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Characterization of 2-Primal Near-Rings

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Abstract In 1999, Kim and Kwak asked one question that "Is a ring R 2-primal if $O_P \subseteq P$ for each $P \in m\mathrm{Spec}(R)$?". In this paper, we prove that if O_P has the IFP for each $P \in m\mathrm{Spec}(N)$, then $O_P \subseteq P$ for each $P \in m\mathrm{Spec}(N)$ if and only if N is a 2-primal near-ring and also we give characterization of 2-primal near- rings by using its minimal 0-prime ideals.

Keywords 2-primal; completely prime; completely semiprime.

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1. Introduction

Throughout this paper, N stands for a zero-symmetric right near-ring. For basic terminology in near-ring, we refer to Pilz [5]. If N is any near-ring, we use $\mathcal{P}_0(N)$, $\mathcal{N}^*(N)$ and $\mathcal{N}(N)$ to denote the 0-prime radical, nil radical and the set of all nilpotent elements of N, respectively. Recall that a near-ring N is called 2-primal if $\mathcal{P}_0(N) = \mathcal{N}(N)$. Kim et al. [4] characterized 2-primal rings in terms of their minimal prime ideals. In this paper, we give some characterization of 2-primal near-rings in terms of their minimal 0-prime ideals.

2. Preliminaries

Definition 2.1 *An ideal P of a near-ring N is 0-prime if for any two ideals A and B of N, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Definition 2.2 *An ideal P of N is said to be completely prime (resp. completely semiprime) if $ab \in P$ implies $a \in P$ or $b \in P$ (resp. $a^2 \in P$ implies $a \in P$) for any $a, b \in N$.*

Definition 2.3 *An ideal P of a near-ring N is a minimal 0-prime ideal if P is minimal among 0-prime ideals of N.*

Definition 2.4 *A subset M of N is called an m-system if for any a, b in M there exists $a_1 \in \langle a\rangle$ and $b_1 \in \langle b\rangle$ such that $a_1b_1 \in M$.*

Definition 2.5 *An ideal I of N is said to have the insertion of factors property (or) simply IFP if $xy \in I$ implies $xNy \subseteq I$ for $x, y \in N$.*

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Observe that every completely semiprime ideal of N has the IFP.

Definition 2.6 *Let N be a near-ring and $(m)\mathrm{Spec}(N)$ the set of all (minimal) 0-prime ideals of N. For $P \in \mathrm{Spec}(N)$, we put*

$$\begin{aligned}
&O\,(P) = \{a \in N | aN\langle b\rangle = 0 \text{ for some } b \in N \setminus P\}\,,\\
&\overline{O}\,(P) = \{a \in N | a^m \in O\,(P) \text{ for some positive integer } m\}\,,\\
&O_P = \{a \in N | ab = 0 \text{ for some } b \in N \setminus P\}\,,\\
&\overline{O}_P = \{a \in N | a^m \in O_P \text{ for some positive integer } m\}\,,\\
&N\,(P) = \{a \in N | aN\langle b\rangle \subseteq \mathcal{P}_0(N) \text{ for some } b \in N \setminus P\}\,,\\
&\overline{N}\,(P) = \{a \in N | a^m \in N\,(P) \text{ for some positive integer } m\},\\
&N_P = \{a \in N | ab \in \mathcal{P}_0(N) \text{ for some } b \in N \setminus P\}\,,\\
&\overline{N}_P = \{a \in N | a^m \in N_P \text{ for some } b \in N \setminus P\}\,.
\end{aligned}$$

3. Characterization of 2-primal near-rings

In this section, we give some characterization of 2-primal near-ring by using its 0-prime ideals.

Proposition 3.1 *For each $P \in \mathrm{Spec}(N)$, $O(P)$ and $N(P)$ are ideals of N.*

Proof Let P be a 0-prime ideal of N and let $a_1, a_2 \in O(P)$. Then $a_1N\langle b_1\rangle = 0$ for some $b_1 \in N \setminus P$ and $a_2N\langle b_2\rangle = 0$ for some $b_2 \in N \setminus P$. Since $b_1, b_2 \in N \setminus P$ and $N \setminus P$ is an m-system, there exist $b_1' \in \langle b_1\rangle$ and $b_2' \in \langle b_2\rangle$ such that $b_1'b_2' \in N \setminus P$. Let $b_3 = b_1'b_2'$. For any $n \in N$ and $x \in \langle b_3\rangle$, $(a_1 - a_2)nx = 0$ implies $a_1 - a_2 \in O(P)$. Let $x \in O(P)$. Then $xN\langle b\rangle = 0$. Thus for $n, n', n_1 \in N$ and $b' \in \langle b\rangle$, we have $(n(n' + x) - nn')n_1b' = 0$ implies $n(n' + x) - nn' \in O(P)$ and $(xn)n_1b' = 0$ implies $xn \in O(P)$. Thus $O(P)$ is an ideal of N. Similarly, $N(P)$ is an ideal of N.

