Journal of Mathematical Research with Applications
Jan., 2012, Vol. 32, No. 1, pp. 33–42
DOI:10.3770/j.issn:2095-2651.2012.01.005
Http://jmre.dlut.edu.cn

Split Left GC-Lpp Semigroups

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Abstract A left GC-lpp semigroup S is called split if the natural homomorphism $\gamma^\flat$ of S onto S/γ induced by γ is split. It is proved that a left GC-lpp semigroup is split if and only if it has a left adequate transversal. In particular, a construction theorem for split left GC-lpp semigroups is established.

Keywords left GC-lpp semigroup; left regular band; left ample semigroup.

MR(2010) Subject Classification 20M10

1. Introduction

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are generalizations of the usual Green's relations $\mathcal{L}$ and $\mathcal{R}$, respectively; elements a and b of a semigroup S are related by $\mathcal{L}^*$ (resp., $\mathcal{R}^*$) if and only if they are related by $\mathcal{L}$ (resp., $\mathcal{R}$) in some oversemigroup of S. S is called *left abundant* if each $\mathcal{R}^*$-class contains at least one idempotent. Right abundant semigroups can be dually defined. Following [4], a semigroup is called abundant if it is both left abundant and right abundant. A left abundant semigroup S is called left adequate [3] if $E(S)$ (the set of idempotents of S) forms a semilattice. Right adequate semigroup is dually defined. A semigroup is called adequate if it is both left adequate and right adequate. It is not difficult to see that each $\mathcal{R}^*$-class of a left adequate semigroup contains exactly one idempotent. For a left adequate semigroup S, we shall use $a^\dagger$ to denote the idempotent in the $\mathcal{R}^*$-class of S containing a. Moreover, a left adequate semigroup S is said to be left ample, also known as left type A, if for all $a \in S$ and $e \in E(S)$, $ae = (ae)^\dagger a$. For (left, right) adequate semigroups, one can refer to [3].

As an application of left ample semigroups, Guo-Guo-Shum [10] introduced left GC-lpp semigroups. In precise, a left GC-lpp semigroup is defined as a left abundant semigroup in which

(1) $E(S)$ is a left regular band (that is, a band satisfying the identity $xy = xyx$); and

(2) For all $a \in S$ and $e \in E(S)$, $ae = (ae)^\dagger a$,

where $a^\dagger$ is the idempotent in the $\mathcal{R}^*$-class of S containing a. Indeed, left GC-lpp semigroups

Received May 24, 2010; Accepted January 12, 2011
Supported by the National Natural Science Foundation of China (Grant No. 10961014), the Natural Science Foundation of Jiangxi Province and the Science Foundation of the Education Department of Jiangxi Province (Grant No. GJJ11388).
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are common generalizations of left ample semigroups and $\mathcal{R}$-unipotent semigroups. In [10], the authors established the construction of left GC-lpp semigroups. After then, Guo-Shum [14] considered some special cases of left GC-lpp semigroups. In [5], the second author investigated proper abundant left GC-lpp semigroups, called abundant left C-lpp proper semigroups. Guo-Ni-Shum [11] studied left GC-lpp monoids which are F-rpp and obtained the construction of such left GC-lpp semigroups. Recently, Guo-Shum [12] gave a structure theorem for proper left GC-lpp semigroups.

By an orthodox semigroup, we mean a regular semigroup whose set of idempotents forms a band. $\mathcal{R}$-unipotent semigroups are just orthodox semigroups each of whose $\mathcal{R}$-classes contains exactly one idempotent. As the analogue of orthodox semigroups in the range of abundant semigroups, El-Qallali and Fountain [2] defined quasi-adequate semigroups. The so-called quasi-adequate semigroups are abundant semigroups in which the set of idempotents constitutes a band. For quasi-adequate semigroups, see [6–9] and others.

Recall that an $\mathcal{R}^*$-homomorphism of a semigroup S into another T is a homomorphism ϕ preserving the $\mathcal{R}^*$-classes, that is, for all $a, b \in S$, if $a\mathcal{R}^*b$, then $a\phi\, \mathcal{R}^*\, b\phi$. It is worth to mention that not all homomorphisms on a semigroup are $\mathcal{R}^*$-homomorphisms but any homomorphism on a regular semigroup is $\mathcal{R}^*$-homomorphic. A congruence ρ on S is called $\mathcal{R}^*$-homomorphic if the natural homomorphism $\rho^\natural$ of S onto S/ρ induced by ρ is $\mathcal{R}^*$-homomorphic. An $\mathcal{R}^*$-homomorphism of S onto T is said to be split if there exists an $\mathcal{R}^*$-homomorphism ψ of T into S such that $\psi\phi = \mathrm{id}_T$, where id_T is the identity mapping on T. An orthodox semigroup is called split if the homomorphism induced by the smallest inverse semigroup congruence is split. In [16], McAlister and Blyth researched split orthodox semigroups. Analogously, we can define split quasi-adequate semigroups. El-Qallali [1] and Guo-Peng [13] investigated split quasi-adequate semigroups.

