Journal of Mathematical Research with Applications
Jan., 2012, Vol. 32, No. 1, pp. 1–10
DOI:10.3770/j.issn:2095-2651.2012.01.001
Http://jmre.dlut.edu.cn

Tchebyshev Approximation by $S_1^0(\Delta)$ over Some Special Triangulations

Ren Hong WANG[1,*], **Wei DAN**[1,2]
1. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
2. *School of Mathematical and Computational Sciences, Guangdong University of Business Studies, Guangdong* 510320*, P. R. China*

Abstract The critical point set plays a central role in the theory of Tchebyshev approximation. Generally, in multivariate Tchebyshev approximation, it is not a trivial task to determine whether a set is critical or not. In this paper, we study the characterization of the critical point set of $S_1^0(\Delta)$ in geometry, where Δ is restricted to some special triangulations (bitriangular, single road and star triangulations). Such geometrical characterization is convenient to use in the determination of a critical point set.

Keywords Tchebyshev approximation; bivariate splines; $S_1^0(\Delta)$.

MR(2010) Subject Classification 41A15; 41A52

1. Introduction

The theory of Tchebyshev approximation for functions in one variable by polynomials is well-known [5]. As we know, for each $f \in C[a,b]$ and its unique best approximation $g \in \mathcal{P}_n$, the error $f-g$ has $n+2$ alternating extremal points, where $\mathcal{P}_n$ is the linear space of all polynomials of degree at most n. This theory has been extended in various directions. Tchebyshev approximation for functions in one variable by univariate splines can be found in [1, 3]. Tchebyshev approximation for functions in several variables was studied in [2, 7]. The author in [2] proposed the concept of the critical point set in multivariate Tchebyshev approximation and showed a characterization theorem for best approximations in terms of the critical point set. However, in practical applications, determining the critical point set as defined in [2] is not a trivial task. In multivariate Tchebyshev approximation, the number of points in the critical point set is not unique. The authors in [7] investigated the geometrical characterization of the critical point set in Tchebyshev approximation by multivariate polynomials, which is convenient to use in many applications, for example, in the determination of a best approximation.

In 1975, Wang [4] established the basic frame on multivariate splines using the conformality of smoothing cofactor method. With the development of computer, the theory and applications

Received March 20, 2010; Accepted May 28, 2010
Supported by the National Natural Science Foundation of China (Grant Nos. 10271022; 60373093; 60533060; 11101366; 61100130) and the Innovation Foundation of the Key Laboratory of High-Temperature Gasdynamics of Chinese Academy of Sciences.
* Corresponding author
E-mail address: renhong@dlut.edu.cn (R. H. WANG); lwdan@yahoo.cn (W. DAN)

of multivariate splines have been developed rapidly [6]. In view of the variety and complexity of the objectives, multivariate splines are often used as approximations. We need to study multivariate Tchebyshev approximation by multivariate splines in practice. The geometrical characterization of the critical point set of multivariate splines heavily depends on the properties of the piecewise algebraic curves. This makes Tchebyshev approximation by multivariate splines extremely difficult. In this paper, we restrict ourselves to the simplest cases, i.e., Tchebyshev approximation by $S_1^0(\Delta)$ over three special triangulations.

The rest of the paper is organized as follows. Section 2 reviews the notations and preliminaries, which will be used throughout this paper. Section 3 studies the uniqueness of the best approximation in multivariate Tchebyshev approximation. The geometrical characterization of the critical point set of $S_1^0(\Delta)$ is given in Section 4. A brief conclusion is presented in Section 5.

