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Jordan Semi-Triple Multiplicative Maps on the Hermitian Matrices

Si Qing YAN, Run Ling AN*, Jin Chuan HOU
Department of Mathematics, Taiyuan University of Technology, Shanxi 030024*, P. R. China*

Abstract In this paper, we show that every injective Jordan semi-triple multiplicative map on the Hermitian matrices must be surjective, and hence is a Jordan ring isomorphism.

Keywords Hermitian matrices; Jordan semi-triple multiplicative map; Jordan ring isomorphism.

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1. Introduction

It is an interesting problem to study the interrelation between the multiplicative and the additive structure of a ring or algebra. Matindale in [2] proved that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Thus the multiplicative structure determines the ring structure for some rings. Besides ring homomorphisms between rings, sometimes one has to consider Jordan ring homomorphisms. Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings and $\Phi:\mathcal{R}\to\mathcal{R}'$ be a transformation. The map Φ is called a Jordan homomorphism if it is additive and satisfies the condition

$$\Phi(AB+BA)=\Phi(A)\Phi(B)+\Phi(B)\Phi(A)$$

for every A,B in $\mathcal{R}$. If the ring $\mathcal{R}'$ is 2-torsion free (i.e, $2x=0$ implies $x=0$), then each Jordan homomorphism $\Phi:\mathcal{R}\to\mathcal{R}'$ is a Jordan semi-triple homomorphism [1], i.e., an additive map satisfying

$$\Phi(ABA)=\Phi(A)\Phi(B)\Phi(A)$$

for all A,B in $\mathcal{R}$. Without the assumption of additivity, such map is called a Jordan semi-triple multiplicative map.

Note that Jordan algebra has important applications in the mathematical foundations of Quantum mechanics. So it is also interesting to ask when the Jordan multiplicative structure

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* Corresponding author
E-mail address: runlingan@yahoo.com.cn (R. L. AN)

determines the Jordan ring structure of Jordan rings (or algebras). The question of when a Jordan semi-triple multiplicative map is additive was studied by several authors. For example, Molnár in [5] proved the additivity of bijective Jordan semi-triple multiplicative maps on standard operator algebras, and recently the additivity of injective Jordan semi-triple multiplicative maps on matrix algebras was given in [4]. The additivity of bijective Jordan semi-triple multiplicative maps on the real Jordan algebras of all self-adjoint operators was considered by An and Hou in [3]. In this paper, motivated by the result in [4], we study injective Jordan semi-triple multiplicative maps on the Hermitian matrices $\mathcal{H}_n(\mathbb{C})$, and show that such maps must be surjective, and hence are Jordan ring isomorphisms.

Let us recall and fix some notations in this paper. Recall that $P \in \mathcal{H}_n(\mathbb{C})$ is called a projection if $P^2 = P$. We define the order $\leq$ between projections as follows: $P \leq Q$ if and only if $PQ = QP = P$ for any projections $P, Q \in \mathcal{H}_n(\mathbb{C})$. For any $1 \leq j, k \leq n$, let E_{jk} be the matrix with 1 in the position (j, k) and zeros elsewhere, and I_j be the unit of $\mathcal{H}_j(\mathbb{C})$.

2. Main result and its proof

Firstly, we give some properties of injective Jordan semi-triple multiplicative maps on $\mathcal{H}_n(\mathbb{C})$.

Lemma 2.1 *Let $\mathcal{H}_n(\mathbb{C})$ be the Hermitian matrices and $\Phi : \mathcal{H}_n(\mathbb{C}) \to \mathcal{H}_n(\mathbb{C})$ be an injective Jordan semi-triple multiplicative map. Then Φ sends idempotents to tripotents and moreover,*

(1) $\Phi(I_n)^2$ is an idempotent and

$$\begin{aligned}\Phi(A) =&\Phi(I_n)\Phi(A)\Phi(I_n) = \Phi(I_n)^3\Phi(A)\Phi(I_n) = \Phi(I_n)\Phi(A)\Phi(I_n)^3\\ =&\Phi(I_n)^2\Phi(A) = \Phi(A)\Phi(I_n)^2\end{aligned}$$

for all $A \in \mathcal{H}_n(\mathbb{C})$ (in particular $\Phi(I_n)^2\Phi(A)\Phi(I_n)^2 = \Phi(A)$);