Left GC-lpp semigroups can be thought as some kind of orthodox semigroups. Also, any left GC-lpp semigroup has the smallest left ample semigroup congruence. Now, natural questions arise:

(1) Can we define split left GC-lpp semigroups?

(2) What can we say about this kind of semigroups?

The aim of this paper is to answer the above questions.

Throughout this paper, we use notations and terminology in Fountain [4] and the book of Howie [15]. For bands, one can refer to the book of Petrich [17]. Here we recall some known results used in the sequel. To begin with, we provide some results on $\mathcal{L}^*$ and the dual for the relation $\mathcal{R}^*$.

Lemma 1.1 ([4]) *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*b$.

(2) For all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

Lemma 1.2 ([4]) *Let S be a semigroup and $e^2 = e$, $a \in S$. Then the following statements are* *equivalent:*

(1) $a\mathcal{L}^*e$*;*

(2) $ae = a$ *and for all* $x, y \in S^1$, $ax = ay$ *implies that* $ex = ey$.

It is well known that $\mathcal{R}^*$ is a left congruence while $\mathcal{L}^*$ is a right congruence. In general, $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{L} \subseteq \mathcal{L}^*$. But when a, b are regular elements of S, $a\mathcal{R}^*\ [\mathcal{L}^*]\ b$ if and only if $a\mathcal{R}\ [\mathcal{L}]\ b$. For the sake of convenience, we denote by $E(S)$ the set of idempotents of S and by $\mathrm{Reg}(S)$ the set of regular elements of S. We use a^* [resp., $a^\dagger$] to denote the typical idempotents related to a by $\mathcal{L}^*$ [resp., $\mathcal{R}^*$]. And, K_a stands for the $\mathcal{K}$-class of S containing a if $\mathcal{K}$ is an equivalence on S.

A band B is called a left regular band [17] if for all $x, y \in B, xy = xyx$. The band B is called left normal if it satisfies the identity: $xyz = xzy$. It is not difficult to show that a left normal band is a left regular band. For a left abundant semigroup T whose set of idempotents constitutes a left regular band, each $\mathcal{R}^*$-class of S contains exactly one idempotent. In fact, if $e, f \in E(S)$ such that $e\mathcal{R}^*a\mathcal{R}^*f$, then $e\mathcal{R}f$, and $f = ef = efe = ee = e$, as required. This fact will be repeatedly used.

The following lemma is due to [11].

Lemma 1.3 *Let* S *be a left GC-lpp semigroup.*

(1) For all $a, b \in S$, $(ab^\dagger)^\dagger = (ab)^\dagger = a^\dagger(ab)^\dagger$.

(2) The relation $\gamma = \{(a, b) \in S \times S : a = eb,\ e \in E(b^\dagger)\}$ *is the smallest left ample semigroup congruence by which the natural homomorphism induced is* $\mathcal{R}^*$*-homomorphic, where* $E(b^\dagger) = \{f \in E(S) : f\mathcal{D}^{E(S)}b^\dagger\}$.

Let U be a left abundant subsemigroup of a left abundant semigroup of S. Then we call U a right *-subsemigroup of S if for all $a \in U$, there exists $e \in E(U)$ such that $a\mathcal{R}^*(S)e$. Now let S° be a left adequate right *-subsemigroup of S and E° be the idempotent semilattice of S°. Then S° is called a left adequate transversal for S if for any element $x \in S$, there exist a unique element $x^\circ \in S^\circ$ and an idempotent $e \in E(S)$ such that $x = ex^\circ$ where $e\mathcal{L}^*(x^\circ)^\dagger$ for $(x^\circ)^\dagger \in E^\circ$. In this case, e can be uniquely determined by x, and $e\mathcal{R}^*x$. We shall denote by e_x the unique idempotent e.

Lemma 1.4 *Let* S *be a left abundant semigroup. If* $a, b \in S$ *and* $b = ea$ *with* $e = ea^\dagger e$*, then* $e\mathcal{R}^*b$.

Proof Suppose that $b = ea$ with $e = ea^\dagger e$. Then $eb = b$ and for all $x, y \in S^1$, $xb = yb$ which implies that $xea^\dagger = yea^\dagger$, so $xea^\dagger e = yea^\dagger e$ and $xe = ye$. Thus $e\mathcal{R}^*b$.