The following results might be helpful for the criterion for a certain class of rings to be 2-primal.

Theorem 3.2 *For a near-ring N, the following statements are equivalent:*

(i) N *is 2-primal;*

(ii) $\mathcal{P}_0(N)$ *is a completely semiprime ideal of* N*;*

(iii) $N(P)$ *is a completely semiprime ideal of N for each* $P \in m\mathrm{Spec}(N)$*;*

(iv) $\overline{N}_P = \overline{N}\,(P) = N\,(P)$ *for each* $P \in m\mathrm{Spec}(N)$*;*

(v) $N\,(P) = N_P$ *for each* $P \in m\mathrm{Spec}(N)$*;*

(vi) $N_P \subseteq P$ *for some* $P \in m\mathrm{Spec}(N)$*;*

(vii) $N_{P/\mathcal{P}_0(N)} \subseteq P/\mathcal{P}_0(N)$ *for each* $P \in m\mathrm{Spec}(N)$.

Proof (i)$\Rightarrow$(ii). Since $\mathcal{P}_0(N) = \mathcal{N}(N)$, for any x in $N, x^2 \in \mathcal{P}_0(N)$ implies x^2 is nilpotent and hence $x \in \mathcal{N}(N) = \mathcal{P}_0(N)$. Therefore, $\mathcal{P}_0(N)$ is a completely semiprime ideal of N.

(ii)$\Rightarrow$(iii). Let P be a minimal 0-prime ideal of N. Let $x \in N$ be such that $x^2 \in N(P)$. Then

$x^2N\langle b\rangle \subseteq \mathcal{P}_0(N)$ for some $b \in N \setminus P$. Since $\mathcal{P}_0(N)$ is a completely semiprime ideal of N, it has the IFP. So $xNxN\langle b\rangle \subseteq \mathcal{P}_0(N)$ which implies $xN\langle b\rangle \subseteq \mathcal{P}_0(N)$. Thus $x \in N(P)$ and hence $N(P)$ is completely semiprime.

(iii)⇒(i). Let $a \in \mathcal{N}(N)$. Then $a^n = 0$ for some positive integer n. If $a \notin \mathcal{P}_0(N)$, then there exists a minimal 0-prime ideal P of N such that $a \notin P$. Since $N(P)$ is a completely semiprime ideal, $a^n = 0 \in N(P)$ implies $a \in N(P) \subseteq P$, a contradiction. Hence $a \in \mathcal{P}_0(N)$.

(ii)⇒(iv). Let P be a minimal 0-prime ideal of N and let $a \in \overline{N}_P$. Then $a^n \in N_P$ for some positive integer n. Thus $a^nb \in \mathcal{P}_0(N)$ for some $b \in N \setminus P$. Since $\mathcal{P}_0(N)$ is completely semiprime ideal of N, it has the IFP. By [3, Lemma 2.1], $ab \in \mathcal{P}_0(N)$. Therefore, $aN\langle b\rangle \subseteq \mathcal{P}_0(N)$ for some $b \in N \setminus P$ and so $a \in N(P)$. Thus $\overline{N}_P \subseteq N(P)$. But $N(P) \subseteq N_P \subseteq \overline{N}_P$ and $\overline{N}(P) \subseteq \overline{N}_P$. Therefore, $\overline{N}_P = \overline{N}(P) = N(P)$ for each $P \in m\mathrm{Spec}(N)$.

(iv)⇒(v)⇒(vi). These are obvious.

(vi)⇒(vii). Let P be a minimal 0-prime ideal of N. Let $\overline{N} = N/\mathcal{P}_0(N)$ and $\overline{P} = P/\mathcal{P}_0(N)$. Let $\overline{a} = a+\mathcal{P}_0(N) \in N_{\overline{P}}$ for some $a \in N$. Then there exists $\overline{b} \in \overline{N} \setminus \overline{P}$ such that $\overline{a}\overline{b} \in \mathcal{P}_0(\overline{N}) = \overline{0}$. Thus $ab \in \mathcal{P}_0(N)$ and so $a \in N_P \subseteq P$. Therefore, $\overline{a} \in \overline{P}$ and hence $N_{\overline{P}} \subseteq \overline{P}$.

