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General Decay of Solutions in a Viscoelastic Equation with Nonlinear Localized Damping

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Abstract In this paper, we consider the following viscoelastic equation

$$u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+a(x)u_t+u\,|u|^r=0$$

with initial condition and Dirichlet boundary condition. The decay property of the energy function closely depends on the properties of the relaxation function $g(t)$ at infinity. In the previous works of [3, 7, 11], it was required that the relaxation function $g(t)$ decay exponentially or polynomially as $t\to+\infty$. In the recent work of Messaoudi [12, 13], it was shown that the energy decays at a similar rate of decay of the relaxation function, which is not necessarily dacaying in a polynomial or exponential fashion. Motivated by [12, 13], under some assumptions on $g(x)$, $a(x)$ and r, and by introducing a new perturbed energy, we also prove the similar results for the above equation.

Keywords general decay; viscoelastic equation; relaxation function.

MR(2010) Subject Classification 35L15; 35L70

1. Introduction

In this paper we are concerned with the following equation with a temporal nonlocal term

$$\begin{aligned}&u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+a(x)u_t+u\,|u|^r=0,\\&u(x,t)=0,\ x\in\partial\Omega,\ t\ge 0,\\&u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega,\end{aligned}\tag{1.1}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, the relaxation function $g(x)$ is a positive nonincreasing function, and the coefficient $a(x)$ of the weak frictional damping is supposed to be positive. The equation in (1.1) describes the motion of a viscoelastic body. It is well known that the viscoelastic materials exhibit nature damping, which is due to the special property of these materials to keep memory of their past history. These damping effects are represented by memory term such as in (1.1) (see [9] and the references therein).

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One important question is at which rate the energy of solutions of the damped equation approaches 0 as time tends to ∞. This problem has already been investigated in several papers (see [3, 7, 8, 11]) under different additional assumptions for g. It is known to us that Cavalcanti et al. in [7] firstly studied the above problem. Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometric restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

when $\int_0^\infty g(s)\mathrm{d}s$ is sufficiently small, an exponential rate of decay $E(t) \leq Ce^{-\beta t}$ was obtained for some positive constants C, β, where $E(t)$ will be specified in Theorem (1.5). This work extended the result of Zuazua [16], in which (1.1) was considered with $g = 0$ and the linear damping was localized.

Later, Berrimi and Cavalcanti in [3] considered

$$u_{tt} - k_0\Delta u + \int_0^t \mathrm{div}[a(x)g(t-\tau)\nabla u(\tau)]\mathrm{d}\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \delta > 0$ and improved the result in [7]. They established an exponential stability when g is decaying exponentially and h is linear, and a polynomial stability when g is decaying polynomially and h is nonlinear.

In [3], Berrimi and Messaoudi studied the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + a(x)|u_t|^m u_t + |u|^r u = 0, \tag{1.2}$$

in a bounded domain. Under the condition that

$$g'(t) \leq -\xi g(t), \quad t \geq 0$$

for some positive constant ξ, the authors also proved an exponential decay under weaker conditons on both a and g, where a is allowed to vanish on any part of Ω (including Ω itself). Then the geometric restriction imposed on $\partial\Omega$ by Cavalcanti et al [7] can be dropped.

Liu [11] also considered problem (1.2) under condition

$$g'(t) \leq -\xi g^p(t), \quad t \geq 0,\ 1 \leq p < 3/2,$$

where ξ is a positive constant. They showed the exponential decay when $p = 1$, and polynomial decay when $1 < p < 3/2$. This result extended the work in [3] where only exponential decay was established.

In [2], Alabau-Boussouira et al. developed a unified method to derive decay estimates for the abstract integro-differential evolution equation

$$u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\mathrm{d}s = \nabla F\big(u(t)\big), \quad t \in (0, \infty),$$

in a Hilbert space X, where $A : D(A) \subset X \to X$ is an accretive self-adjoint linear operator with dense domain, and ∇F denotes the gradient of a Găteaux differentiable functional $F : D(\sqrt{A}) \to R$. Depending on the properties of convolution kernel β at infinity, they showed that the energy of solution decays exponentially or polynomially as $t \to \infty$.

Recently, in [12] Messaoudi studied the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = 0$$

where the relaxation function g is assumed as follows

(i) $g : R_+ \to R_+$ is a nonincreasing differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0.$$

(ii) There exists a differentiable function ξ satisfying

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0,$$

$$|\frac{\xi'(t)}{\xi(t)}t| \le k, \quad \xi(t) > 0, \ \xi'(t) \le 0, \ \forall t > 0.$$

He proved that the solution energy decays at the same rate of decay of the relaxation function, which is not necessarily polynomial or exponential decay.