Let B be a band and assume that $B = \cup_{\alpha \in Y} B_\alpha$ is the semilattice decomposition of B into rectangular bands B_α with $\alpha \in Y$. A subset $E = \{x_\alpha : \alpha \in Y\}$ of B is called a skeleton if $x_\alpha \in B_\alpha$ for any $\alpha \in Y$ and $x_\alpha x_\beta = x_{\alpha\beta} = x_\beta x_\alpha$ for all $\alpha, \beta \in Y$. It is easy to see that any skeleton of B is isomorphic to the structure semilattice Y of B.

Lemma 1.5 ([16]) *A band is split if and only if it has a skeleton.*

2. Definition and characterizations

Definition 2.1 *A left GC-lpp semigroup S is called split if the natural homomorphism $\gamma^\natural$ of S onto S/γ induced by γ is split.*

Let S be a left GC-lpp semigroup. By Guo-Guo-Shum [10], the smallest left ample semigroup congruence γ is idempotent-pure and $\mathcal{R}^*$-homomorphic, and so the restriction of γ to $E(S)$ is just $\mathcal{D}^{E(S)}$. We now assume that the natural homomorphism $\gamma^\flat$ induced by γ is split and that the $\mathcal{R}^*$-homomorphism φ is the one such that $\varphi\gamma^\flat = \mathrm{id}_{S/\gamma}$. Then the natural homomorphism from $E(S)$ onto $E(S)/\mathcal{D}^{E(S)}$ induced by $\mathcal{D}^{E(S)}$ is split. In fact, for all $e \in E(S)$, we have $e\gamma \subseteq E(S)$ and $e\gamma = e\gamma|_{E(S)}$, so $e\gamma \in E(S/\gamma)$. Suppose that $e\gamma\varphi = a$, then $a\gamma e$ and so $(e\gamma)\varphi|_{E(S)/\gamma}\gamma|^\flat_{E(S)} = a\gamma|^\flat_{E(S)} = a\gamma|_{E(S)} = e\gamma|_{E(S)} = e\gamma$. This shows that $\varphi|_{E(S)/\gamma}\gamma|^\flat_{E(S)} = \mathrm{id}_{E(S)/\gamma}$. Thus $\gamma|^\flat_{E(S)}$ is split. By Lemma 1.5, we deduce the following corollary.

Corollary 2.2 *If S is a split left GC-lpp semigroup, then $E(S)$ has a skeleton.*

Let S be a left GC-lpp semigroup with the left regular band B of idempotents. If E is a skeleton of B, then we define the span of E by

$$\mathrm{span}(E) = \{a \in S|\ (\exists e \in E)\ e\mathcal{R}^*a\} = \{a \in S|\ a^\dagger \in E\}.$$

Lemma 2.3 *Let S be a left GC-lpp semigroup with a left regular band B of idempotents. If E is a skeleton of B, then* $\mathrm{span}(E)$ *meets every γ-class of S exactly once.*

Proof Pick $a \in S$. Then there exists $e_1 \in B$ such that $e_1\mathcal{R}^*a$. On the other hand, since E is a skeleton of B, there exists uniquely $e \in E$ such that $e \in E(e_1)$. Since B is a left regular band, we know that $E(e_1)$ is a left zero band. Thereby $e\mathcal{L}e_1$. Suppose that $\tilde{a} = ea$. Then $\tilde{a}\gamma a$ and by Lemma 1.4, $\tilde{a}\mathcal{R}^*e$. Consequently, $\tilde{a} \in \mathrm{span}(E)$ and hence $\mathrm{span}(E)$ meets every γ-class at least once.

In order to show that $\mathrm{span}(E)$ meets every γ-class precisely once, it suffices to verify that if $a \in S$ and $b \in \mathrm{span}(E)$ with $a\gamma b$, then $\tilde{a} = b$. Suppose that there exists uniquely $u \in E$ such that $u\mathcal{R}^*b$. Then by Lemma 1.4, $e \in E(u)$ and whence $e = u$ because $e, u \in E$ and E is a skeleton of B. Thus $\tilde{a}\mathcal{R}^*e = u\mathcal{R}^*b$ so that $(\tilde{a}, b) \in \gamma \cap \mathcal{R}^*$. Since $\gamma \cap \mathcal{R}^* = \mathrm{id}_S$, $\tilde{a} = b$. Consequently, $\mathrm{span}(E)$ meets every γ-class exactly once.

