jian Ren LONG, Peng Cheng WU*
School of Mathematics and Computer Science, Guizhou Normal University, Guizhou 550001*, P. R. China*

Abstract We study the boundedness and compactness of the weighted composition followed and proceeded by differentiation operators from $Q_k(p,q)$ space to weighted α-Bloch space and little weighted α-Bloch space. Some necessary and sufficient conditions for the boundedness and compactness of these operators are given.

Keywords $Q_k(p,q)$ space; weighted α-Bloch space; weighted composition operators; differentiation operators; boundedness; compactness.

Document code A
MR(2010) Subject Classification 47B38; 30D45
Chinese Library Classification O175

1. Introduction

Let Δ be the open unit disc in the complex plane and $H(\Delta)$ the class of all analytic functions on Δ. The α-Bloch space B^α ($\alpha>0$) is the space of all analytic functions f on Δ such that

$$\|f\|_{B^\alpha}=|f(0)|+\sup_{z\in\Delta}(1-|z|^2)^\alpha|f'(z)|<\infty.$$

Under the above norm, B^α is a Banach space. When $\alpha=1$, $B^1=B$ is the well-known Bloch space. Let B_0^α denote the subspace of B^α consisting of those for which

$$(1-|z|^2)^\alpha|f'(z)|\to 0 \ \text{ as } |z|\to 1.$$

This space is called the little α-Bloch space.

An $f\in H(\Delta)$ is said to belong to the weighted α-Bloch space $B_{\log}^\alpha$ (see [2–4]), if

$$\|f\|_{B_{\log}^\alpha}=|f(0)|+\sup_{z\in\Delta}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|f'(z)|<\infty.$$

$B_{\log}^\alpha$ is a Banach space with the norm $\|\cdot\|_{B_{\log}^\alpha}$, when $\alpha=1$, $B_{\log}^1=B_{\log}$ is the weighted Bloch

Received August 10, 2010; Accepted November 20, 2010
Supported by the National Natural Science Foundation of China(Grant No. 11171080) and the Foundation of Science and Technology Department of Guizhou Province (Grant No. 2010[07]).
* Corresponding author.
E-mail address: longjianren2004@163.com (J. R. LONG); wupc@gznu.edu.cn (P. C. WU)

space [1]. From the inequality

$$(1-|z|^2)^\alpha \log \frac{2}{1-|z|^2}|f'(z)| \geq \log 2((1-|z|^2)^\alpha)|f'(z)|,$$

and $f_0(z) = (1-z)^{1-\alpha} \in B^\alpha$, but $f_0(z)\overline{\in} B^\alpha_{\log}$, we obtain $B^\alpha_{\log} \subset B^\alpha$. Let $B^\alpha_{\log,0}$ denote the subspace of $B^\alpha_{\log}$ consisting of those for which

$$(1-|z|^2)^\alpha \log \frac{2}{1-|z|^2}|f'(z)| \to 0 \quad \text{as } |z| \to 1.$$

This space is called the little weighted α Bloch space.

Throughout this paper, we assume that K is a right continuous and nonnegative nondecreasing function on $[0,\infty)$. For $0 < p < \infty$, $-2 < q < \infty$, we say that a function $f \in H(\Delta)$ belongs to the space $Q_k(p,q)$ (see [5]), if

$$\|f\| = \Big(\sup_{z\in\Delta}\int_\Delta |f'(z)|^p(1-|z|^2)^q K(g(z,a))\mathrm{d}A(z)\Big)^{\frac{1}{p}} < \infty,$$

where $\mathrm{d}A$ denotes the normalized Lebesgue area measure in Δ (i.e., $A(\Delta) = 1$) and $g(z,a)$ is the Green function with logarithmic singularity at a, that is, $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ ($\varphi_a(z)$ is a conformal automorphism defined by $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ for $a \in \Delta$). If $K(x) = x^s$, $s \geq 0$, the space $Q_k(p,q)$ equals to $F(p,q,s)$, which was introduced by Zhao in [6]. Moreover from [6], we have that, $F(p,q,s) = B^{\frac{q+2}{p}}$ and $F_0(p,q,s) = B_0^{\frac{q+2}{p}}$ for $s > 1$, $F(p,q,s) \subseteq B^{\frac{q+2}{p}}$ and $F_0(p,q,s) \subseteq B_0^{\frac{q+2}{p}}$ for $0 \leq s < 1$. When $p \geq 1$, $Q_k(p,q)$ is a Banach space under the norm

$$\|f\|_{Q_k(p,q)} = |f(0)| + \|f\|.$$

From [5], we know that $Q_k(p,q) \subseteq B^{\frac{q+2}{p}}$, $Q_k(p,q) = B^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 K(\log\frac{1}{r})(1-r^2)^{-2} r\mathrm{d}r < \infty.$$