(vii)⇒(i). Suppose that $\overline{N} = N/\mathcal{P}_0(N)$ is not reduced. Then there exists $\overline{a} \in \overline{N}$ such that $\overline{a}^2 = \overline{0}$ and $\overline{a} \neq \overline{0}$. Thus $a \notin \mathcal{P}_0(N)$ and hence $a \notin P$ for some $P \in m\mathrm{Spec}(N)$. Then $\overline{a} \notin \overline{P}$ and so $\overline{a} \in \overline{N} \setminus \overline{P}$. But since $\overline{a}^2 = \overline{0}$, we obtain $\overline{a} \in N_{\overline{P}} \subseteq \overline{P}$, which is a contradiction. Therefore $\mathcal{P}_0(N) = \mathcal{N}(N)$ and hence N is 2-primal. □

Corollary 3.3 *For a near-ring N, assume that N is 2-primal. If $P = N(P)$ for each $P \in \mathrm{Spec}(N)$, then P is completely prime ideal of N.*

Proof Suppose that N is a 2-primal near-ring. Let $xy \in P = N(P)$. Then there exists $b \in N \setminus P$ such that $(xy)N\langle b\rangle \subseteq \mathcal{P}_0(N)$. Since $\mathcal{P}_0(N)$ has the IFP, we have $(xNy)N\langle b\rangle \subseteq \mathcal{P}_0(N) \subseteq P$ and so $xNy \subseteq P$ since $b \notin P$. Hence $x \in P$ or $y \in P$ since P is a 0-prime ideal of N. Therefore, P is a completely prime ideal of N. □

Proposition 3.4 *For a near-ring N, we have the following:*

(i) $\mathcal{N}(N) \subseteq \bigcap_{P\in\mathrm{Spec}(N)} \overline{O}(P) \subseteq \bigcap_{Q\in m\mathrm{Spec}(N)} \overline{O}(Q)$;

(ii) $\mathcal{P}_0(N) \subseteq \bigcap_{P\in\mathrm{Spec}(N)} N(P) = \bigcap_{Q\in m\mathrm{Spec}(N)} N(Q)$.

Proof (i) Let $a \in \mathcal{N}(N)$. Then $a^n = 0$ for some positive integer n. Let P be any 0-prime ideal and let $b \in N \setminus P$. Since $a^n = 0$, $a^nN\langle b\rangle = 0$. Thus $a^n \in O(P)$ and hence $a \in \overline{O}(P)$. Therefore, $a \in \bigcap_{P\in\mathrm{Spec}(N)} \overline{O}(P)$. The other inclusion is obvious.

(ii) Let $a \in \mathcal{P}_0(N)$. Let P be any 0-prime ideal of N. Then $aN\langle b\rangle \subseteq \mathcal{P}_0(N)$ for any $b \in N \setminus P$ which implies that $a \in N(P)$ and so $a \in \bigcap_{P\in\mathrm{Spec}(N)} N(P)$. Therefore,

$$\mathcal{P}_0(N) \subseteq \bigcap_{P\in\mathrm{Spec}(N)} N(P).$$

But $\bigcap_{P\in\mathrm{Spec}(N)} N(P) \subseteq \bigcap_{Q\in m\mathrm{Spec}(N)} N(Q)$ always. Since $N(Q) \subseteq Q$ for each $Q \in m\mathrm{Spec}(N)$, $\bigcap_{Q\in m\mathrm{Spec}(N)} N(Q) \subseteq \mathcal{P}_0(N)$. □

Our next result indicates that our characterization of minimal 0-prime ideals P in terms of $N(P)$ holds.

Theorem 3.5 *For a near-ring N, assume that N is 2-primal. Then for each $P \in \mathrm{Spec}(N)$, the following statements are equivalent:*

(i) $P \in m\mathrm{Spec}(N)$;

(ii) $N(P) = P$.