Lemma 2.4 *Let S be a split left GC-lpp semigroup with a left regular band B of idempotents and φ be an $\mathcal{R}^*$-homomorphism of S/γ into S such that $\varphi\gamma^\flat = \mathrm{id}_{S/\gamma}$. Denote $E = B\gamma^\flat\varphi$ (Certainly, E is a skeleton of B). Then* $\mathrm{span}(E)$ *is a left adequate right *-subsemigroup of S.*

Proof Let $S^\circ = (S/\gamma)\varphi$. Then for each element $s \in S^\circ$, there exists $a \in S$ such that $(a\gamma)\varphi = s$, so $a\gamma = s\gamma$. Since S is a left GC-lpp semigroup and φ is an $\mathcal{R}^*$-homomorphism, there exists $e \in E(S)$ such that $s\mathcal{R}^*e$, hence $a\gamma = s\gamma\mathcal{R}^*e\gamma$ and $(a\gamma)\varphi\mathcal{R}^*(e\gamma)\varphi$, i.e. $s\mathcal{R}^*(e\gamma)\varphi$ where $(e\gamma)\varphi \in E(S^\circ)$, so S° is a right *-subsemigroup and furthermore it is a left adequate right *-subsemigroup S° of S having E as a semilattice of idempotents.

Since $\varphi\gamma^\flat = \mathrm{id}_{S/\gamma}$, we have $\gamma^\flat\varphi\gamma^\flat = \gamma^\flat$ and so $[(a\gamma)\varphi]\gamma^\flat = a\gamma$ for all $a \in S$. This shows that

S° meets each γ-class of S. On the other hand, assume that $a, b \in S^\circ$ and $a\gamma = b\gamma$. Take $x\varphi = a$ and $y\varphi = b$ with $x, y \in S/\gamma$. Then

$$x = x\varphi\gamma^\flat = a\gamma = b\gamma = y\varphi\gamma^\flat = y \text{ and } a = x\varphi = y\varphi = b.$$

This shows that S° meets a γ-class of S at most one. Consequently, S° meets each γ-class of S exactly once.

It remains to show that $S^\circ = \mathrm{span}(E)$. Given $a \in S^\circ$. Then, by using the arguments of the above proof, S° is a right *-subsemigroup of S, and so $e\mathcal{R}^*a$ for some idempotents $e \in E(S^\circ)$. Since E is a skeleton of B, $a \in \mathrm{span}(E)$ and whence, $S^\circ \subseteq \mathrm{span}(E)$. For any $u \in \mathrm{span}(E)$, by the above proof again, there exists $v \in S^\circ$ such that $u\gamma v$. Because $S^\circ \subseteq \mathrm{span}(E)$, we have $v \subseteq \mathrm{span}(E)$. But since $\mathrm{span}(E)$ meets every γ-class exactly once, we have $u = v$ and consequently $u \in S^\circ$. This leads to $\mathrm{span}(E) \subseteq S^\circ$. We have now proved that $\mathrm{span}(E) = S^\circ$, as required.

Theorem 2.5 *If S is a left GC-lpp semigroup, then S is split if and only if S has a left adequate transversal.*

Proof Suppose that S is split and that φ is an $\mathcal{R}^*$-homomorphism of S/γ into S such that $\varphi\gamma^\flat = \mathrm{id}_{S/\gamma}$. Then $E^\circ = E(S)\gamma^\flat\varphi$ is a skeleton of $E(S)$. By Lemma 2.4, $\mathrm{span}(E^\circ)$ is a left adequate right *-subsemigroup of S. Since $\mathrm{span}(E^\circ)$ meets every γ-class exactly once, for any $a \in S$, there exists a unique $a^\circ \in \mathrm{span}(E^\circ)$ such that $a\gamma a^\circ$. By the definition of γ, $a = ea^\circ$ for some $e \in E((a^\circ)^\dagger)$ with $(a^\circ)^\dagger \in E^\circ$. Obviously, $e(a^\circ)^\dagger\mathcal{L}(a^\circ)^\dagger$. This shows that S has a left adequate transversal $\mathrm{span}(E^\circ)$.

Conversely, assume that S has a left adequate transversal S°. Denote by E° the set of idempotents of S°. Then for any $a \in S$, there are a unique element $a^\circ \in S^\circ$ and an idempotent $e \in E(S)$ such that $a = ea^\circ$ with $e\mathcal{L}^*(a^\circ)^\dagger$ for $(a^\circ)^\dagger \in E^\circ$. Clearly, $a\gamma a^\circ$. For any $b \in S$, we suppose that $b^\circ \in S^\circ$ has the similar property as a°. It follows that if $a^\circ\gamma b^\circ$ and $a^\circ = mb^\circ$ with $m \in E((b^\circ)^\dagger)$. Furthermore, $a^\circ = mb^\circ = m(b^\circ)^\dagger b^\circ$. We can also easily show that $m(b^\circ)^\dagger\mathcal{L}(b^\circ)^\dagger$. Observe that S° is a left adequate transversal of S. Hence $a^\circ = b^\circ$. On the other hand, S° meets every γ-class of S exactly once. This shows that

$$\varphi: \ S/\gamma \to S; \ a\gamma \mapsto a^\circ$$

is well defined.