Moreover, $\|f\|_{B^{\frac{q+2}{p}}} \leq C\|f\|_{Q_k(p,q)}$ (see, [5, Theorem 2.1]). Throughout the paper we assume that

$$\int_0^1 K(\log\frac{1}{r})(1-r^2)^{q} r\mathrm{d}r < \infty,$$

otherwise $Q_k(p,q)$ consists only of constant functions [5].

Let φ denote a nonconstant analytic self-map of Δ and ϕ be an analytic function of Δ. We can define the linear operators (called weighted composition followed and proceeded by differentiation operators)

$$\phi C_\varphi Df = \phi(f'o\varphi) \quad \text{and } \phi DC_\varphi f = \phi(fo\varphi)', \text{ for } f \in H(\Delta),$$

where C_φ and D are composition and differentiation operators, respectively. The boundedness and compactness of DC_φ on the Hardy space were investigated by Hibschweiler and Portnoy in [7] and by Ohno in [8]. In [9], Li and Stević studied the boundedness and compactness of the operator DC_φ on the α-Bloch spaces, while in [10] they studied these operators between H^∞ and α-Bloch spaces.

In this paper, we study the operators ϕDC_φ and $\phi C_\varphi D$ from $Q_k(p,q)$ space to weighted α Bloch space and little weighted α-Bloch space. Some necessary and sufficient conditions for the boundedness and compactness of these operators are given.

Throughout this paper, we denote by C the positive constants, which may differ from one occurrence to the other. The notation $A \approx B$ means that there is a positive constant C such that $\frac{B}{C} \leq A \leq CB$.

2. Statement of the main results

Theorem 2.1 *Let φ be an analytic self-map of Δ and ϕ be an analytic function of Δ. Suppose $p>0$, $q>-2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K(\log\frac{1}{r})(1-r)^{\min\{-1,q\}}(\log\frac{1}{1-r})^{\chi_{-1}(q)} r\mathrm{d}r < \infty, \tag{2.1}$$

where $\chi_A(x)$ denotes the characteristic function of the set A. Then ϕDC_φ: $Q_k(p,q) \to B^\alpha_{\log}$ is bounded if and only if

$$\begin{aligned} &\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} < \infty, \\ &\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty. \end{aligned} \tag{2.2}$$

Theorem 2.2 *Let φ be an analytic self-map of Δ and ϕ be an analytic function of Δ. Suppose $p>0$, $q>-2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.1) holds. Then $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log}$ is compact if and only if $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log}$ is bounded, and*

$$\begin{aligned} &\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0, \\ &\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0. \end{aligned} \tag{2.3}$$

Theorem 2.3 *Let φ be an analytic self-map of Δ and ϕ be an analytic function of Δ. Suppose $p>0$, $q>-2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.1) holds. Then $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log,0}$ is compact if and only if*

$$\begin{aligned} &\lim_{|z|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0, \\ &\lim_{|z|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0. \end{aligned} \tag{2.4}$$

Similarly to the proofs of Theorems 2.1–2.3, we can get the following results. We omit the proof.

Theorem 2.4 *Let φ be an analytic self-map of Δ and ϕ be an analytic function of Δ. Suppose $p>0$, $q>-2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.1) holds. Then* *(i)* $\phi C_\varphi D : Q_k(p,q) \to B^\alpha_{\log}$ *is bounded if and only if*

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} < \infty,$$

and

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty.$$

(ii) $\phi C_\varphi D : Q_k(p,q) \to B^\alpha_{\log}$ *is compact if and only if* $\phi C_\varphi D : Q_k(p,q) \to B^\alpha_{\log}$ *is bounded,*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0,$$

and

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0.$$

(iii) $\phi C_\varphi D : Q_k(p,q) \to B^\alpha_{\log,0}$ *is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0,$$

and

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0.$$

3. Proofs of the main results

In this section, we give proofs of the main results. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way [11, Proposition 3.11].