2. Notations and preliminaries

Let Δ be a regular triangulation of a simply connected polygonal domain D in $\mathbb{R}^2$, i.e., a set of closed triangles such that the intersection of any two triangles is empty, a common edge, or a vertex. Each of triangles in Δ is called a cell. Denote all cells of Δ by $D_i, i = 1, 2, \ldots, N$, where N is the number of the cells of Δ. A cell is called an edge cell if it has a vertex which is not a vertex of any other cells. The line segments that form the triangles are called the mesh segments. The vertices of the triangles are called the mesh points. The number of all mesh segments with a certain mesh point V as a common endpoint is called degree of the mesh point V. The space of bivariate splines is defined by

$$S_k^\mu(\Delta) := \{s \in C^\mu(D) \big| \ s\,|_{D_i} \in \mathcal{P}_k, \quad i = 1, \ldots, N\},$$

where $\mathcal{P}_k$ is the collection of all bivariate polynomials with real coefficients and total degree at most k.

If $s(x) \in S_k^\mu(\Delta)$ is a nontrivial bivariate spline, then the curve

$$\Gamma : s(x) = 0, \quad s(x) \in S_k^\mu(\Delta)$$

is called a piecewise algebraic curve. It is obvious that the piecewise algebraic curve is a generalization of the classical algebraic curve.

Let $C(D)$ denote the space of continuous functions on D, equipped with the supremum norm. The Tchebyshev approximation problem in this paper is stated as follows. Given $f \in C(D)$, determine $s^* \in S_1^0(\Delta)$ such that

$$\|f - s^*\|_\infty \le \|f - s\|_\infty$$

for all $s \in S_1^0(\Delta)$. Such an s^* is said to be a best approximation to f. The points satisfying

$$|f(x) - s(x)| = \|f - s\|_\infty$$

are said to be extremal points. The set of extremal points is divided into two parts as follows.

$$\mathfrak{P} = \{x \big| \ x \in D, \quad f(x) - s(x) = \|f - s\|_\infty\},$$

$$\mathfrak{N} = \{x \big| \ x \in D, \quad f(x) - s(x) = -\|f - s\|_\infty\}.$$

The points in these two sets are called positive points and negative points, respectively. These designations may also be used when s is not a best approximation to f.

Let $\mathcal{H}$ be a k dimensional linear subspace of $C(D)$, and suppose points $x_1, x_2, \ldots, x_k \in D$. If for any given real numbers $a_1, a_2, \ldots, a_k$, there exists a unique $p(x) \in \mathcal{H}$ such that

$$p(x_i) = a_i, \quad i = 1, 2, \ldots k,$$

then the point set $\{x_1, x_2, \ldots, x_k\}$ is called a properly posed knot set of $\mathcal{H}$.

3. Uniqueness

In general, the best approximation $s \in S_1^0(\Delta)$ to f is not unique. However, under some assumptions of s, the uniqueness of the best approximation can be guaranteed. This will be discussed in this section. In this paper, we adopt the concept of the critical point set in [7] because it is more convenient to use in our analysis than the concept in [2].

Definition 1 ([7]) *A point set $\mathfrak{S} = \{x_1, x_2, \ldots, x_{p+1}\}$, $p \leq k$, is said to be an uninterpolated point set of the space $\mathcal{H}$ if the following two conditions hold.*

1) $\mathfrak{S}$ is not a properly posed knot set of any subspace of $\mathcal{H}$.

2) If any point is deleted from $\mathfrak{S}$, then the new point set is a properly posed knot set of some subspace of $\mathcal{H}$.

Now suppose the point set $\{x_1, x_2, \ldots, x_{p+1}\}$ is an uninterpolated point set of the space $\mathcal{H}$, then the point set $\{x_1, x_2, \ldots, x_p\}$ is a properly posed point set of some subspace of $\mathcal{H}$. Let $L_1(x), L_2(x), \ldots, L_p(x)$ be the corresponding Lagrange interpolation functions based on the points $x_1, x_2, \ldots, x_p$. There exist unique (without considering a constant multiple) nonzero constants $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$ such that

$$\alpha_1 L_j(x_1) + \alpha_2 L_j(x_2) + \cdots + \alpha_{p+1} L_j(x_{p+1}) = 0, \quad j = 1, 2, \ldots, p. \tag{1}$$

Definition 2 ([7]) *An uninterpolated point set $\{x_1, x_2, \ldots, x_{p+1}\}$ of $\mathcal{H}$ is said to be a critical point set of $\mathcal{H}$ if each point possesses a corresponding sign of $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$.*

Remark Critical point sets play a central role in Tchebyshev approximation in several variables. The sign distribution of the points in an uninterpolated point set determines whether it is critical or not.