Let $a, e^2 = e \in S$ and $e\mathcal{R}^*a$. Since S° is a left adequate transversal of S, $a = e_a a^\circ$ with e_a and a° having the same meanings as in Section 1, and whence $e_a\mathcal{L}(a^\circ)^\dagger$, $e_a\mathcal{R}^*a$. Since S is a left GC-lpp semigroup, $e_a = e$ and $(e_a)^\circ = e^\circ$. Consider $e_a\mathcal{L}(a^\circ)^\dagger$, then $(e_a)^\circ = (a^\circ)^\dagger$, hence $(e_a)^\circ\mathcal{R}^*a^\circ$. So $e^\circ\mathcal{R}^*a^\circ$. Thereby, these imply that for any $a, b \in S$, if $a\mathcal{R}^*b$, then $a^\circ\mathcal{R}^*b^\circ$.

We next prove that for all $a\gamma, b\gamma \in S/\gamma$, if $a\gamma\mathcal{R}^*b\gamma$, then $a^\circ\mathcal{R}^*b^\circ$. Let $a\gamma\mathcal{R}^*b\gamma$. Then there exist $e, f \in E(S)$ such that $e\mathcal{R}^*a$ and $f\mathcal{R}^*b$. By the above proof, $e^\circ\mathcal{R}^*a^\circ$. By Lemma 1.3 (2), $e\gamma\mathcal{R}^*a\gamma$ and $f\gamma\mathcal{R}^*b\gamma$. It follows that $e\gamma\mathcal{R}^*f\gamma$. But since S/γ is left ample semigroup, $e\gamma = f\gamma$. Hence, $e^\circ = f^\circ$ because S° meets every γ-class exactly once. Thus, we have proved that $a^\circ\mathcal{R}^*b^\circ$.

Since S° is a left adequate transversal of S, φ is a bijection. By the above proof, φ is an $\mathcal{R}^*$-homomorphism if φ is a homomorphism. Also, we can easily see that $\varphi\gamma^\flat = \mathrm{id}_{S/\gamma}$. Now, to show that $\gamma^\flat$ is split, we need only to prove that φ is a homomorphism. For this, we need only to show that $(ab)^\circ = a^\circ b^\circ$ for all $a, b \in S$.

In fact, since $a\gamma a^\circ$ and $b\gamma b^\circ$, $ab\gamma a^\circ b^\circ$. Also, we have $ab\gamma(ab)^\circ$. Thus $S^\circ \cap \gamma_{ab} = \{a^\circ b^\circ, (ab)^\circ\}$, where γ_{ab} is the γ-class of S containing ab. But since S° meets every γ-class exactly once, we have $a^\circ b^\circ = (ab)^\circ$. Thus the proof is completed. $\Box$

It is a natural question whether the converse of Corollary 2.2 is true. We cannot answer this question. For left GC-lpp semigroups with left normal bands of idempotents, we have

Theorem 2.6 *If S is a left GC-lpp semigroup with a left normal band B of idempotents, then S is split if and only if B is split.*

Proof We only prove the sufficiency because the necessity is trivial. For this purpose, we assume that B is split and E is a skeleton of B. Then, by the proof of the necessary part of Theorem 2.5, span(E) is a left adequate transversal of S if span(E) is a left adequate right *-subsemigroup of S. By the definition of the span of E, it can be easily seen that span(E) is a left adequate right *-subsemigroup of S if span(E) is a subsemigroup of S.

We now proceed to show that span(E) is a subsemigroup of S. We only need to prove that $ab \in \mathrm{span}(E)$ for all $a, b \in \mathrm{span}(E)$. Since $a, b \in \mathrm{span}(E)$, we have $a^\dagger, b^\dagger \in E$. Observe that $a^\dagger(ab)^\dagger = (ab)^\dagger$ since S is a left GC-lpp semigroup. Since E is a skeleton of S, we deduce that $k \in E$ such that $k\mathcal{D}^B(ab)^\dagger$. This shows that $(ab)^\dagger k(ab)^\dagger = (ab)^\dagger$ and $k(ab)^\dagger k = k$. Hence

$$(ab)^\dagger = a^\dagger(ab)^\dagger a^\dagger = a^\dagger(ab)^\dagger k(ab)^\dagger a^\dagger = a^\dagger k(ab)^\dagger a^\dagger = a^\dagger(k(ab)^\dagger k a^\dagger) = a^\dagger k a^\dagger = a^\dagger k \in E,$$

since B is a left normal band. Therefore $ab \in \mathrm{span}(E)$. Thus, span(E) is indeed a subsemigroup of S. $\Box$

3. A construction theorem

Consider

Y a semilattice;

T a left type A semigroup with semilattice Y of idempotents; and

L a left regular band having Y as a skeleton.