(2) $\Phi(I_n)$ commutes with $\Phi(A)$ for every $A \in \mathcal{H}_n(\mathbb{C})$;

(3) $\Phi(P)^2 = \Phi(I_n)\Phi(P)$ is an projection for each projection $P \in \mathcal{H}_n(\mathbb{C})$;

(4) A map $\Psi : \mathcal{H}_n(\mathbb{C}) \to \mathcal{H}_n(\mathbb{C})$ defined by $\Psi(A) = \Phi(I_n)\Phi(A)$ for all $A \in \mathcal{H}_n(\mathbb{C})$, is a Jordan semi-triple multiplicative map, which is injective if and only if Φ is injective.

For Ψ defined in Lemma 2.1, we can see that $\Psi(P)^2 = \Psi(P)$, and $P \leq Q \Rightarrow \Psi(P) \leq \Psi(Q)$ for any projections $P, Q \in \mathcal{H}_n(\mathbb{C})$. Therefore we have:

Corollary 2.1 *Let $n, m \in \mathbb{N}$ and $\Phi : \mathcal{H}_n(\mathbb{C}) \to \mathcal{H}_m(\mathbb{C})$ be an injective Jordan semi-triple multiplicative map. Then $m \geq n$. In the case $m = n$, for each projection $P \in \mathcal{H}_n(\mathbb{C})$ the rank of projection $\Psi(P)$ is equal to the rank of P. In particular, $\Psi(0) = \Phi(0) = 0$, $\Psi(I) = \Phi(I)^2 = I$ and $\Psi(A^2) = \Psi(A)^2$.*

The following is the main result of this paper:

Theorem 2.1 *A map $\Phi : \mathcal{H}_n(\mathbb{C}) \to \mathcal{H}_n(\mathbb{C})$ is an injective Jordan semi-triple multiplicative map if and only if there exist an element $\xi \in \mathbb{R}$, $\xi = \pm 1$, and a unitary matrix $U \in M_n(\mathbb{C})$ such that $\Phi(A) = \xi UAU^*$.*

The main idea is to use the induction on n, the dimension of the matrix algebra, after proving the result for 2×2 matrices.

Proof In order to prove Theorem 2.1, it suffices to characterize Ψ. Note that if $\Psi(A) = UAU^*$, then $\Phi(I_n)^2 = \Psi(I_n) = I_n$, that is $\Phi(I_n)$ is invertible and $\Phi(I_n) = \Phi(I_n)^{-1}$. By Lemma 2.1, $\Phi(I_n)$ commutes with $\Phi(A)$ for all $A \in \mathcal{H}_n(\mathbb{C})$. It follows that $\Phi(I_n)$ commutes with $\Psi(A)$ for all $A \in \mathcal{H}_n(\mathbb{C})$. Therefore if $\Psi(A) = UAU^*$, $\Phi(I_n)$ must be a scalar matrix. As $\Phi(I_n)^2 = I_n$, thus $\Phi(I_n) = I_n$ or $\Phi(I_n) = -I_n$ and hence Φ has the desired form.

Therefore, we mainly characterize Ψ. The proof is given in two steps.

Step 1. The proof for $\mathcal{H}_2(\mathbb{C})$.

The matrix E_{11} is a projection of rank one. By Corollary 2.1, $\Psi(E_{11})$ is a rank one projection. There exists a 2×2 unitary matrix U such that $\Psi(E_{11}) = UE_{11}U^*$. Without loss of generality, we may assume that $\Psi(E_{11}) = E_{11}$.