Proof (i)$\Rightarrow$(ii). Let P be a minimal 0-prime ideal of N and let $a \in P$. Suppose $a \notin N(P)$. Let $S = \{a, a^2, a^3, \ldots\}$. If $0 \in S$, then $a^k = 0$ for some positive integer k and hence $a \in \mathcal{N}(N) = \mathcal{P}_0(N)$, which implies that $a \in N(P)$ by Proposition 3.4, a contradiction. So $0 \notin S$. Thus S is a multiplicative system that does not contain 0. Let $L = N \setminus P$, i.e., L is an m-system. Let T be the set of all non zero elements of N of the form $a^{t_0}x_1a^{t_1}x_2\cdots a^{t_{n-1}}x_na^{t_n}$, where $x_i \in L$ and the t_i's are positive integers with t_0 and t_n allowed to be zero. Clearly, $L \subseteq T$. Let $M = T \cup S$. We show that M is an m-system. Let $x, y \in M$. If $x, y \in S$, then $xy \in S \subseteq M$ and we are done. Let $x \in S$ and $y \in T$, say $x = a^s$ and $y = a^{t_0}y_1a^{t_1}y_2a^{t_2}\cdots y_na^{t_n}$. If $xy \neq 0$, then $xy \in T$. Suppose $xy = 0$. Since $y_1, y_2 \in L$, there exist $y_1' \in \langle y_1\rangle$ and $y_2' \in \langle y_2\rangle$ such that $y_1'y_2' \in L$. Since $y_1'y_2', y_3 \in L$, there exist $y_{12}' \in \langle y_1'y_2'\rangle \subseteq \langle\langle y_1\rangle\langle y_2\rangle\rangle$ and $y_3' \in \langle y_3\rangle$ such that $y_{12}'y_3' \in L$. Continuing this process, we get $y_{123\ldots n-2}'y_{n-1}'$, $y_n \in L$. Then there exist $y_{123\ldots n-1}' \in \langle y_{123\ldots n-2}'y_{n-1}'\rangle \subseteq \langle\cdots\langle\langle\langle y_1\rangle\langle y_2\rangle\rangle\langle y_3\rangle\rangle\cdots\langle y_{n-1}\rangle\rangle$ and $y_n' \in \langle y_n\rangle$ such that $w = y_{123\ldots n-1}'y_n' \in L$. Since $xy = 0$, $xy \in \mathcal{P}_0(N)$. Thus $a^sa^{t_0}y_1a^{t_1}y_2\cdots y_na^{t_n} \in \mathcal{P}_0(N)$. Since $\mathcal{P}_0(N) = \mathcal{N}(N)$, $\mathcal{P}_0(N)$ is completely semiprime ideal of N and hence $y_1y_2\cdots y_na^{s+t_0+t_1+\cdots+t_n} \in \mathcal{P}_0(N)$. Choose $m = s+t_0+t_1+\cdots+t_n$. Then $y_1y_2\cdots y_na^m \in \mathcal{P}_0(N)$. Since $\mathcal{P}_0(N)$ has the IFP, $\langle y_1\rangle\langle y_2\rangle\cdots\langle y_n\rangle\langle a^m\rangle \subseteq \mathcal{P}_0(N)$. Continuing this process, we obtain $\langle\cdots\langle\langle\langle y_1\rangle\langle y_2\rangle\rangle\langle y_3\rangle\rangle\cdots\langle y_{n-1}\rangle\rangle\langle y_n\rangle\langle a^m\rangle \subseteq \mathcal{P}_0\,(N)$ and so $y_{123\ldots n-1}'y_n'a^m \in \mathcal{P}_0(N)$. Hence $wa^m \in \mathcal{P}_0(N)$, where $w = y_{123\ldots n-1}'y_n'$. Since $\mathcal{P}_0(N)$ is a completely semiprime ideal, $(aw)^m \in \mathcal{P}_0(N)$ and hence $aw \in \mathcal{P}_0(N)$. Thus $a \in N_P = N(P)$, which is a contradiction. Therefore, if $x \in S$, $y \in T$, then $xy \neq 0$ and so $xy \in T$.

Similarly, one can show that if $x, y \in T$ then $xy \neq 0$ and $xy \in T$. This shows that M is an m-system that is disjoint from (0). Hence, by [5, Proposition 2.81] there is a 0-prime ideal Q that is disjoint from M such that $a \notin Q$ and $Q \subseteq P$. Since P is a minimal 0-prime ideal, $P = Q$. Therefore, $a \notin P$, which is a contradiction. Consequently $a \in N(P)$.