Theorem 1 ([7]) *$h^* \in \mathcal{H}$ is a best approximation to f if and only if the set of extremal points of $f - h^*$ contains a critical point set of $\mathcal{H}$.*

Applying Theorem 1 to the class $S_1^0(\Delta)$, we obtain the following result.

Corollary 1 *$s^* \in S_1^0(\Delta)$ is a best approximation to f if and only if the set of extremal points of $f - s^*$ contains a critical point set of $S_1^0(\Delta)$.*

Theorem 2 ([2]) *Let $s_1, s_2 \in S_1^0(\Delta)$ be two best approximations to f, and suppose $\mathfrak{R}$ is a* *critical point set belonging to the set of extremal points of $f - s_1$. Then $s_2(x) = s_1(x)$ at all points in the set $\mathfrak{R}$.*

Combining Theorem 2 and the definition of the properly posed knot set, we get the following uniqueness theorem of the best approximation.

Theorem 3 *Let $h^* \in \mathcal{H}$ be a best approximation to f. If the union of the critical point sets belonging to the set of extremal points of $f - h^*$ contains a properly posed knot set of $\mathcal{H}$, then h^* is the unique best approximation to f.*

Corollary 2 *Let $h^* \in \mathcal{H}$ be a best approximation to f. If the set of extremal points of $f - h^*$ contains a critical point set of $k+1$ points, then h^* is the unique best approximation to f.*

4. Geometrical characterization

If the points in a critical point set of $S_1^0(\Delta)$ are distributed over all cells of Δ, then the critical point set is said to be a global critical point set. Otherwise, it is said to be a local critical point set. A local critical point set of $S_1^0(\Delta)$ is a global critical point set of $S_1^0(\Delta_1)$, where Δ_1 is a subset of Δ. So we mainly focus on the global critical point sets.

Theorem 4 *If there exists an edge cell containing only one point of the point set $\{x_0, x_1, \ldots, x_k\}$, then the point set is not a critical point set of $S_1^0(\Delta)$.*

Proof Without loss of generality, we suppose the point x_0 lies within the edge cell. If the point set $\{x_0, x_1, \ldots, x_k\}$ is a critical point set of $S_1^0(\Delta)$, then there exists a subspace of $S_1^0(\Delta)$, denoted by M, such that the point set $\{x_1, \ldots, x_k\}$ is a properly posed knot set of M.

Let A be the vertex of the edge cell which is not a vertex of any other cell, and $B_A(x)$ be the nontrivial bivariate linear spline function determined by $B_A(A) = 1$ and $B_A(x) = 0$ at any other vertices. Clearly, $B_A(x) = 0$ at the points $x_1, x_2, \ldots, x_k$. Therefore, $B_A(x)$ is not in the space M. It is easy to check that the point set $\{x_0, x_1, \ldots, x_k\}$ is a properly posed knot set of the space $N = \mathrm{span}(M, B_A)$. This is a contradiction. □

Lemma 1 *Suppose Δ_1 is a connected subset of Δ.*

1) If $s \in S_1^0(\Delta)$ and $s_1 = s|_{\Delta_1}$, then $s_1 \in S_1^0(\Delta_1)$.

2) If $s_1 \in S_1^0(\Delta_1)$, then there exists a spline function $s \in S_1^0(\Delta)$ such that $s|_{\Delta_1} = s_1$.

Proof It is well known that the bivariate linear polynomial on a triangle is uniquely determined by its values at the three vertices. This lemma is obvious. □

Lemma 2 ([2]) *A point set $\{x_1, x_2, \ldots, x_k\}$ is a critical point set of $S_1^0(\Delta)$ if and only if there is no spline function $s \in S_1^0(\Delta)$ such that the sign of the point x_i is the same as the sign of $s(x_i)$ for all $i = 1, 2, \ldots, k$.*

From Lemmas 1 and 2, we can obtain the following theorem.

