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Some New Translation Surfaces in 3-Minkowski Space

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Abstract In this paper we study translation surfaces of some new types in 3-Minkowski space $\mathbb{E}_1^3$ and give some classifications of such surfaces whose mean curvature and Gauss curvature satisfy certain conditions.

Keywords Minkowski space; translation surface; Weingarten surface.

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1. Introduction

For the study of the surfaces theory in 3-Euclidean space $\mathbb{E}^3$ or 3-Minkowski space $\mathbb{E}_1^3$, it is a very important and interesting problem to construct or classify the constant mean curvature or constant Gaussian curvature, or even more general, Weingarten surfaces. It is well-known that the translation surface is special and minimal one in 3-Euclidean space $\mathbb{E}^3$ is Scherk surface. Here we consider translation surfaces in 3-Minkowski space. The second author gave some classification results for translation surfaces in [1] and [2]. However according to our recent work [3–6] we know that the results in [1] or [2] are only the Cases 1 and 2 of following 6 types of translation surfaces.

In 3-Minkowski space $\mathbb{E}_1^3$, according to the spacelike direction, timelike direction and lightlike direction, the translation surfaces can be considered as the following six types

Type 1. Along spacelike direction and spacelike direction;

Type 2. Along spacelike direction and timelike direction;

Type 3. Along lightlike direction and lightlike direction;

Type 4. Along lightlike direction and spacelike direction;

Type 5. Along timelike direction and lightlike direction;

Type 6. Along timelike direction and timelike direction.

As we know that they are really different under Lorentz transformation in $\mathbb{E}_1^3$. Using certain coordinate frames, we can express them in the different way [3, 6].

Let $\mathbb{E}_1^3$ be the 3-Minkowski space with the inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3.$$

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Translation surface S_a of Types 5 and 6 can be written as

$$S_a : x(u,v) = \{X(u,v), Y(u,v), Z(u,v)\} = \{f(u+av) + g(v), u, v\}.$$

(i) When $|a| = 1$, the surface S_a is translation surface of Type 5.

(ii) When $|a| > 1$, the surface S_a is translation surface of Type 6.

With $x_u = \frac{\partial x(u,v)}{\partial u}$, etc., the first fundamental form I of the surface S_a is given by

$$\begin{aligned} I &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2, \\ E &= \langle x_u, x_u\rangle = f_u^2 + 1, \\ F &= \langle x_u, x_v\rangle = f_u(af_v + g_v), \\ G &= \langle x_v, x_v\rangle = (af_v + g_v)^2 - 1. \end{aligned}$$

For spacelike or timelike surface in $\mathbb{E}_1^3$, we have $EG - F^2 > 0$ or $EG - F^2 < 0$. The second fundamental form II of S_a is given by

$$\begin{aligned} II &= L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2, \\ L &= \frac{1}{\sqrt{|EG-F^2|}}\det(x_u, x_v, x_{uu}) = \frac{f_{uu}}{\sqrt{|(af_v+g_v)^2 - f_u^2 - 1|}}, \\ M &= \frac{1}{\sqrt{|EG-F^2|}}\det(x_u, x_v, x_{uv}) = \frac{af_{uv}}{\sqrt{|(af_v+g_v)^2 - f_u^2 - 1|}}, \\ N &= \frac{1}{\sqrt{|EG-F^2|}}\det(x_u, x_v, x_{vv}) = \frac{a^2 f_{vv} + g_{vv}}{\sqrt{|(af_v+g_v)^2 - f_u^2 - 1|}}. \end{aligned}$$

The Gauss curvature K and the mean curvature H of S_a are given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{f_{uu}(a^2 f_{vv} + g_{vv}) - a^2 f_{uv}^2}{((af_v+g_v)^2 - f_u^2 - 1)|(af_v+g_v)^2 - f_u^2 - 1|}, \tag{1}$$

$$\begin{aligned} H &= \frac{EN - 2FM + GL}{2(EG - F^2)} \\ &= \frac{(f_u^2+1)(a^2 f_{vv} + g_{vv}) - 2af_u f_{uv}(af_v + g_v) + f_{uu}((af_v+g_v)^2 - 1)}{2((af_v+g_v)^2 - f_u^2 - 1)\sqrt{|(af_v+g_v)^2 - f_u^2 - 1|}}. \end{aligned} \tag{2}$$