(ii)$\Rightarrow$(i). If $Q \subseteq P$ for $Q \in m\mathrm{Spec}(N)$, then $N(P) \subseteq N(Q) \subseteq Q \subseteq P = N(P)$. Therefore, $P \in m\mathrm{Spec}(N)$. □

Theorem 3.6 *For a near-ring N, the following statements are equivalent:*

(i) N *is 2-primal;*

(ii) $\overline{O}_P \subseteq P$ *for each* $P \in m\mathrm{Spec}(N)$;

(iii) $\mathcal{N}\,(N) = \bigcap_{P\in m\mathrm{Spec}(N)} \overline{O}_P = \mathcal{P}_0(N)$.

Proof (i)$\Rightarrow$(ii). Note that $\overline{O}_P \subseteq \overline{N}_P$ for each $P \in m\mathrm{Spec}(N)$. By Theorem 3.2, we have $\overline{N}_P = N\,(P) \subseteq P$ and therefore, $\overline{O}_P \subseteq P$ for each $P \in m\mathrm{Spec}(N)$.

(ii)⇒(iii). Since $\overline{O}_P \subseteq P$ for each $P \in m\mathrm{Spec}(N)$, $\bigcap_{P\in m\mathrm{Spec}(N)} \overline{O}_P \subseteq \mathcal{P}_0(N)$. Let $a \in \mathcal{N}(N)$. Then $a^m = 0 \in O(P)$ for some integer m and any $P \in m\mathrm{Spec}(N)$. Hence $a \in \bigcap_{P\in m\mathrm{Spec}(N)} \overline{O}_P$. Thus $\mathcal{N}(N) \subseteq \bigcap_{P\in m\mathrm{Spec}(N)} \overline{O}_P \subseteq \mathcal{P}_0(N) \subseteq \mathcal{N}(N)$.

(iii)⇒(i). It is obvious. □

Proposition 3.7 *Assume that $O(P)$ is a 0-prime ideal of near-ring N for each $P \in m\mathrm{Spec}(N)$. Then $O(P)$ has the IFP for each $P \in m\mathrm{Spec}(N)$ if and only if N is a 2-primal near-ring.*

Proof Assume that N is a 2-primal near-ring. Let P be a minimal 0-prime ideal of N such that $O(P)$ is a 0-prime ideal of N. Let $xy \in O(P)$ for $x, y \in N$. This implies that $xyN\langle z\rangle = 0$ for $z \in N \setminus P$. Then $xyN\langle z\rangle \subseteq P$. Since $z \notin P$ and P is 0-prime, $xy \in P$. Therefore, $O(P) \subseteq P$. Since P is a minimal 0-prime ideal, $O(P) = P$. Since N is 2-primal and $P \in m\mathrm{Spec}(N)$, $N(P) = P$ by Theorem 3.5. Therefore, P is completely prime by Corollary 3.3. Since $P = O(P)$, $O(P)$ is completely prime. In particular, $O(P)$ has the IFP.

Conversely, suppose that $O(P)$ has the IFP for each $P \in m\mathrm{Spec}(N)$. Let $x \in \mathcal{N}(N)$. This implies that $x^n = 0$ for some positive integer n. So that $x^n \in O(P)$. If $x \notin \mathcal{P}_0(N)$, then there exists a minimal 0-prime ideal P of N such that $x \notin P$. Since P is a 0-prime ideal, there exist $r_1, r_2, \ldots, r_{n-1} \in N$ such that $xr_1x\cdots xr_{n-1}x \notin P$. But since $O(P)$ has the IFP, $xr_1x\cdots xr_{n-1}x \in O(P)$. Since $O(P) \subseteq P$, $xr_1x\cdots xr_{n-1}x \in P$, a contradiction. Thus $x \in \mathcal{P}_0(N)$. Therefore, $\mathcal{N}(N) \subseteq \mathcal{P}_0(N)$. Always $\mathcal{P}_0(N) \subseteq \mathcal{N}(N)$. Hence $\mathcal{N}(N) = \mathcal{P}_0(N)$. □

Proposition 3.8 *If $O(P)$ has the IFP for each $P \in m\mathrm{Spec}(N)$, then for every $P \in m\mathrm{Spec}(N)$, $O(P)$ is a 0-prime ideal if and only if $O(P)$ is a completely prime ideal of N.*