Lemma 3.1 *Let φ be an analytic self-map of Δ and ϕ be an analytic function of Δ. Suppose $p>0$, $q>-2$. Then ϕDC_φ (or $\phi C_\varphi D$) : $Q_k(p,q)\to B^\alpha_{\log}$ is compact if and only if ϕDC_φ (or $\phi C_\varphi D$) : $Q_k(p,q)\to B^\alpha_{\log}$ is bounded and for any bounded sequence $\{f_n\}_{n\in N}$ in $Q_k(p,q)$ which converges to zero uniformly on compact subsets of Δ as $n\to\infty$, one has $\|\phi DC_\varphi f_n\|_{B^\alpha_{\log}}\to 0$ (or $\|\phi C_\varphi D f_n\|_{B^\alpha_{\log}}\to 0$) as $n\to\infty$.*

Lemma 3.2 *A closed set K in $B^\alpha_{\log,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|f'(z)| = 0. \tag{3.1}$$

Proof First suppose that K is compact and let $\varepsilon>0$. Choose an $\frac{\varepsilon}{2}$-net $f_1, f_2, \ldots, f_n$ in K. There is an r for $0<r<1$, such that $(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|f_i'(z)| < \frac{\varepsilon}{2}$, if $|z|>r$, $1\le i\le n$. If $f\in K$, $\|f-f_i\|_{B^\alpha_{\log}} < \frac{\varepsilon}{2}$ for some f_i and so

$$(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|f'(z)| \le \|f-f_i\|_{B^\alpha_{\log}} + (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|f_i'(z)| < \varepsilon,$$

whenever $|z|>r$. This establishes (3.1).

On the other hand, if K is a closed bounded set which satisfies (3.1) and $\{f_n\}$ is a sequence in K, then by Montel's theorem there is a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of Δ to some analytic function f, then also $\{f'_{n_k}\}$ converges uniformly to f' on compact subsets of Δ. By (3.1), if $\varepsilon > 0$, there is an r, $0 < r < 1$, such that for all $g \in K$, $(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. It follows that $(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|f'(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. Since $\{f_{n_k}\}$ converges uniformly to f and $\{f'_{n_k}\}$ converges uniformly to f' on $|z| \le r$, it follows that $\lim_{k\to\infty}\sup\|f_{n_k} - f\|_{B^\alpha_{\log}} \le \varepsilon$. Since $\varepsilon > 0$, $\lim_{k\to\infty}\|f_{n_k} - f\|_{B^\alpha_{\log}} = 0$ and K is compact.

Lemma 3.3 ([14]) *Let $\alpha > 0$. Then for $f \in H(\Delta)$ the following are equivalent:*

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha|f'(z)| \approx |f'(0)| + \sup_{z\in\Delta}(1-|z|^2)^{\alpha+1}|f''(z)|.$$

Proof of Theorem 2.1 Suppose that the conditions in (2.2) hold. Then for any $z \in \Delta$ and $f \in Q_k(p,q)$, making use of the fact $\|f\|_{B^{\frac{q+2}{p}}} \le C\|f\|_{Q_k(p,q)}$ and Lemma 3.3, we have

$$\begin{aligned}
&(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|(\phi DC_\varphi f)'(z)| \\
&= (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|\phi'(z)\varphi'(z)f'(\varphi(z)) + \phi(z)[f''(\varphi(z))(\varphi'(z))^2 + f'(\varphi(z))\varphi''(z)]| \\
&\le (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|\phi(z)f''(\varphi(z))(\varphi'(z))^2| + \\
&\quad (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|[\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)]f'(\varphi(z)| \\
&\le (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}C\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}\|f\|_{B^{\frac{q+2}{p}}} + \\
&\quad (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\|f\|_{B^{\frac{q+2}{p}}} \\
&\le \{(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}C\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} + \\
&\quad (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\}\|f\|_{Q_k(p,q)}.
\end{aligned} \tag{3.2}$$

Taking the supremum in (3.2) for $z \in \Delta$, and then employing (2.2), we obtain

$$\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log}$$

is bounded.