Then in [13], the same author studied the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = u|u|^\gamma,$$

where the relaxation function g is assumed as follows

(i) $g : R_+ \to R_+$ is a nonincreasing differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0.$$

(ii) There exists a differentiable function ξ satisfying

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0,$$

$$|\frac{\xi'(t)}{\xi(t)}| \le k, \quad \xi(t) > 0, \ \xi'(t) \le 0, \ \forall t > 0.$$

(iii) For the nonlinear term, assume that

$$0 < \gamma \le \frac{2}{n-2}, \quad n \ge 3, \ \gamma > 0, \ n = 1, 2.$$

It was also proved that the solution energy decays at the same rate of decay of the relaxation function, which is not necessarily polynomial or exponential decay.

Motivated by the above work of [12, 13], in this paper we also concern with problems (1.1) and (1.2). By using Lyapunov type technique for some perturbed energy, which was introduced in [12, 13], we show that the solution energy decays at a similar rate of decay of the relaxation function, which is not necessarily the decay in an exponential or polynomial fashion. Therefore, our result allows a larger class of relaxation functions and improves earlier results in the literature [3, 7, 11]. Our assumptions on the function $g(x)$, $a(x)$, and r are as follows.

(A1) $g : R_+ \to R_+$ is a nonincreasing differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0.$$

(A2) There exists a differentiable function ξ satisfying

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0,$$
$$|\frac{\xi'(t)}{\xi(t)}| \leq k, \quad \xi(t) > 0, \ \xi'(t) \leq 0, \ \forall\, t > 0.$$

(A3) Assume that $a(x)$ is a nonnegative and bounded function such that

$$a(x) \geq a_0 > 0.$$

(A4) For the nonlinear term, we assume

$$0 < r < \frac{2}{n-2}, \quad n \geq 3; \ r > 0, \ n = 1, 2.$$

Remark 1.1 There are many functions satisfying the assumptions (A1) and (A2), and examples have been given in [12], such as

$$\begin{aligned} g_1(t) &= a(1+t)^v, \quad v < -1, \\ g_2(t) &= ae^{-b(t+1)^p}, \quad 0 < p \leq 1, \\ g_3(t) &= \frac{ae^{-bt}}{(1+t)^n}, \quad n = 1, 2, \end{aligned}$$

for $a, b > 0$ to be chosen properly.

Remark 1.2 Since ξ is nonincreasing, $\xi(t) \leq \xi(0) = M$.

Remark 1.3 Condition (A1) is necessary to guarantee the hyperbolicity of the system (1.1).

We will also use the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq 2n/n-2$, if $n \geq 3$ and $q \geq 2$ if $n = 1, 2$; and $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$ and we will use the same embedding constant denoted by C, i.e.,

$$\|u\|_q \leq C\|\nabla u\|_2, \quad \|u\|_q \leq C\,\|u\|_r\,.$$

The existence of global solution of problem (1.1) can be easily obtained by making use of the Faedo-Galerkin method, and we refer to [5] for details.

Proposition 1.4 *Let $(u_0, u_t) \in H_0^1(\Omega) \times L^2(\Omega)$. Assume (A1)–(A4) hold, then problem (1.1) has a unique global solution*

$$u \in C^0([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Our main result is stated as follows.

Theorem 1.5 *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that (A1)–(A4) hold, then for each $t_0 > 0$, there exist strictly positive constants K and λ such that the solution of (1.1) satisfies*

$$E(t) \leq Ke^{-\lambda \int_{t_0}^t \xi(s)\mathrm{d}s}, \quad t \geq t_0,$$

where

$$E(t) = \frac{1}{2}\Big(1 - \int_0^t g(s)\mathrm{d}s\Big)\,\|\nabla u\|_2^2 + \frac{1}{2}\,\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{r+2}\,\|u\|_{r+2}^{r+2} \tag{1.3}$$

and

$$(g \circ v)(t) = \int_0^t g(t-s) \|v(t) - v(s)\|_2^2 \mathrm{d}s.$$

Remark 1.6 Our method used in this paper is applicable to the equation (1.2) in a bounded domain with $0 \le \max\{m, r\} \le \frac{2}{n-2}$, if $n \ge 3$. Thus we can extend the result in [11].

The rest of this paper is organized as follows. In next section, we present some Lemmas needed for our work. Section 3 contains the proof of our main result.

2. Preliminaries

In this section we will prove some lemmas.