2. Main results

By a transformation

$$\begin{cases} y = u + av, \\ z = v, \end{cases}$$

and $\frac{\partial(y,z)}{\partial(u,v)} \neq 0$, from (1) and (2) we get

$$K = \frac{f_{yy} g_{zz}}{\varepsilon((a^2 f_y + g_z)^2 - f_y^2 - 1)^2}, \tag{3}$$

$$H = \frac{g_{zz}(1 + f_y^2) + f_{yy}(a^4 - 1 + g_z^2)}{2\varepsilon(\varepsilon((a^2 f_y + g_z)^2 - f_y^2 - 1))^{\frac{3}{2}}}, \tag{4}$$

where $\varepsilon = \pm 1$. In the following, we will consider translation surfaces of Types 5 and 6 whose Gauss curvature K and mean curvature H satisfy certain conditions. They are usually called

Weingarten surfaces.

Theorem 1 *Let S_a be a translation surface of Type 6 in $\mathbb{E}_1^3$. If S_a is minimal, it is congruent to a plane or the functions f and g satisfy*

$$\begin{cases} f = -\frac{1}{c}\log|\sec(-c(u+av)+c_1)| + c_2, \\ g = \frac{1}{c}\log|\sec(c\sqrt{a^4-1}v + c_1)| + c_2, \end{cases}$$

where c, c_1, c_2 are constants and $c \neq 0$.

Proof Let S_a be a translation surface of Type 6 in $\mathbb{E}_1^3$. By a transformation in $\mathbb{E}_1^3$, the translation surface S_a can be written as

$$x(u,v) = \{f(u+av) + g(v), u, v\}, \quad |a| > 1.$$

From (4), putting $H = 0$ gives

$$g_{zz}(1+f_y^2) + f_{yy}(a^4 - 1 + g_z^2) = 0.$$

Hence

$$\frac{g_{zz}}{a^4-1+g_z^2} = -\frac{f_{yy}}{1+f_y^2} = c,$$

where c is constant.

i) When $c = 0$, we have

$$g_{zz} = 0 \text{ and } f_{yy} = 0.$$

Then the surface is a plane.

ii) When $c \neq 0$, we have

$$\begin{cases} f = -\frac{1}{c}\log|\sec(-c(u+av)+c_1)| + c_2, \\ g = \frac{1}{c}\log|\sec(c\sqrt{a^4-1}v + c_1)| + c_2, \end{cases}$$

where c_1, c_2 are constants. This completes the proof of Theorem (1). □

Theorem 2 *Let S_a be a translation surface of Type 6 with constant mean curvature $H \neq 0$ in $\mathbb{E}_1^3$. Then*

(i) If S_a is spacelike, it is congruent to the following surfaces or an open part of them in $\mathbb{E}_1^3$

(a) $X(u,v) = -\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2v^2-1} - a^2cv + c(u+av)$, $c \in R$,

(b) $X(u,v) = -\frac{\sqrt{c^2+a^4-1}}{2H\sqrt{a^4-1}}\sqrt{\frac{4H^2}{a^4-1}(u+av)^2-1} - \frac{a^2c}{a^4-1}u + \frac{a^4-a^3-1}{a^4-1}cv$, $c \in R$;

(ii) If S_a is timelike, it is congruent to the following surfaces or an open part of them in $\mathbb{E}_1^3$

(c) $X(u,v) = -\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2v^2+1} - a^2cv + c(u+av)$, $c \in R$,

(d) $X(u,v) = -\frac{\sqrt{c^2+a^4-1}}{2H\sqrt{a^4-1}}\sqrt{\frac{4H^2}{a^4-1}(u+av)^2+1} - \frac{a^2c}{a^4-1}u + \frac{a^4-a^3-1}{a^4-1}cv$, $c \in R$.