Conversely, suppose that $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log}$ is bounded, that is, there exists a constant C such that $\|\phi DC_\varphi f\|_{B^\alpha_{\log}} \le C\|f\|_{Q_k(p,q)}$ for all $f \in Q_k(p,q)$. Taking the functions $f(z) \equiv z$, and $f(z) \equiv \frac{z^2}{2}$, which belong to $Q_k(p,q)$, we get

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)| < \infty, \tag{3.3}$$

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|\phi(z)(\varphi'(z))^2 + \varphi(z)[\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)]| < \infty. \tag{3.4}$$

From (3.3), (3.4), and the boundedness of the function $\varphi(z)$, it follows that

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|\phi(z)(\varphi'(z))^2| < \infty. \tag{3.5}$$

For $w \in \Delta$, let

$$f_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^{\frac{q+2}{p}}}.$$

By some direct calculation we have that

$$f_w'(w) = \frac{q+2}{p}\frac{\bar{w}}{(1-|w|^2)^{\frac{q+2}{p}}},\ f_w''(w) = \frac{q+2}{p}\frac{p+q+2}{p}\frac{\bar{w}^2}{(1-|w|^2)^{\frac{p+q+2}{p}}}.$$

From [13], we know that $f_w \in Q_k(p,q)$, for each $w \in \Delta$. Moreover there is a positive constant C such that $\sup_{w\in\Delta}\|f_w\|_{Q_k(p,q)} \le C$. Hence, we have

$$\begin{aligned} C\|\phi DC_\varphi\| &\ge \|\phi DC_\varphi f_{\varphi(z)}\|_{B^\alpha_{\log}} \\ &\ge -\frac{q+2}{p}\frac{p+q+2}{p}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} + \\ &\quad \frac{q+2}{p}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}, \end{aligned} \tag{3.6}$$

for $z \in \Delta$. Therefore, we obtain

$$\begin{aligned} &(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} \\ &\le C\|\phi DC_\varphi\| + \frac{p+q+2}{p}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}. \end{aligned} \tag{3.7}$$

Next, for $w \in \Delta$, let

$$g_w(z) = \frac{(1-|w|^2)^2}{(1-\bar{w}z)^{\frac{p+q+2}{p}}} - \frac{p+q+2}{q+2}\frac{1-|w|^2}{(1-\bar{w}z)^{\frac{q+2}{p}}}.$$

Then from [13], we see that $g_w(z) \in Q_k(p,q)$ and $\sup_{w\in\Delta}\|g_w\|_{Q_k(p,q)} < \infty$. Since

$$g'_{\varphi(z)}(\varphi(z)) = 0,\ |g''_{\varphi(z)}(\varphi(z))| = \frac{p+q+2}{p}\frac{|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}},$$

we have

$$\begin{aligned} \infty > C\|\phi DC_\varphi\| &\ge \|\phi DC_\varphi g_{\varphi(z)}\|_{B^\alpha_{\log}} \\ &\ge \frac{p+q+2}{p}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}. \end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned} &\sup_{|\varphi(z)|>\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \\ &\le \sup_{|\varphi(z)|>\frac{1}{2}} 4(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \le C\|\phi DC_\varphi\| < \infty. \end{aligned} \tag{3.9}$$

Inequality (3.5) gives

$$\sup_{|\varphi(z)|\le\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}$$
$$\le(\frac{4}{3})^{\frac{p+q+2}{p}}\sup_{|\varphi(z)|\le\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|\phi(z)(\varphi'(z))^2|<\infty. \tag{3.10}$$

Therefore, the first inequality in (2.2) follows from (3.9) and (3.10). From (3.7) and (3.8), we obtain

$$\sup_{z\in\Delta}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}<\infty. \tag{3.11}$$

Inequalities (3.3) and (3.11) imply

$$\sup_{|\varphi(z)|>\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}$$
$$\le 2\sup_{|\varphi(z)|>\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}<\infty, \tag{3.12}$$
$$\sup_{|\varphi(z)|\le\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}$$
$$\le(\frac{4}{3})^{\frac{q+2}{p}}\sup_{|\varphi(z)|\le\frac{1}{2}}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|<\infty. \tag{3.13}$$