By Corollary 2.1 and from the following fact

$$\Psi((E_{12} + E_{21})^2) = I_2 = \Psi(E_{12} + E_{21})^2$$

and

$$E_{11}\Psi(E_{12} + E_{21})E_{11} = \Psi(E_{11}(E_{12} + E_{21})E_{11}) = \Psi(0) = 0$$

we conclude that $\Psi(E_{12} + E_{21}) = (\begin{matrix} 0 & a \\ \bar{a} & 0 \end{matrix})$ with $|a| = 1$. Let $V = \begin{pmatrix} a^{-\frac{1}{2}} & 0 \\ 0 & \bar{a}^{-\frac{1}{2}} \end{pmatrix}$ by replacing Ψ with $V^*\Psi(.)V$ if necessary, we may assume that $\Psi(E_{12} + E_{21}) = E_{12} + E_{21}$ and $\Psi(E_{11}) = E_{11}$ since $V^*\Psi(E_{11})V = E_{11}$. For E_{22}, since $\Psi(E_{22})$ is a rank one projection satisfying $\Psi(E_{11})\Psi(E_{22})\Psi(E_{11}) = 0$ and $\Psi(E_{12} + E_{21})\Psi(E_{22})\Psi(E_{12} + E_{21}) = \Psi(E_{11}) = E_{11}$, we have $\Psi(E_{22}) = E_{22}$. Now for any $A = (a_{ij}) \in \mathcal{H}_2(\mathbb{C})$, let $B = (b_{ij}) = \Psi(A)$. Then $b_{ii}E_{ii} = E_{ii}BE_{ii} = \Psi(E_{ii})\Psi(A)\Psi(E_{ii}) = \Psi(E_{ii}AE_{ii}) = \Psi(a_{ii}E_{ii})$. Thus, the (i,i)th entry of $\Psi(A)$ depends on the (i,i)th entry of A only. For $aE_{12} + \bar{a}E_{21}$, applying similar argument as above, we have $\Psi(aE_{12} + \bar{a}E_{21}) = bE_{12} + \bar{b}E_{21}$. Therefore, there exist injective functionals $f, g, h : \mathbb{R} \to \mathbb{R}$ such that f, g satisfy respectively $f(a^2b) = f(a)^2f(b)$ and $g(a^2b) = g(a)^2g(b)$, and $\Psi\begin{pmatrix} a_{11} & a_{12} \\ \bar{a_{12}} & a_{22} \end{pmatrix} = \begin{pmatrix} f(a_{11}) & h(a_{12}) \\ h(\bar{a_{12}}) & g(a_{22}) \end{pmatrix}$. From $f(1) = g(1) = 1$, it is easy to verify that f, g is multiplicative. Next we prove that $f(x) = g(x) = x$. First we prove $f = g$.

Let $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Since $(E_{12} + E_{21})A(E_{12} + E_{21}) = A$ and $A^2 = A$, we have $(E_{12} + E_{21})\Psi(A)(E_{12} + E_{21}) = \Psi((E_{12} + E_{21})A(E_{12} + E_{21})) = \Psi(A)$ and $\Psi(A)^2 = \Psi(A^2) = \Psi(A)$, hence $\Psi(A) = A$ or $\Psi(A) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$. If $\Psi(A) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\Psi(A)\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = A.$$

So we assume that $\Psi(A) = A$. Thus $f(\frac{1}{2}) = g(\frac{1}{2}) = \frac{1}{2}$. Let $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $AJA = J$,

$A\Psi(J)A = \Psi(A)\Psi(J)\Psi(A) = \Psi(AJA) = \Psi(J)$, and $\Psi(J) = \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}$, we have $\Psi(J) = J$. For any $a \in \mathbb{R}$, since

$$f(a)J = J(f(a)E_{11})J = \Psi(J)\Psi(aE_{11})\Psi(J) = \Psi(aJE_{11}J) = \Psi(aJ) = \begin{pmatrix} f(a) & b \\ \bar{b} & g(a) \end{pmatrix},$$

we have $f(a) = g(a)$.

Next we prove that f is additive. Since $\Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2) = \Psi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2$, and we have proved $\Psi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix}$. For any $a, b \in \mathbb{R}$, let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\Psi(A) = \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix}$. By the fact that $JAJ = (a+b)J$, one can get that

$$\begin{aligned} \begin{pmatrix} f(a)+f(b) & 0 \\ 0 & f(a)+f(b) \end{pmatrix} &= J\Psi(A)J = \Psi(JAJ) = \Psi((a+b)J) \\ &= \begin{pmatrix} f(a+b) & c \\ \bar{c} & f(a+b) \end{pmatrix} \end{aligned}$$

and $f(a+b) = f(a) + f(b)$. Thus f is identity map of $\mathbb{R}$.