Proof Let S_a be a translation surface of Type 6 with constant mean curvature $H \neq 0$ in $\mathbb{E}_1^3$.

We assume that $f_{yy}g_{zz} \neq 0$. Differentiating (4) with respect to y and z, we obtain

$$(a^4-1)\frac{\left(\frac{\left(\frac{\left(\frac{g_{zzz}}{g_{zz}}\right)_z}{g_{zz}}\right)_z}{g_{zz}}\right)_z}{g_{zz}} = 3\frac{\left(\frac{\left(\frac{f_{yy}}{f_y^2+1}\right)_y}{f_{yy}}\right)_y}{f_{yy}} = 3H.$$

That is

$$\begin{cases} f_{yy} = (\frac{H}{2}f_y^2 + c_1 f_y + c_2)(f_y^2+1), \\ g_{zz} = \frac{k}{24}g_z^4 + k_1 g_z^3 + k_2 g_z^2 + k_3 g_z + k_4, \end{cases}$$

where $k = \frac{3H}{a^4-1}$, c_1, c_2, k_1, k_2, k_3, k_4 are constants. Putting f_{yy} into (4) and considering the coefficient of f_y^4, we can get $H = 0$ or $g(z) =$ constant, which contradicts $H \neq 0$.

By a transformation in $\mathbb{E}_1^3$ we can assume that $f_{yy} = 0$ and write $f(y) = cy$. From (4) we have

$$(c^2+1)g_{zz} = 2H((a^2c + g_z)^2 - c^2 - 1)^{\frac{3}{2}} \tag{5}$$

or

$$(c^2+1)g_{zz} = -2H(c^2+1-(a^2c+g_z)^2)^{\frac{3}{2}}. \tag{6}$$

Solving these equations, we obtain the following surfaces, respectively

$$g(z) = -\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2(z+c_1)^2-1} - a^2cz + c_2, \quad c_1, c_2, c \in R, \tag{7}$$

which is spacelike and congruent to the surface (a) given by Theorem (2);

$$g(z) = -\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2(z+c_1)^2+1} - a^2cz + c_2, \quad c_1, c_2, c \in R, \tag{8}$$

which is timelike and congruent to the surface (c) given by Theorem (2).

When $g_{zz} = 0$ we assume that $g(z) = cz$. By (4) we have

$$(a^4+c^2-1)f_{yy} = 2H((a^2f_y + c)^2 - f_y^2 - 1)^{\frac{3}{2}} \tag{9}$$

or

$$(a^4+c^2-1)f_{yy} = -2H(f_y^2+1-(a^2f_y+c)^2)^{\frac{3}{2}}. \tag{10}$$

Solving these equations, we obtain the following surfaces, respectively

$$f(y) = -\frac{\sqrt{a^4+c^2-1}}{2H\sqrt{a^4-1}}\sqrt{\frac{4H^2}{a^4-1}(y+c_1)^2-1} - \frac{a^2c}{a^4-1}y + c_2, \quad c_1, c_2, c \in R, \tag{11}$$

which is spacelike and congruent to the surface (b) given by Theorem (2);

$$f(y) = -\frac{\sqrt{a^4+c^2-1}}{2H\sqrt{a^4-1}}\sqrt{\frac{4H^2}{a^4-1}(y+c_1)^2+1} - \frac{a^2c}{a^4-1}y + c_2, \quad c_1, c_2, c \in R, \tag{12}$$

which is timelike and congruent to the surface (d) given by Theorem (2). This completes the proof of Theorem (2). □

Theorem 3 *Let $S_a : x(u,v) = \{f(u+av)+g(v), u, v\}$ be a translation surface of Type 5 or 6 with Gauss curvature $K=0$ in $\mathbb{E}_1^3$. Then the functions f and g satisfy*

$$\begin{cases} f(u+av) = c_1(u+av)+c_2, \quad c_1, c_2 \in R, \\ g(v) \text{ is any function,} \end{cases} \tag{13}$$

or

$$\begin{cases} g = c_1 v + c_2, \quad c_1, c_2 \in R, \\ f(u+av) \text{ is any function.} \end{cases} \tag{14}$$