Theorem 5 *Suppose that Δ_1 is a connected subset of Δ, and points $x_1, x_2, \ldots, x_k \in \Delta_1$. Then* *the point set $\{x_1, x_2, \ldots, x_k\}$ is a critical point set of $S_1^0(\Delta)$ if and only if it is a critical point set of $S_1^0(\Delta_1)$.*

Theorem 6 *For an even star triangulation Δ, denote all cells of Δ by $\Delta_1, \Delta_2, \ldots, \Delta_{2n}$ in a counter-clockwise order, where $2n$ is the number of the cells of Δ. Suppose that points $x_1, x_2, \ldots, x_{2n}$ are exactly the intersections of two distinct piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$, where $s_1, s_2 \in S_1^0(\Delta)$, and $x_i \in \Delta_i$, $i = 1, 2, \ldots, 2n$. For any spline function $s \in S_1^0(\Delta)$ such that $s(x_i) = 0$ for $i = 1, 2, \ldots, 2n-1$, then $s(x_{2n}) = 0$.*

Proof Let $\hat{s}_{ij}(x)$ be the restriction on Δ_j of the spline $s_i(x)$, and $\hat{s}_j(x)$ be the restriction on Δ_j of the spline $s(x)$, where $i = 1, 2$, and $j = 1, 2, \ldots, 2n$. Suppose l is the common mesh segment of the cells Δ_1 and Δ_2.

Because the lines $\hat{s}_{11} = 0$, $\hat{s}_{21} = 0$ and $\hat{s}_1 = 0$ pass through the same point x_1, there exist the constants λ, μ such that $\hat{s}_1(x) = \lambda\hat{s}_{11}(x) + \mu\hat{s}_{21}(x)$. Similarly, we have $\hat{s}_2(x) = \hat{\lambda}\hat{s}_{12}(x) + \hat{\mu}\hat{s}_{22}(x)$. By the conformality condition, the equality

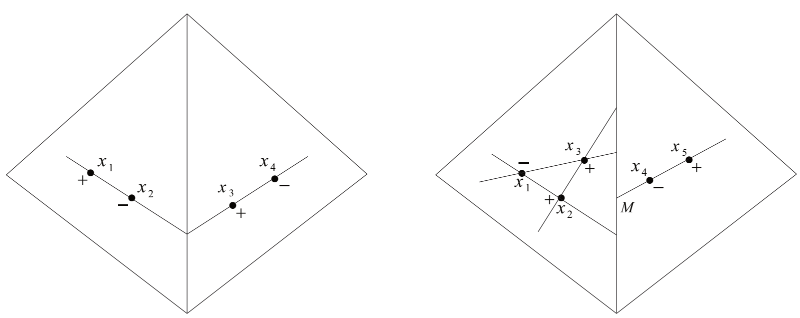
$$(\lambda - \hat{\lambda})\hat{s}_{11}(x) + (\mu - \hat{\mu})\hat{s}_{21}(x) = 0 \tag{2}$$

holds at all points in the line l. Since the piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$ are distinct, we get $\lambda = \hat{\lambda}$, $\mu = \hat{\mu}$ from Eq. (2).

With the same arguments, we have $\hat{s}_j(x) = \lambda\hat{s}_{1j}(x) + \mu\hat{s}_{2j}(x)$ $(j = 1, 2, \ldots, 2n-1)$. Using the global conformality conditions of $s(x), s_1(x), s_2(x)$ gives $\hat{s}_{2n}(x) = \lambda\hat{s}_{1(2n)}(x) + \mu\hat{s}_{2(2n)}(x)$, which means $s(x_{2n}) = 0$. The proof is completed. □

4.1 Bitriangular triangulations

A triangulation is said to be a bitriangular triangulation if it consists of two triangles having a common edge. Now in this subsection, let Δ be a bitriangular triangulation as shown in Figure 1. Since $\dim S_1^0(\Delta) = 4$, the critical point set of $S_1^0(\Delta)$ has three cases.