Lemma 2.1 *If u is a solution of (1.1), then energy $E(t)$ satisfies $E'(t) \le 0$.*

Proof By multiplying equation (1.1) by u_t and integrating over Ω, then using integration by parts and hypotheses (A1) and (A2), after some manipulation, we obtain.

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u\|_2^2 - \int_\Omega a(x) |u_t|^2 \mathrm{d}x \le 0. \quad \square \tag{2.1}$$

Remark 2.2 It follows form Lemma 2.1 that the energy is uniformly bounded (by $E(0)$) and deceasing in t, which also implies that

$$l \|\nabla u\|_2^2 \le 2E(0). \tag{2.2}$$

Then we define the perturbed energy functional

$$F(t) = E(t) + \varepsilon_1 \psi(t) + \varepsilon_2 \phi(t), \tag{2.3}$$

where ε_1 and ε_2 are positive constants and

$$\psi(t) = \xi(t) \int_\Omega u u_t \mathrm{d}x,$$

$$\phi(t) = -\xi(t) \int_\Omega u_t \int_0^t g(t-s)(u(t) - u(s)) \mathrm{d}s \mathrm{d}x. \tag{2.4}$$

Lemma 2.3 *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t) - u(s)) \mathrm{d}s \Big)^2 \mathrm{d}x \le (1-l) C^2 (g \circ \nabla u)(t).$$

Proof

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t) - u(s)) \mathrm{d}s \Big)^2 \mathrm{d}x = \int_\Omega \Big(\int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} (u(t) - u(s)) \mathrm{d}s \Big)^2 \mathrm{d}x.$$

By applying Cauchy-Schwarz inequality and Poincaré's inequality, we easily see that

$$\begin{aligned} &\int_\Omega \Big(\int_0^t g(t-s)(u(t) - u(s)) \mathrm{d}s \Big)^2 \mathrm{d}x \\ &\quad \le \int_\Omega \Big(\int_0^t g(t-s) \mathrm{d}s \Big) \Big(\int_0^t g(t-s) \left(u(t) - u(s)\right)^2 \mathrm{d}s \Big) \mathrm{d}x \le (1-l) C^2 (g \circ \nabla u)(t). \quad \square \end{aligned}$$

Lemma 2.4 *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t) \tag{2.5}$$

holds for two positive constant α_1 and α_2.

Proof After straightforward computation, we can see that

$$\begin{aligned} F(t) \le & E(t) + \frac{\varepsilon_1}{2}\xi(t)\Big(\int_\Omega |u|^2\mathrm{d}x + \int_\Omega |u_t|^2\mathrm{d}x\Big) + \frac{\varepsilon_2}{2}\xi(t)\int_\Omega |u_t|^2\mathrm{d}x + \\ & \frac{\varepsilon_2}{2}\xi(t)\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\Big)^2\mathrm{d}x \\ \le & E(t) + (\frac{\varepsilon_1+\varepsilon_2}{2})M\int_\Omega |u_t|^2\mathrm{d}x + \frac{\varepsilon_1}{2}MC^2\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\varepsilon_1}{2}MC^2(1-l)(g\circ\nabla u)(t) \\ \le & \alpha_2 E(t), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} F(t) \ge & E(t) - \frac{\varepsilon_1}{2}\xi(t)\int_\Omega |u_t|^2\mathrm{d}x - \frac{\varepsilon_1}{2}\xi(t)C^2\int_\Omega |\nabla u|^2\mathrm{d}x - \\ & \frac{\varepsilon_2}{2}\xi(t)\int_\Omega |u_t|^2\mathrm{d}x - \frac{\varepsilon_2}{2}\xi(t)C^2(1-l)(g\circ\nabla u)(t) \\ \ge & (\frac{1}{2} - \frac{M}{2}(\varepsilon_1+\varepsilon_2))\int_\Omega |u_t|^2\mathrm{d}x + \frac{1}{2}(l - \frac{\varepsilon_1}{2}MC^2)\int_\Omega |\nabla u|^2\mathrm{d}x + \\ & (\frac{1}{2} - \frac{\varepsilon_2}{2}M)C^2(1-l)(g\circ\nabla u)(t) + \frac{1}{r+2}\|u\|_{r+2}^{r+2}\mathrm{d}x \\ \ge & \alpha_1 E(t), \end{aligned} \tag{2.7}$$

for ε_1 and ε_2 small enough. □

Lemma 2.5 *Let u be the solution of problem (1.1) derived in Proposition (1.4). Then we have*

$$\begin{aligned} \psi'(t) \le & \Big[1 + \frac{C^2(k+\|a\|_\infty)^2}{l}\Big]\xi(t)\int_\Omega |u_t|^2\mathrm{d}x + \frac{1-l}{2l}\xi(t)(g\circ\nabla u)(t) - \\ & \frac{l}{4}\xi(t)\int_\Omega |\nabla u|^2\mathrm{d}x - \xi(t)\int_\Omega |u|^{r+2}\mathrm{d}x. \end{aligned} \tag{2.8}$$