Moreover, assume that $L = \cup_{\alpha\in Y} L_\alpha$ is the semilattice decomposition of L into left zero bands L_α with $\alpha \in Y$. Denote by End(L) the semigroup of endomorphisms (on the left) on L. Now define

$$\phi:\ T \to \mathrm{End}(L);\ t \mapsto \phi t = \phi_t.$$

Then, we call the above quadruple $(Y, T, L; \phi)$ a GC-system if the following conditions are satisfied:

$(GC1)$ ϕ is a semigroup homomorphism.

$(GC2)$ For all $a \in T$ and $x \in L_\alpha$, we have $\phi_a x \in L_{(a\alpha)^\dagger}$.

Given a $GC-system(Y,T,L;\phi)$, and put

$$GC = GC(Y,T,L;\phi) = \{(e,a) \in L \times T | \ a \in T.e \in L_{a^\dagger}\}.$$

Define a multiplication on GC by the rule that

$$(e,a) \circ (g,b) = (e(\phi_a g), ab).$$

Since $\mathcal{R}^*$ is a left congruence on T, $ab\mathcal{R}^*ab^\dagger$ and so $(ab)^\dagger = (ab^\dagger)^\dagger$ as T is a left ample semigroup. By $(GC2)$, it follows that $\phi_a g \in L_{(ab)^\dagger}$. On the other hand, since $a^\dagger(ab)^\dagger = (ab)^\dagger$, $e(\phi_a g) \in L_{a^\dagger}L_{(ab)^\dagger} \subseteq L_{a^\dagger(ab)^\dagger} = L_{(ab)^\dagger}$. This shows that $\circ$ is well defined.

Lemma 4.1 *If $(Y,T,L;\phi)$ is a GC-system, then $GC(Y,T,L;\phi)$ is a semigroup with respect to the above operation $\circ$.*

Proof If $(e,a),(f,b),(g,c) \in GC$, then

$$\begin{aligned}(e,a)\circ[(f,b)\circ(g,c)] &= (e,a)\circ(f\phi_b(g),bc) = (e\phi_a(f\phi_b(g)),abc)\\ &= (e\phi_a(f)\phi_{ab}(g),abc) = (e\phi_a(f),ab)\circ(g,c)\\ &= [(e,a)\circ(f,b)]\circ(g,c)\end{aligned}$$

and $(GC,\circ)$ satisfies the associative law. Thus $(GC,\circ)$ is a semigroup. □

Lemma 4.2 *Let $(Y,T,L;\phi)$ be a GC-system. Then the following statements hold for $GC = GC(Y,T,L;\phi)$:*

(1) $(e,a) \in E(G)$ if and only if $a \in E(T)$. Moreover, $E(G)$ is a left regular band.

(2) $(e,a)\mathcal{R}^*(f,b)$ if and only if $e = f$ and $a\mathcal{R}^*b$.

(3) $(GC,\circ)$ is a left GC-lpp semigroup.

Proof (1) If $(e,a) \in E(GC)$, then $(e(\phi_a e),a^2) = (e,a)$ and so $a^2 = a$. Conversely, if $a \in Y$, then by (GC2), $e,\phi_a e \in L_a$, and $e(\phi_a e) = e$ since L_a is a left zero band. This shows that $(e,a)^2 = (e(\phi_a e),a^2) = (e,a)$.

If $(e,a),(f,b) \in E(GC)$, then $a,b \in E(T)$ and

$$(e,a)(f,b)(e,a) = (e\phi_a(f),ab)(e,a) = (e(\phi_a f)(\phi_{ab}e),aba).$$

Since $f \in L_b$, we have $\phi_a f \in L_{(ab)^\dagger} = L_{ab}$, and so $\phi_{ab}e \in L_{(aba)^\dagger} = L_{ab}$, thereby $(\phi_a f)(\phi_{ab}e) = \phi_a f$. Thus $(e,a)(f,b)(e,a) = (e\phi_a f,ab) = (e,a)(f,b)$, and whence $E(GC)$ is a left regular band.