(a) (b)

Figure 1 The global critical point sets of $S_1^0(\Delta)$ over the bitriangular triangulation

Case 1 The points x_1, x_2, x_3, x_4 are distributed over the triangulation Δ as shown in Figure 1(a).

The point set $\{x_1, x_2, x_3, x_4\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following

conditions are satisfied simultaneously.

(a) The lines x_1x_2, x_3x_4 and the common mesh segment are parallel or concurrent.

(b) The signs of x_1, x_2, x_3, x_4 are determined by (1).

Case 2 The points x_1, x_2, x_3, x_4, x_5 are distributed over the triangulation Δ as shown in Figure 1(b).

Let M be the intersection of the lines x_4x_5 and the common mesh segment.

The point set $\{x_1, x_2, x_3, x_4, x_5\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following conditions are satisfied simultaneously.

(a) The arbitrary three points among M, x_1, x_2, x_3 are not colinear.

(b) The signs of x_1, x_2, x_3, x_4, x_5 are determined by (1).

Case 3 The local critical point sets have been discussed in [7].

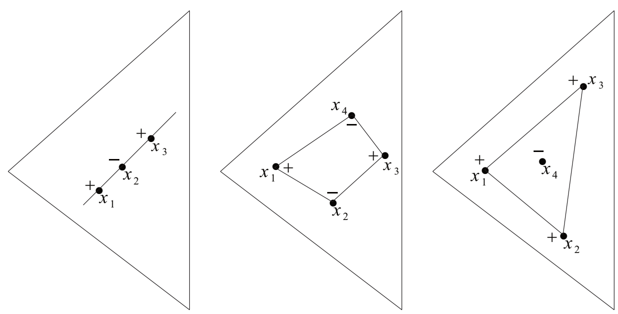


Figure 2 The local critical point sets of $S_1^0(\Delta)$ over the bitriangular triangulation

In this case, the point set is a critical point set of $S_1^0(\Delta)$ if and only if its elements are distributed over the cell as shown in Figure 2.

Theorem 7 *$s^* \in S_1^0(\Delta)$ is a best approximation to f if and only if the set of extremal points of $f - s^*$ contains a critical point set as shown in Figures 1 or 2.*

4.2 Single road triangulations

A triangulation is said to be a single road triangulation if it satisfies the following two conditions simultaneously.

1) It contains two edge cells.

2) Each cell except the two edge cells has two adjacent cells.

Now in this subsection, let Δ be a single road triangulation as shown in Figure 3. Since $\dim S_1^0(\Delta) = k+2$, the global critical point set of $S_1^0(\Delta)$ has three cases.

Case 1 The points $x_1, x_2, \ldots, x_{k+2}$ are distributed over the triangulation Δ as shown in Figure 3.

The point set $\{x_1, x_2, \ldots, x_{k+2}\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following conditions are satisfied simultaneously.

(a) There exists a piecewise algebraic curve $s = 0, s \in S_1^0(\Delta)$ passing through the points

$x_1, x_2, \ldots, x_{k+2}$.

(b) The piecewise algebraic curve $s = 0$ in (a) does not pass through any vertex whose degree is ≥ 3.

(c) The signs of $x_1, x_2, \ldots, x_{k+2}$ are determined by (1).

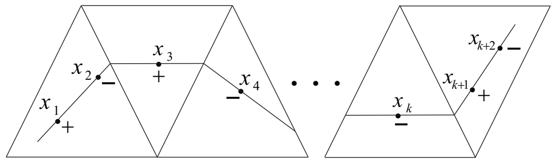


Figure 3 A global critical point set of $S_1^0(\Delta)$ over the single road triangulation

Case 2 The points $x_1, x_2, \ldots, x_{k+3}$ are distributed over the triangulation Δ as shown in Figure 4(a).