Proof Suppose that $O(P)$ is a 0-prime ideal for every $P \in m\mathrm{Spec}(N)$. Let $xy \in O(P)$ for $x, y \in N$. If $x \in O(P)$, we have done. Suppose $x \notin O(P)$. Since $xy \in O(P)$ and $O(P)$ has the IFP, $xNy \subseteq O(P)$. This implies that $xNyN\langle z\rangle = 0$ for $z \in N \setminus P$. This implies that $xNyN\langle z\rangle \subseteq P$. Since P is 0-prime, $xNy \subseteq P$ and therefore $x \in P$ or $y \in P$. By Proposition 3.7, $P = O(P)$. Since $x \notin O(P)$, $x \notin P$. Therefore $y \in P = O(P)$. Hence $O(P)$ is completely prime. The Converse is obvious. □

Proposition 3.9 *Let N be a near-ring with unity. Let $O(P)$ be a 0-prime ideal of N for each $P \in m\mathrm{Spec}(N)$. Then the following are equivalent:*

(i) *N is a 2-primal near-ring;*

(ii) *$O(P)$ has the IFP for each $P \in m\mathrm{Spec}(N)$;*

(iii) *$O(P)$ is a completely semiprime ideal for each $P \in m\mathrm{Spec}(N)$;*

(iv) *$O(P)$ is a symmetric ideal for each $P \in m\mathrm{Spec}(N)$;*

(v) *$xy \in O(P)$ implies $yNx \subseteq O(P)$ for $x, y \in N$ and for each $P \in m\mathrm{Spec}(N)$.*

Proof (i)⇒(ii). It follows from Proposition 3.7.

(ii)⇒(iii). By Proposition 3.8, $O(P)$ is a completely prime ideal and hence $O(P)$ is completely semiprime.

(iii)⇒(iv). Suppose that $O(P)$ is a completely semiprime ideal for each $P \in m\mathrm{Spec}(N)$. Therefore, it has the IFP. Let a, b, $c \in N$ be such that $abc \in O(P)$. We shall prove that $acb \in O(P)$. Since $abc \in O(P)$, there exists $s \in N \setminus P$ such that $abcN\langle s\rangle = 0$. So that $abcN\langle s\rangle \subseteq O(P)$. Since $O(P)$ has the IFP, $acbcN\langle s\rangle \subseteq O(P)$. Suppose that $cN\langle s\rangle \nsubseteq O(P)$. If $acb \notin O(P)$, since $O(P)$ is 0-prime, there exists some $n \in N$ such that $acbncN\langle s\rangle \nsubseteq O(P)$, which contradicts the IFP of $O(P)$. Therefore, $acb \in O(P)$.

Suppose that $cN\langle s\rangle \subseteq O(P)$. Since $O(P)$ has the IFP, $cbN\langle s\rangle \subseteq O(P)$. Since $O(P)$ is 0-prime and $s \notin P = O(P)$, $cb \in O(P)$. Therefore, $acb \in O(P)$. Hence $O(P)$ is a symmetric ideal in N.

(iv)⇒(v). Suppose that $xy \in O(P)$ for $P \in m\mathrm{Spec}(N)$. Since $O(P)$ is symmetric and N has unity, $yx \in O(P)$. Since $O(P)$ has the IFP, $yNx \subseteq O(P)$.

(v)⇒(i). Let $x \in \mathcal{N}(N)$. Then $x^r = 0$ for some r. So that $x^r \in O(P)$ for $P \in m\mathrm{Spec}(N)$. Suppose that $x \notin \mathcal{P}_0(N)$. Since $\mathcal{P}_0(N) = \bigcap_{P\in \mathrm{Spec}(N)} P$, $x \notin P$. Since P is a 0-prime ideal, there exist $n_1, n_2, \ldots, n_{r-1} \in N$ such that $xn_1x \cdots xn_{r-1}x \notin P$. Since $xy \in O(P)$, by hypothesis $yNx \subseteq O(P)$. Therefore, $xn_1x \cdots xn_{r-1}x \in O(P) \subseteq P$, a contradiction. Thus $x \in \mathcal{P}_0(N)$. Hence $\mathcal{N}(N) \subseteq \mathcal{P}_0(N)$. Always $\mathcal{P}_0(N) \subseteq \mathcal{N}(N)$ and consequently N is a 2-primal near-ring. □

Theorem 3.10 *Let $O(P)$ be a 0-prime ideal for each $P \in m\mathrm{Spec}(N)$. Then the following are equivalent;*

(i) N is a 2-primal near-ring;

(ii) $O(P)$ has the IFP;

(iii) Every minimal 0-prime ideal of N is a completely prime ideal of N.