(2) We first prove that $(e,a)\mathcal{R}^*(e,a^\dagger)$. We can easily see that $(e,a^\dagger)(e,a) = (e(\phi_{a^\dagger}e),a^\dagger a) = (e,a)$. If $(g,c),(h,d) \in (GC)^1$ such that $(g,c)(e,a) = (h,d)(e,a)$, then $(g\phi_c(e),ca) = (h\phi_d(e),da)$. By comparing components, $g\phi_c(e) = h\phi_d(e)$ and $ca = da$. The second equality derives that $ca^\dagger = da^\dagger$. Thus $(g\phi_c(e),ca^\dagger) = (h\phi_d(e),da^\dagger)$, that is, $(g,c)(e,a^\dagger) = (h,d)(e,a^\dagger)$. Therefore $(e,a)\mathcal{R}^*(e,a^\dagger)$. Moreover,

$$\begin{aligned}(e,a)\mathcal{R}^*(f,b) &\Leftrightarrow (e,a^\dagger)\mathcal{R}(f,b^\dagger)\\ &\Leftrightarrow (e,a^\dagger)(f,b^\dagger) = (f,b^\dagger) \text{ and } (f,b^\dagger)(e,a^\dagger) = (e,a^\dagger)\\ &\Leftrightarrow e(\phi_{a^\dagger}f) = f,\ f(\phi_{b^\dagger}e) = e,\ a^\dagger b^\dagger = b^\dagger \text{ and } b^\dagger a^\dagger = a^\dagger\end{aligned}$$

$$\Leftrightarrow a^\dagger = b^\dagger \text{ and } e = f$$
$$\Leftrightarrow a\mathcal{R}^* b \text{ and } e = f.$$

(3) By (1) and (2), we only need to prove that $(e,a)(f,b^\dagger) = ((e,a)(f,b^\dagger))^\dagger(e,a)$ for all $(f,b^\dagger) \in E(G)$. In fact,

$$\begin{aligned}((e,a)(f,b^\dagger))^\dagger(e,a) &= (e(\phi_a f), ab^\dagger)^\dagger(e,a) = (e(\phi_a f), (ab^\dagger)^\dagger)(e,a)\\ &= (e(\phi_a f)\phi_{(ab)^\dagger}(e), (ab^\dagger)^\dagger a).\end{aligned}$$

Since $\phi_a(f), \phi_{(ab)^\dagger}(e) \in L_{(ab)^\dagger}$ and T is left ample, $(\phi_a f)(\phi_{(ab)^\dagger} e) = \phi_a f$ and $(ab^\dagger)^\dagger a = ab^\dagger$. So

$$((e,a)(f,b^\dagger))^\dagger(e,a) = (e\phi_a(f), ab^\dagger) = (e,a)(f,b^\dagger).$$

Thus GC is a left GC-lpp semigroup. □

Theorem 4.3 *If $(Y,T,L;\phi)$ is a GC-system, then $GC^\circ = \{(a^\dagger, a)|\ a \in T\}$ is a left adequate transversal of $GC(Y,T,L;\phi)$. Moreover, GC is split.*

Proof It is easy to check that the mapping

$$\psi:\ GC^\circ \to T;\ (a^\dagger, a) \mapsto a$$

is a semigroup isomorphism. Hence GC° is a left adequate semigroup. On the other hand, by Lemma 4.2, $(a^\dagger, a^\dagger)\mathcal{R}^*(a^\dagger, a)$ and so GC° is a right *-subsemigroup of GC. Thus GC° is a left adequate *-subsemigroup of GC.

Now let $(e,a) \in GC$. It is not difficult to find that $(e,a) = (e,a^\dagger)(a^\dagger,a)$. Since $(e,a^\dagger)(a^\dagger,a^\dagger) = (e\phi_{a^\dagger}a^\dagger, a^\dagger) = (e,a^\dagger)$ and $(a^\dagger,a^\dagger)(e,a^\dagger) = (a^\dagger,a^\dagger)$, we have $(e,a^\dagger)\mathcal{L}(a^\dagger,a^\dagger)$. On the other hand, if $(b^\dagger, b) \in GC^\circ$ such that $(e,a) = (x,\alpha)(b^\dagger,b)$, where $(x,\alpha) \in E(G)$ with $(x,\alpha)\mathcal{L}(b^\dagger,b^\dagger)$, then $(x,\alpha)(b^\dagger,b^\dagger) = (x,\alpha)$ and $(b^\dagger,b^\dagger)(x,\alpha) = (x,\alpha)$, so $\alpha b^\dagger = \alpha$ and $b^\dagger\alpha = b^\dagger$, thereby $\alpha = b^\dagger$ since T is a left ample semigroup. Now, from the fact that $(e,a) = (x,\alpha)(b^\dagger,b)$, we can show that $a = b$. Thus $(a^\dagger,a) = (b^\dagger,b)$ and whence GC° is a left adequate transversal of GC.

The rest follows from Theorem 2.5.

We conclude this paper with proving that any split left GC-lpp semigroup is isomorphic to some $GC(Y,T,L;\phi)$. In what follows, we always assume that S is a split left GC-lpp semigroup with left regular band E of idempotents. By Theorem 2.5, we let S° be a left adequate transversal for S.