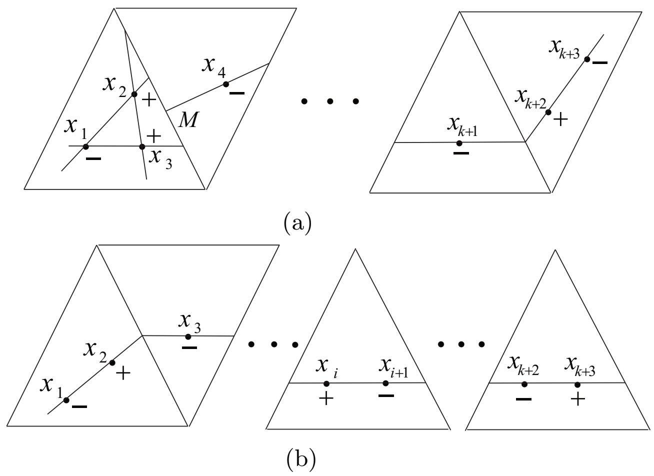


Figure 4 The global critical point sets of $S_1^0(\Delta)$ over the single road triangulation

Let $s = 0, s \in S_1^0(\Delta)$ be a piecewise algebraic curve passing through the points $x_4, x_5, \ldots, x_{k+3}$. Suppose that M is the intersection of $s = 0$ and the mesh segment l, where l is the mesh segment as shown in Figure 4(a).

The point set $\{x_1, x_2, \ldots, x_{k+3}\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following conditions are satisfied simultaneously

(a) The arbitrary three points among M, x_1, x_2, x_3 are not colinear.

(b) The piecewise algebraic curve $s = 0$ in (a) does not pass through any vertex whose degree is ≥ 3.

(c) The signs of $x_1, x_2, \ldots, x_{k+3}$ are determined by (1).

Case 3 The points $x_1, x_2, \ldots, x_{k+3}$ are distributed over the triangulation Δ as shown in Figure 4(b).

Let $s_1 = 0, s_1 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_2, x_4, \ldots, x_i, x_{i+2}, \ldots, x_{k+3}$, $s_2 = 0, s_2 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_2, x_4, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+3}$, and $s_3 = 0, s_3 \in S_1^0(\Delta)$ be the piecewise algebraic curves passing through the points $x_1, x_2, x_4, \ldots, x_{k+2}$.

The point set $\{x_1, x_2, \ldots, x_{k+3}\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following conditions are satisfied simultaneously.

(a) The piecewise algebraic curves $s_1 = 0$, $s_2 = 0$ and $s_3 = 0$ are distinct.

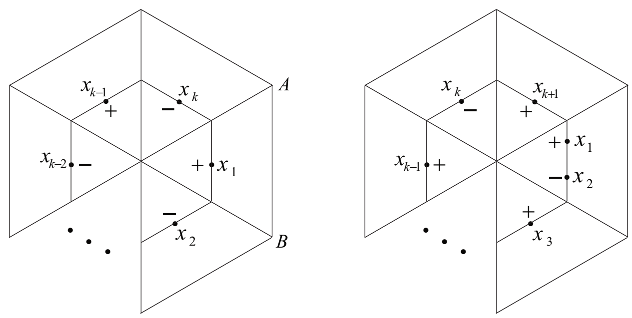
(b) The point x_3 does not lie within the piecewise algebraic curves $s_1 = 0$, $s_2 = 0$ and $s_3 = 0$.

(c) The signs of $x_1, x_2, \ldots, x_{k+3}$ are determined by (1).

4.3 Star triangulations

A triangulation is said to be a star triangulation if it is the union of all triangles with a certain interior mesh point as a common vertex. Now in this subsection, let Δ be a star triangulation with k cells as shown in Fig. 5. Since $\dim S_1^0(\Delta) = k+1$, the global critical point set of $S_1^0(\Delta)$ has four cases.

Case 1 The points $x_1, x_2, \ldots, x_k$ are distributed over the triangulation Δ as shown in Figure 5(a).