Proof From (3), putting $K=0$, we get

$$f_{yy} g_{zz} = 0.$$

i) When $f_{yy}=0$, we have

$$\begin{cases} f = c_1 y + c_2 = c_1(u+av)+c_2, \quad c_1, c_2 \in R, \\ g(v) \text{ is any function.} \end{cases} \tag{15}$$

ii) When $g_{zz}=0$, we get

$$\begin{cases} g = c_1 z + c_2 = c_1 v + c_2, \quad c_1, c_2 \in R, \\ f(u+av) \text{ is any function.} \quad \square \end{cases} \tag{16}$$

Theorem 4 *There is no translation surface of Type 5 or 6 with constant Gauss curvature $K \neq 0$ in $\mathbb{E}_1^3$.*

Proof Let S_a be a translation surface of Type 6 with constant Gauss curvature $K \neq 0$ in $\mathbb{E}_1^3$. From (3) we have $f_{yy} g_{zz} \neq 0$. Differentiating (3) with respect to y and z, we obtain

$$g_{zzz}((a^4-1)f_y + g_z) - 2a^2 g_{zz}^2 = 0. \tag{17}$$

If $g_{zzz}=0$ and $a \neq 0$, then $g_{zz}=0$, which contradicts the assumption $K \neq 0$. So when $g_{zzz} \neq 0$ we have

$$(a^4-1)f_y = \frac{2a^2 g_{zz}^2}{g_{zzz}} - g_z = c,$$

that is

$$\begin{cases} (a^4-1)f_y = c, \\ \dfrac{2a^2 g_{zz}^2}{g_{zzz}} - g_z = c. \end{cases} \tag{18}$$

By (18) we get that $f_{yy}=0$. That means $K=0$. Therefore, there is no translation surface of Type 6 with constant Gauss curvature $K \neq 0$ in $\mathbb{E}_1^3$. The proof of translation surface of Type 5 is similar. This completes the proof of Theorem (4). □

With the same methods we can also obtain the following results. We omit the proofs.

Theorem 5 *Let $x(u,v) = \{f(u+av)+g(v), u, v\}$ be a translation surface of Type 5 which is* *minimal in* $\mathbb{E}_1^3$. *Then the surface is a plane or the functions* f *and* g *satisfy*

$$\begin{cases} f=\dfrac{1}{c}\log|\sec(c(u+av)+c_1)|+c_2, \\ g=\dfrac{1}{c}\log|cv+c_1|+c_2, \end{cases} \tag{19}$$

where c_1, c_2, c *are constants and* $c\neq 0$.

Theorem 6 *Let* S_a *be a translation surface of Type 5 with constant mean curvature* $H\neq 0$ *in* $\mathbb{E}_1^3$. *Then*

(i) *If* S_a *is spacelike, it is congruent to the following surfaces or an open part of them in* $\mathbb{E}_1^3$

(a) $X(u,v)=-\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2v^2-1}+cu, \quad c\in R,$

(b) $X(u,v)=-\frac{c}{8H^2}\frac{1}{u+v}+\frac{1-c^2}{2c}u+\frac{c^2+1}{2c}v$, $c\neq 0$ *and* $c\in R$;

(ii) *If* S_a *is timelike, it is congruent to the following surfaces or an open part of them in* $\mathbb{E}_1^3$

(c) $X(u,v)=-\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2v^2+1}+cu$,$c\in R$,

(d) $X(u,v)=\frac{c}{8H^2}\frac{1}{u+v}+\frac{1-c^2}{2c}u+\frac{c^2+1}{2c}v$, $c\neq 0$ *and* $c\in R$.

Theorem 7 *Let* S_a *be a translation surface of Type 5 or 6 in* $\mathbb{E}_1^3$ *whose Gauss curvature* K *and mean curvature* H *satisfy* $bH+cK=0$ $(bc\neq 0)$. *Then it is congruent to a plane or an open part of it.*

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