For $t \in S^\circ$, define

$$\varphi_t:\ E \to E;\ x \mapsto \varphi_t x = (tx)^\dagger.$$

If $x,y \in E$, then since S is a left GC-lpp semigroup,

$$\varphi_t(xy) = (txy)^\dagger\mathcal{R}^* txy = (tx)^\dagger ty\mathcal{R}^*(tx)^\dagger(ty)^\dagger = (\varphi_t x)(\varphi_t y)$$

and whence $\varphi_t(xy) = (\varphi_t x)(\varphi_t y)$ since each $\mathcal{R}^*$-class of a left GC-lpp semigroup contains exactly one idempotent. Thus φ_t is a homomorphism.

Now let $s,t \in S^\circ$. Then for all $x \in E$, since S is a left GC-lpp semigroup,

$$\varphi_{st}x = (stx)^\dagger\mathcal{R}^* stx\mathcal{R}^* s(tx)^\dagger\mathcal{R}^*(s(tx)^\dagger)^\dagger = \varphi_s\varphi_t(x).$$

Since each $\mathcal{R}^*$-class of a left GC-lpp semigroup contains precisely one idempotent, it follows that $\varphi_{st}x = \varphi_s(\varphi_t x)$, and whence $\varphi_{st} = \varphi_s\varphi_t$. Thus the mapping

$$\varphi : \ S^\circ \to \mathrm{End}(E);\ t \mapsto \varphi_t$$

is a homomorphism of S° into $\mathrm{End}(E)$. On the other hand, note that $E(S^\circ)$ is a skeleton of E, we observe that E has the semilattice decomposition into left zero bands E_α with $\alpha \in E(S^\circ)$, such that $\alpha \in E_\alpha$. If $x \in E_\alpha$ and $t \in S^\circ$, then $\varphi_t x = (tx)^\dagger \mathcal{R}^* tx = tx\alpha = (tx)^\dagger t\alpha \mathcal{R}^* (\varphi_t x)(t\alpha)^\dagger$, thereby $\varphi_t x = (\varphi_t x)(t\alpha)^\dagger$ since each $\mathcal{R}^*$-class of a left GC-lpp semigroup contains precisely one idempotent. Similarly, $(t\alpha)^\dagger = (t\alpha)^\dagger(\varphi_t x)$. Thus $\varphi_t x \mathcal{L} (t\alpha)^\dagger$, that is, $\varphi_t x \in E_{(t\alpha)^\dagger}$. This means that φ_t satisfies Condition $(GC2)$. So, we have

Lemma 4.4 *$(E(S^\circ), S^\circ, E; \varphi)$ is a GC-system.*

Theorem 4.5 *S is isomorphic to $GC(E(S^\circ), S^\circ, E; \varphi)$.*

Proof We need only to prove that the mapping

$$\theta : \ S \to GC(E(S^\circ), S^\circ, E; \varphi);\ a \mapsto (e_a, a^\circ),$$

where a° has the same meaning as before, is a semigroup isomorphism. Since S° is a left adequate transversal of S, θ is a bijection.

Now, it remains to verify that θ is a homomorphism. In fact, for $a, b \in S$,

$$\theta(a)\theta(b) = (e_a, a^\circ)(e_b, b^\circ) = (e_a(\varphi_{a^\circ} e_b), a^\circ b^\circ) = (e_a(a^\circ e_b)^\dagger, a^\circ b^\circ).$$

It is easy to see that $e_a(a^\circ e_b)^\dagger a^\circ b^\circ = e_a a^\circ e_b b^\circ = ab$. On the other hand, since $e_b \mathcal{R}^* b$, we have $(a^\circ e_b)^\dagger \mathcal{R}^* a^\circ e_b \mathcal{R}^* a^\circ b$, and $e_a(a^\circ e_b)^\dagger \mathcal{R}^* e_a a^\circ b = ab$. By the uniqueness of e_{ab} and $(ab)^\circ$, $e_{ab} = e_a(a^\circ e_b)^\dagger$ and $(ab)^\circ = a^\circ b^\circ$. Therefore, $\theta(ab) = (e_a(a^\circ e_b)^\dagger, a^\circ b^\circ) = \theta(a)\theta(b)$. Thus θ is a homomorphism. $\Box$

Summing up Theorems 4.3 and 4.5, we obtain the construction theorem for split left GC-lpp semigroups.

Theorem If $(Y, T, L; \phi)$ is a GC-system, then $GC = GC(Y, T, L; \phi) = \{(e, a) \in L \times T \mid a \in T, e \in L_{a^\dagger}\}$ is a split left GC-lpp semigroup whose band of idempotents is isomorphic to L. Conversely, any split left GC-lpp semigroup can be constructed in this way.

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