(a) (b)

Figure 5 The global critical point sets of $S_1^0(\Delta)$ over the star triangulation

Suppose k is odd. Let $s_1 = 0, s_1 \in S_1^0(\Delta)$ be the piecewise curve passing through the points $A, x_1, x_2, \ldots, x_{k-1}$, and $s_2 = 0, s_2 \in S_1^0(\Delta)$ be the piecewise curve passing through the points $\{B, x_1, x_2, \ldots, x_{k-1}\}$. It is well known that $s_1 = 0$, $s_2 = 0$ cannot pass through the point x_k simultaneously, which means that the point set $\{x_1, x_2, \ldots, x_k\}$ must be a properly posed knot set of some subspace of $S_1^0(\Delta)$. Therefore, the point set $\{x_1, x_2, \ldots, x_k\}$ is not a critical point set of $S_1^0(\Delta)$.

Suppose k is even. The point set $\{x_1, x_2, \ldots, x_k\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following conditions are satisfied simultaneously.

(a) The points $x_1, x_2, \ldots, x_k$ are the intersections of two distinct piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$, where $s_1, s_2 \in S_1^0(\Delta)$.

(b) The signs of $x_1, x_2, \ldots, x_k$ are determined by (1).

Case 2 The points $x_1, x_2, \ldots, x_{k+1}$ are distributed over the triangulation Δ as shown in Figure 5(b).

The point set $\{x_1, x_2, \ldots, x_{k+1}\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following

conditions are satisfied simultaneously.

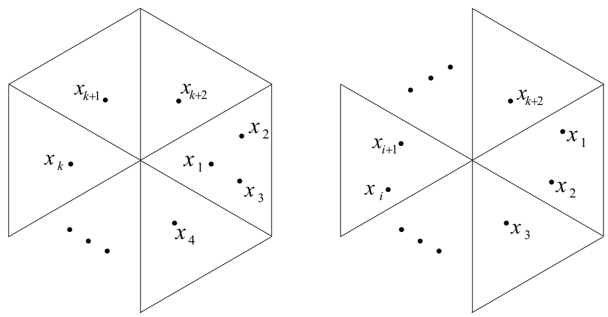
(a) There exists a piecewise algebraic curve $s = 0$, $s \in S_1^0(\Delta)$ passing through the points $x_1, x_2, \ldots, x_{k+1}$.

(b) The points $x_1, x_3, \ldots, x_{k+1}$ are not the intersections of any two distinct piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$, where $s_1, s_2 \in S_1^0(\Delta)$.

(c) The points $x_2, x_3, \ldots, x_{k+1}$ are not the intersections of any two distinct piecewise algebraic curves $s_3 = 0$ and $s_4 = 0$, where $s_3, s_4 \in S_1^0(\Delta)$.

(d) The signs of $x_1, x_2, \ldots, x_{k+1}$ are determined by (1).

Case 3 The points $x_1, x_2, \ldots, x_{k+2}$ are distributed over the triangulation Δ as shown in Figure 6(a).



(a) (b)

Figure 6 The global critical point sets of $S_1^0(\Delta)$ over the star triangulation

Let $s_1 = 0, s_1 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_2, x_4, \ldots, x_{k+1}$, $s_2 = 0, s_2 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_3, x_4, \ldots, x_{k+1}$, and $s_3 = 0, s_3 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_2, x_3, \ldots, x_{k+1}$.

The point set $\{x_1, x_2, \ldots, x_{k+2}\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following conditions are satisfied simultaneously.

(a) The piecewise algebraic curves $s_1 = 0, s_2 = 0$ and $s_3 = 0$ are distinct.

(b) The point x_{k+2} does not lie within the piecewise algebraic curves $s_1 = 0, s_2 = 0$ and $s_3 = 0$.

(c) The signs of $x_1, x_2, \ldots, x_{k+2}$ are determined by (1).

Case 4 The points $x_1, x_2, \ldots, x_{k+2}$ are distributed over the triangulation Δ as shown in Figure 6(b).