Finally, we prove $\Psi\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ for any $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$. Let $R = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$ and $\Psi(R) = \begin{pmatrix} 1 & b \\ \bar{b} & 0 \end{pmatrix}$. By the fact that $JRJ = J$, we get $J\Psi(R)J = \Psi(JRJ) = \Psi(J) = J$ and $\Psi(R)^2 = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}$, thus $\Psi(R) = R$ or $\Psi(R) = \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix}$. By a unitary transformation, we may assume that $\Psi(R) = R$. For any $A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$, let $\Psi(A) = \begin{pmatrix} a & \lambda \\ \bar{\lambda} & c \end{pmatrix}$. From $J\Psi(A)J = \Psi(JAJ)$ and $R\Psi(A)R = \Psi(RAR)$, we have the real and imaginary parts of b and λ are equal, thus $\Psi(A) = A$.

Step 2. The induction.

Let $P = I_{n-1} \oplus [0]$. Then P is a rank $n-1$ projection, so is $\Psi(P)$ by Corollary 2.1. There exists a unitary matrix U such that $\Psi(P) = UPU^*$. Replacing Ψ by the map $A| \to U^*\Psi(A)U$, we may assume that $\Psi(P) = P$.

For any $\hat{A} \in \mathcal{H}_{n-1}(\mathbb{C})$, let $A = \hat{A} \oplus [0]$. Then $PAP = A$ implies

$$P\Psi(A)P = \Psi(P)\Psi(A)\Psi(P) = \Psi(PAP) = \Psi(A).$$

It follows that $\Psi(\hat{A} \oplus [0]) = \Psi(A) = \hat{X} \oplus [0]$ for some matrix $\hat{X} \in \mathcal{H}_{n-1}(\mathbb{C})$. Define the map $\hat{\Psi}$ on $\mathcal{H}_{n-1}(\mathbb{C})$ by $\hat{\Psi}(\hat{A}) = \hat{X}$. It is easy to check that $\hat{\Psi}$ is an injective Jordan triple multiplicative map on $\mathcal{H}_{n-1}(\mathbb{C})$. Furthermore, $\Psi(P) = P$ implies that $\hat{\Psi}(I_{n-1}) = I_{n-1}$. By the induction hypothesis there is a unitary matrix $\hat{U}$ such that $\hat{\Psi}(\hat{A}) = \hat{U}\hat{A}\hat{U}^*$. Let $U = \hat{U} \oplus [1]$. Without loss of generality, we assume that $\hat{\Psi}(\hat{A}) = \hat{A}$ for all $\hat{A} \in \mathcal{H}_{n-1}(\mathbb{C})$. This is equivalent to $\Psi(\hat{A} \oplus [0]) = \hat{A} \oplus [0]$. Also for any $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in H_n(\mathbb{C})$ with $A_{11} \in \mathcal{H}_{n-1}(\mathbb{C})$ we have $PAP = A_{11} \oplus [0]$. Thus

$$P\Psi(A)P = \Psi(P)\Psi(A)\Psi(P) = \Psi(PAP) = A_{11} \oplus [0]. \qquad (*)$$

Let us define matrices R_i for each $i \in \{1, 2 \ldots, n-1\}$ by $R_i = I_n - E_{ii} - E_{nn} + E_{in} + E_{ni}$. For an arbitrary i, from $(*)$ we have $P\Psi(R_i)P = (I_{n-1} - E_{ii}) \oplus [0]$. Then there exist $x \in \mathbb{C}^{n-1}$ and $y \in \mathbb{R}$ such that $\Psi(R_i) = \begin{pmatrix} I_{n-1} - E_{ii} & x \\ \bar{x}^t & y \end{pmatrix}$. From the equalities $\Psi(R_i)^2 = \Psi(R_i^2) = \Psi(I_n) = I_n$, we get that $I_{n-1} - E_{ii} + x\bar{x}^t = I_{n-1}$ and $\bar{x}^t x + y^2 = 1$. These equalities imply that $x\bar{x}^t = E_{ii}$ and $y^2 = 1 - \bar{x}^t x = 1 - tr(x\bar{x}^t) = 1 - tr(E_{ii}) = 0$. Hence only the ith entry a_i of x is nonzero and $|a_i| = 1$. It follows that $\Psi(R_i) = I_n - E_{ii} - E_{nn} + a_i E_{in} + \bar{a}_i E_{ni}$. Next, take any two distinct $i, j \in \{1, 2 \ldots, n-1\}$. From $R_i R_j R_i = I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}$ and using $(*)$, we get