Inequlity (3.12) together with (3.13) implies the second inequality of (2.2). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 Suppose $\phi DC_\varphi : Q_k(p,q)\to B^\alpha_{\log}$ is bounded and (2.3) holds. Let $\{f_n\}_{n\in N}$ be a sequence in $Q_k(p,q)$ such that $\sup_{n\in N}\|f_n\|_{Q_k(p,q)}<\infty$, and f_n converges to 0 uniformly on compact subsets of Δ as $n\to\infty$. By the assumption, for any $\varepsilon>0$, there exists a $\delta\in(0,1)$ such that

$$(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}<\varepsilon,$$
$$(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}<\varepsilon,$$

when $\delta<|\ \varphi(z)\ |<1$. Since $\phi DC_\varphi : Q_k(p,q)\to B^\alpha_{\log}$ is bounded, from the proof of Theorem 2.1 we have

$$M_1:=\sup_{z\in\Delta}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|<\infty,$$
$$M_2:=\sup_{z\in\Delta}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|\phi(z)(\varphi'(z))^2|<\infty.$$

Let $K=\{z\in\Delta:|\ \varphi(z)\ |\le\delta\}$. Then we have

$$\|\phi DC_\varphi f_n\|_{B^\alpha_{\log}}=\sup_{z\in\Delta}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|(\phi DC_\varphi f_n)'(z)|+|\phi(0)f_n'(\varphi(0))\varphi'(0)|$$
$$\le\sup_{z\in\Delta}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|\phi(z)f_n''(\varphi(z))(\varphi'(z))^2|+$$

$$
\begin{aligned}
&\sup_{z\in\Delta}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||f_n'(\varphi(z))|+|\phi(0)f_n'(\varphi(0))\varphi'(0)|\\
\le&\sup_{z\in K}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)f_n''(\varphi(z))(\varphi'(z))^2|+\\
&\sup_{z\in K}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||f_n'(\varphi(z))|+\\
&\sup_{z\in(\Delta-K)}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)f_n''(\varphi(z))(\varphi'(z))^2|+\\
&\sup_{z\in(\Delta-K)}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||f_n'(\varphi(z))|+|\phi(0)f_n'(\varphi(0))\varphi'(0)|\\
\le&\sup_{z\in K}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)f_n''(\varphi(z))(\varphi'(z))^2|+\\
&\sup_{z\in K}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||f_n'(\varphi(z))|+|\phi(0)f_n'(\varphi(0))\varphi'(0)|+\\
&C\sup_{z\in(\Delta-K)}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}\|f_n\|_{Q_k(p,q)}+\\
&\sup_{z\in(\Delta-K)}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\|f_n\|_{Q_k(p,q)}\\
\le& M_2\sup_{z\in K}|f_n''(\varphi(z))|+M_1\sup_{z\in K}|f_n'(\varphi(z))|+2C\varepsilon\|f_n\|_{Q_k(p,q)}+|\phi(0)f_n'(\varphi(0))\varphi'(0)|. \qquad (3.14)
\end{aligned}
$$

The assumption that $f_n \to 0$ as $n \to \infty$ on compact subsets of Δ along with Cauchy's estimate gives that $f_n' \to 0$ and $f_n'' \to 0$ as $n \to \infty$ on compact subsets of Δ. Letting $n \to \infty$ in (3.14) and using the fact that ε is an arbitrary positive number, we obtain $\lim_{n\to\infty}\|\phi DC_\varphi f_n\|_{B^{\alpha}_{\log}} = 0$. Applying Lemma 3.1 yields the result.

Now, suppose that $\phi DC_\varphi : Q_k(p,q) \to B^{\alpha}_{\log}$ is compact. Then it is clear that $\phi DC_\varphi : Q_k(p,q) \to B^{\alpha}_{\log}$ is bounded. Let $\{z_n\}$ be a sequence in Δ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let

$$f_n(z)=\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{q+2}{p}}}.$$

Then $\sup_{n\in N}\|f_n\|_{Q_k(p,q)} < \infty$ and f_n converges to 0 uniformly on compact subsets of Δ as $n \to \infty$. Since $\phi DC_\varphi : Q_k(p,q) \to B^{\alpha}_{\log}$ is compact, by Lemma 3.1 we have $\lim_{n\to\infty}\|\phi DC_\varphi f_n\|_{B^{\alpha}_{\log}} = 0$. On the other hand, from (3.6) we have