Proof (i)⇒(ii). It follows from Proposition 3.7.

(ii)⇒(iii). Let P be a minimal 0-prime ideal of N. Let $a, b \in N$ be such that $ab \in P$. If $b \in P$, we have done. Suppose that $b \notin P$. Since $O(P) = P$, $ab \in O(P)$. Since $O(P)$ has the IFP, $aNb \subseteq O(P) = P$. Since P is 0-prime and $b \notin P$, $a \in P$. Hence, P is completely prime ideal.

(iii)⇒(i). Let $x \in \mathcal{N}(N)$. Then $x^r = 0$ for some r. So that $x^r \in P$, where $P \in m\mathrm{Spec}(N)$. Since every minimal 0-prime ideal is completely prime, $x \in P$ for every $P \in m\mathrm{Spec}(N)$. Since $\mathcal{P}_0(N) = \bigcap_{P\in m\mathrm{Spec}(N)} P$, $x \in \mathcal{P}_0(N)$. Thus $\mathcal{N}(N) \subseteq \mathcal{P}_0(N)$. □

Theorem 3.11 *Let $O(P)$ be a 0-prime ideal of N for every $P \in m\mathrm{Spec}(N)$. Then N is a 2-primal near-ring if and only if $P = \overline{O}(P)$ for every minimal 0-prime ideal P of N.*

Proof Suppose that N is a 2-primal near-ring. Then $O(P)$ is a completely prime ideal of N by Proposition 3.7. Let $a \in \overline{O}(P)$. Then $a^m \in O(P)$. Since $O(P)$ is completely prime, $a \in O(P)$. Therefore, $\overline{O}(P) \subseteq O(P)$. Clearly, $O(P) \subseteq \overline{O}(P)$. Thus $O(P) = \overline{O}(P)$. Since $O(P)$ is a 0-prime ideal of N, $P = O(P)$. Hence $P = \overline{O}(P)$.

Conversely, assume that $P = \overline{O}(P)$ for every minimal 0-prime ideal P of N. Let $x \in \mathcal{N}(N)$. This implies that $x^n = 0$ for some n. So $x^n \in P$ for every $P \in m\mathrm{Spec}(N)$. Since $P = \overline{O}(P) = O(P)$, $x^n \in O(P)$. Since $O(P)$ is completely prime, $x \in O(P) = \overline{O}(P) = P$. This implies that

$x \in \mathcal{P}_0(N)$. Thus $\mathcal{N}(N) \subseteq \mathcal{P}_0(N)$ and consequently N is a 2-primal near-ring. $\square$

In [4], Kim and Kwak asked one question that "Is a ring R 2-primal if $O_P \subseteq P$ for each $P \in m\mathrm{Spec}(R)$?". Here we prove the following theorem for near-rings.

Theorem 3.12 *If O_P has the IFP for each $P \in m\mathrm{Spec}(N)$, then $O_P \subseteq P$ for each $P \in m\mathrm{Spec}(N)$ if and only if N is a 2-primal near-ring.*

Proof Let $x \in \mathcal{N}(N)$. Then $x^n = 0$ for some n. So that $x^n \in O(P) \subseteq O_P$. Suppose $x \notin \mathcal{P}_0(N)$. Since $\mathcal{P}_0(N) = \bigcap_{P \in m\mathrm{Spec}(N)} P$, there exists $P \in m\mathrm{Spec}(N)$ such that $x \notin P$. Since P is a 0-prime ideal, there exist $r_1, r_2, \ldots, r_{n-1} \in N$ such that $xr_1x \cdots xr_{n-1}x \notin P$. But since O_P has the IFP, $xr_1x \cdots xr_{n-1}x \in O_P$. Again since $O_P \subseteq P$, $xr_1x \cdots xr_{n-1}x \in P$, a contradiction. Thus $x \in \mathcal{P}_0(N)$. Hence $\mathcal{N}(N) \subseteq \mathcal{P}_0(N)$.

Conversely, assume that N is a 2-primal near-ring. By Theorem 3.6, $\overline{O}_P \subseteq P$ for each $P \in m\mathrm{Spec}(N)$. Since $O_P \subseteq \overline{O}_P$, $O_P \subseteq P$ for each $P \in m\mathrm{Spec}(N)$.

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