$$\begin{aligned} \Psi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) &= \Psi(R_i)\Psi(R_j)\Psi(R_i) \\ &= I_n - E_{ii} - E_{jj} + \bar{a}_i a_j E_{ij} + a_i \bar{a}_j E_{ji}, \end{aligned}$$

which implies that $a_i = a_j$. Let $D = I_{n-2} \oplus a^{-\frac{1}{2}} \oplus \bar{a}^{-\frac{1}{2}}$. Then $D\Psi(R_1)D^* = R_1$, so we may assume that $\Psi(R_1) = R_1$. From the equality $R_1 R_i R_1 = I_n - E_{11} - E_{ii} + E_{1i} + E_{i1}$ and $R_1 \Psi(R_i) R_1 = \Psi(R_1 R_i R_1)$, we can get that $\Psi(R_i) = R_i$.

Next we prove that $\Psi(A) = A$ for any $A \in \mathcal{H}_n(\mathbb{C})$. Let us fix some $i \in \{1, 2 \ldots, n-1\}$. As $n > 2$, there is another $j \in \{1, 2 \ldots, n-1\}$ such that $aE_{ni} + \bar{a}E_{in} = R_j(aE_{ij} + \bar{a}E_{ji})R_j$. Then for any $a \in \mathbb{R}$ and $b \in \mathbb{C}$,

$$\Psi(aE_{nn}) = \Psi(R_1)\Psi(aE_{11})\Psi(R_1) = R_1 aE_{11} R_1 = aE_{nn}$$

and

$$\begin{aligned} \Psi(bE_{ni} + \bar{b}E_{in}) =& \Psi(R_j(bE_{ni} + \bar{b}E_{in})R_j) = \Psi(R_j)\Psi(bE_{ni} + \bar{b}E_{in})\Psi(R_j) \\ =& R_j(bE_{ni} + \bar{b}E_{in})R_j = bE_{ni} + \bar{b}E_{in}. \end{aligned}$$

Let

$$A = \begin{pmatrix} aE_{ii} & x \\ \bar{x}^t & a \end{pmatrix},$$

where $x \in^{n-1}$ has only one nonzero entry in the ith position , and $\Psi(A) = \begin{pmatrix} aE_{ii} & y \\ \bar{y}^t & b \end{pmatrix}$. Then from $E_{nn}AE_{nn} = aE_{nn}$, we have $E_{nn}\Psi(A)E_{nn} = \Psi(E_{nn})\Psi(A)\Psi(E_{nn}) = \Psi(E_{nn}AE_{nn}) = aE_{nn}$ and $b = a$, and with $\Psi(A^2) = \Psi(A)^2$, we have $\Psi(A) = A$. For any $A \in \mathcal{H}_n(\mathbb{C})$, let $A = \begin{pmatrix} \hat{A} & x \\ \bar{x}^t & a \end{pmatrix}$ and $\Psi(A) = \begin{pmatrix} \hat{A} & y \\ \bar{y}^t & b \end{pmatrix}$. By the same argument as above, we have $a = b$. For any $i \in \{1, 2 \ldots, n-1\}$, since $(E_{in} + E_{ni})A(E_{in} + E_{ni}) = B = \begin{pmatrix} \hat{B} & z \\ \bar{z}^t & \alpha \end{pmatrix}$ where $\hat{B} \in \mathcal{H}_{n-1}(\mathbb{C})$ and z has only one nonzero entry a_{nn} and a_{in} in the (i, i)th and ith position, respectively, α is equal to the (i,i)th entry of A , we have $(E_{in}+E_{ni})\Psi(A)(E_{in}+E_{ni}) = \Psi(E_{in}+E_{ni})\Psi(A)\Psi(E_{in}+E_{ni}) = \Psi((E_{in}+E_{ni})A(E_{in}+E_{ni})) = \Psi(B) = B$ and so $x = y$. As desired. □

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