$$
\begin{aligned}
C\|\phi DC_\varphi f_n\|_{B^{\alpha}_{\log}} \ge& -\frac{q+2}{p}\frac{p+q+2}{p}(1-|z_n|^2)^{\alpha}\log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}}+\\
&\frac{q+2}{p}(1-|z_n|^2)^{\alpha}\log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)\varphi''(z_n)+\phi'(z_n)\varphi'(z_n)||\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{q+2}{p}}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lim_{|\varphi(z_n)|\to1}\frac{p+q+2}{p}(1-|z_n|^2)^{\alpha}\log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}}\\
=&\lim_{|\varphi(z_n)|\to1}(1-|z_n|^2)^{\alpha}\log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)\varphi''(z_n)+\phi'(z_n)\varphi'(z_n)||\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{q+2}{p}}}, \qquad (3.15)
\end{aligned}
$$

if one of these two limits exists.

Next, for $n \in N$, set

$$g_n(z) = \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{p+q+2}{p}}} - \frac{p+q+2}{q+2}\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{q+2}{p}}}.$$

Then $\{g_n\}_{n\in N}$ is a sequence in $Q_k(p,q)$. Notice that $g_n'(\varphi(z_n)) = 0$,

$$\mid g_n''(\varphi(z_n)) \mid = \frac{p+q+2}{p}\frac{\mid \varphi(z_n) \mid^2}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}}.$$

And g_n converges to 0 uniformly on compact subsets of Δ as $n \to \infty$. Since $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log}$ is compact, we have $\lim_{n\to\infty} \|\phi DC_\varphi g_n\|_{B^\alpha_{\log}} = 0$. On the other hand, we have

$$\|\phi DC_\varphi g_n\|_{B^\alpha_{\log}} \geq \frac{p+q+2}{p}(1-|z_n|^2)^\alpha \log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}}.$$

Therefore,

$$\begin{aligned}&\lim_{|\varphi(z_n)|\to 1}(1-|z_n|^2)^\alpha \log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}}\\ &= \lim_{|\varphi(z_n)|\to 1}(1-|z_n|^2)^\alpha \log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}} = 0. \end{aligned} \tag{3.16}$$

This along with (3.15) implies

$$\lim_{|\varphi(z_n)|\to 1}(1-|z_n|^2)^\alpha \log\frac{2}{1-|z_n|^2}\frac{|\phi(z_n)\varphi''(z_n)+\phi'(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{q+2}{p}}} = 0. \tag{3.17}$$

From the last two equalities (3.16) and (3.17), the desired result follows.

Proof of Theorem 2.3 Sufficiency. Let $f \in Q_k(p,q)$. By the proof of Theorem 2.1 we have

$$\begin{aligned}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|(\phi DC_\varphi f)'(z)| \leq C\{&(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}+\\ &(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\}\|f\|_{Q_k(p,q)}.\end{aligned} \tag{3.18}$$

Taking the supremum in (3.18) over all $f \in Q_k(p,q)$ such that $\|f\|_{Q_k(p,q)} \leq 1$, then letting $\mid z \mid \to 1$, we get

$$\lim_{|z|\to 1}\sup_{\|f\|_{Q_k(p,q)}\leq 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|(\phi DC_\varphi f)'(z)| = 0.$$

So by Lemma 3.2, we see that the operator $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log,0}$ is compact.

Necessity. Assume that $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log,0}$ is compact. By taking $f(z) \equiv z$ and using the boundedness of $\phi DC_\varphi : Q_k(p,q) \to B^\alpha_{\log,0}$, we get

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)| = 0. \tag{3.19}$$

From this, by taking the test function $f(z) \equiv \frac{z^2}{2}$ and using the boundedness of $\phi DC_\varphi : Q_k(p,q) \to$

$B^{\alpha}_{\log,0}$, it follows that

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)(\varphi'(z))^2| = 0. \tag{3.20}$$

If $\|\varphi\|_{\infty} < 1$, from (3.19) and (3.20), we obtain that

$$\begin{aligned}
&\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}\\
&\le \frac{1}{(1-\|\varphi\|_{\infty})^{\frac{p+q+2}{p}}}\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)(\varphi'(z))^2| = 0,\\
&\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\\
&\le \frac{1}{(1-\|\varphi\|_{\infty})^{\frac{q+2}{p}}}\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)| = 0.
\end{aligned}$$

So the result follows in this case.