Let $s_1 = 0, s_1 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_{k+1}$, $s_2 = 0, s_2 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}$, $s_3 = 0, s_3 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_1, x_3, \ldots, x_{k+1}$, and $s_4 = 0, s_4 \in S_1^0(\Delta)$ be the piecewise algebraic curve passing through the points $x_2, x_3, \ldots, x_{k+1}$.

The point set $\{x_1, x_2, \ldots, x_{k+2}\}$ is a critical point set of $S_1^0(\Delta)$ if and only if the following

conditions are satisfied simultaneously.

(a) The piecewise algebraic curves $s_1 = 0, s_2 = 0, s_3 = 0$ and $s_4 = 0$ are distinct.

(b) The point x_{k+2} does not lie within the piecewise algebraic curves $s_1 = 0, s_2 = 0, s_3 = 0$ and $s_4 = 0$.

(c) The signs of $x_1, x_2, \ldots, x_{k+2}$ are determined by (1).

The approaches for proving the results in the above three subsections are similar. For simplicity, we only give the proof of Case 2 in the Subsection 3.3.

Proof Condition (a) is equivalent to the condition that the point set $\{x_1, x_2, \ldots, x_{k+1}\}$ is not a properly posed knot set of $S_1^0(\Delta)$.

It is obvious that the new point set obtained by deleting any point $x_i,\ 3 \le i \le k+1$ from the point set $\{x_1, x_2, \ldots, x_{k+1}\}$ is a properly posed knot set of some subspace of $S_1^0(\Delta)$.

Condition (b) is equivalent to the condition that the point set $\{x_1, x_3, \ldots, x_{k+1}\}$ is a properly posed knot set of some subspace of $S_1^0(\Delta)$.

Condition (c) is equivalent to the condition that the point set $\{x_2, x_3, \ldots, x_{k+1}\}$ is a properly posed knot set of some subspace of $S_1^0(\Delta)$.

Conditions (a), (b) and (c) yield the point set $\{x_1, x_2, \ldots, x_{k+1}\}$ which is an uninterpolated point set of $S_1^0(\Delta)$. This fact and condition (d) complete the proof. □

5. Conclusion

The critical point set plays a central role in the theory of Tchebyshev approximation. The characterization theorem states that $h \in \mathcal{H}$ is a best approximation to f if and only if the set of extremal points of $f - h$ contains a critical point set of $\mathcal{H}$. In general, it is not a trivial task to determine whether a set is critical or not in multivariate Tchebyshev approximation. It appears impossible to give a simple geometric interpretation of the critical point set in multivariate Tchebyshev approximation. We restricted ourselves to Tchebyshev approximation by $S_1^0(\Delta)$ over some special triangulations (bitriangular, single road and star triangulations) in this paper. The geometrical characterization of the critical point set of $S_1^0(\Delta)$ is proposed. Such geometrical characterization can be used in the determination of a critical point set.

Acknowledgement The authors would like to thank the referees for their comments, which improve the English expression in this paper.

References

[1] G. Nürnberger, M. Sommer. *Alternation for best spline approximations.* Numer. Math., 1983, **41**(2): 207–221.
[2] J. R. Rice. *Tchebycheff approximation in several variables.* Trans. Amer. Math. Soc., 1963, **109**(3): 444–466.
[3] J. R. Rice. *Characterization of Tchebyshev approximation by splines.* SIAM J. Math., 1967 **4**(4): 557–567.
[4] Renhong Wang. *The structural characterization and interpolation for multivariate splines.* Acta Math. Sinica, 1975, **18**: 91–106.
[5] Renhong Wang. *Numerical Approximation.* Higher Education Press, Beijing, 1999.
[6] Renhong Wang. *Multivariate Spline Functions and Their Applications.* Kluwer Pub, New York, 2001.
[7] Renhong Wang, Xuezhang Liang. *Multivariate Function Approximations.* Scientific Press, Beijing, 1988.