Assume that $\|\varphi\|_{\infty} = 1$. Let $\{\varphi(z_n)\}_{n\in N}$ be a sequence such that $\lim_{n\to\infty} |\ \varphi(z_n)\ | = 1$. From the compactness of $\phi DC_{\varphi} : Q_k(p,q) \to B^{\alpha}_{\log,0}$, we see that $\phi DC_{\varphi} : Q_k(p,q) \to B^{\alpha}_{\log}$ is compact. From Theorem 2.2 we get

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0, \tag{3.21}$$

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0. \tag{3.22}$$

From (3.19) and (3.22), we have that for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \varepsilon,$$

when $r < |\varphi(z)| < 1$, and there exists a $\sigma \in (0,1)$ such that

$$(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)| \le \varepsilon(1-r^2)^{\frac{q+2}{p}},$$

when $\sigma < |z| < 1$. Therefore, when $\sigma <|\ z\ |< 1$, and $r < |\varphi(z)| < 1$, we have

$$(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \varepsilon. \tag{3.23}$$

On the other hand, if $\sigma <|\ z\ |< 1$, and $|\ \varphi(z)\ |\le r$, we obtain

$$\begin{aligned}
&(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\\
&< \frac{1}{(1-r^2)^{\frac{q+2}{p}}}(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2}|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)| < \varepsilon.
\end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we get the second equality of (2.4). Similarly to the above arguments, by (3.20) and (3.21), we get the first equality of (2.4). The proof is completed. □

References

[1] ATTLE K. *Toeplitz and Hankel on Bergman one space* [J]. Hokkaido Math., 1992, **21**: 279–293.

[2] GALANOPOULOS P. *On* $B_{\log}$ *to* $Q^p_{\log}$ *pullback* [J].J. Math. Anal. Appl., 2008, **337**(1): 712–725.

[3] LI Haiying, LIU Peide. *Composition operators between generally weighted Bloch space and* $Q^q_{\log}$ *space* [J]. Banach J. Math. Anal., 2009, **3**(1): 99–110.

[4] STEVIĆ S. *On new Bloch-type spaces* [J]. Appl. Math. Comput., 2009, **215**(2): 841–849.

[5] WULAN Hasi, ZHOU Jizhen. Q_K *type spaces of analytic functions* [J]. J. Funct. Spaces Appl., 2006, **4**(1): 73–84.

[6] ZHAO Ruhan. *On a general family of function spaces* [J]. Ann. Acad. Sci. Fenn. Math. Diss., 1996, **105**: 1–56.

[7] HIBSCHWEILER R A, PORTNOY N. *Composition followed by differentiation between Bergman and Hardy spaces* [J]. Rocky Mountain J. Math., 2005, **35**(3): 843–855.

[8] OHNO S. *Products of composition and differentiation between Hardy spaces* [J]. Bull. Austral. Math. Soc., 2006, **73**(2): 235–243.

[9] LI Songxiao, STEVIĆ S. *Composition followed by differentiation between Bloch type spaces* [J]. J. Comput. Anal. Appl., 2007, **9**(2): 195–205.

[10] LI Songxiao, STEVIĆ S. *Composition followed by differentiation between* H^∞ *and* α*-Bloch spaces* [J]. Houston J. Math., 2009, **35**(1): 327–340.

[11] COWEN C C, MACCLUER B D. *Composition Operators on Spaces of Analytic Functions* [M]. CRC Press, Boca Raton, FL, 1995.

[12] SHAPIRO J H. *Composition Operators and Classical Function Theory* [M]. Springer-Verlag, New York, 1993.

[13] KOTILAIINEN M. *On composition operators in* Q_k *type spaces* [J]. J. Funct. Spaces Appl., 2007, **5**(2): 103–122.

[14] ZHU Kehe. *Bloch type spaces of analytic functions* [J]. Rocky Mountain J. Math., 1993, **23**(3): 1143–1177.

[15] MADIGAN K, MATHESON A. *Compact composition operators on the Bloch space* [J]. Trans. Amer. Math. Soc., 1995, **347**(7): 2679–2687.