Proof By using Eq. (1.1), we easily see that

$$\begin{aligned} \psi'(t) = & \xi'(t)\int_\Omega uu_t\mathrm{d}x + \xi(t)\int_\Omega |u_t|^2\mathrm{d}x - \xi(t)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x - \\ & \xi(t)\int_\Omega a(x)uu_t\mathrm{d}x - \xi(t)\int_\Omega |u|^{r+2}\mathrm{d}x. \end{aligned} \tag{2.9}$$

We now estimate the third term in the right hand side of (2.9) as follows

$$\begin{aligned} & \int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x \\ & \le \frac{1}{2}\int_\Omega |\nabla u(t)|^2\mathrm{d}x + \frac{1}{2}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)|\mathrm{d}s\Big)^2\mathrm{d}x \\ & \le \frac{1}{2}\int_\Omega |\nabla u(t)|^2\mathrm{d}x + \frac{1}{2}\int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)| + |u(t)|)\mathrm{d}s\Big)^2\mathrm{d}x. \end{aligned} \tag{2.10}$$

Thanks to Young's inequality, and the fact $\int_0^t g(s)\mathrm{d}s \le \int_0^\infty g(s)\mathrm{d}s = 1-l$, we obtain that for any $\eta > 0$,

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)\mathrm{d}s\Big)^2 \mathrm{d}x \\
&\le \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\mathrm{d}s\Big)^2 \mathrm{d}x + \int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(t)|)\mathrm{d}s\Big)^2 \mathrm{d}x + \\
&\quad 2\int_\Omega \Big(\int_0^t g(t-s)\,|\nabla u(s)-\nabla u(t)|\,\mathrm{d}s\Big)\cdot\Big(\int_0^t g(t-s)\,|\nabla u(t)|\,\mathrm{d}s\Big)\mathrm{d}x \\
&\le (1+\eta)\int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(t)|)\mathrm{d}s\Big)^2 \mathrm{d}x + (1+\frac{1}{\eta})\int_\Omega \Big(\int_0^t g(t-s)\,|\nabla u(s)-\nabla u(t)|\,\mathrm{d}s\Big)^2 \mathrm{d}x \\
&\le (1+\frac{1}{\eta})(1-l)(g\circ\nabla u)(t) + (1+\eta)(1-l)^2\int_\Omega |\nabla u(t)|^2 \mathrm{d}x. \qquad (2.11)
\end{aligned}
$$

Combining (2.10) and (2.11), and using

$$\int_\Omega u u_t \mathrm{d}x \le \alpha C^2 \int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \frac{1}{4\alpha}\int_\Omega u_t^2 \mathrm{d}x, \quad \alpha > 0, \qquad (2.12)$$

$$\int_\Omega a(x) u u_t \mathrm{d}x \le \alpha\, \|a(x)\|_\infty C^2 \int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \frac{1}{4\alpha}\,\|a(x)\|_\infty \int_\Omega u_t^2 \mathrm{d}x, \quad \alpha > 0, \qquad (2.13)$$

we get

$$
\begin{aligned}
\psi'(t) \le & [1+\frac{1}{4\alpha}(\|a\|_\infty + |\frac{\xi'(t)}{\xi(t)}|)]\xi(t)\int_\Omega |u_t|^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\xi(t)(g\circ\nabla u)(t) - \\
& \frac{1}{2}[1-(1+\eta)(1-l)^2 - 2\alpha C^2(\|a\|_\infty + |\frac{\xi'(t)}{\xi(t)}|)]\xi(t)\int_\Omega |\nabla u|^2 \mathrm{d}x - \xi(t)\int_\Omega |u|^{r+2}\mathrm{d}x \\
\le & [1+\frac{1}{4\alpha}(\|a\|_\infty + k)]\xi(t)\int_\Omega |u_t|^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\xi(t)(g\circ\nabla u)(t) - \\
& \frac{1}{2}[1-(1+\eta)(1-l)^2 - 2\alpha C^2(\|a\|_\infty + k)]\xi(t)\int_\Omega |\nabla u|^2 \mathrm{d}x - \xi(t)\int_\Omega |u|^{r+2}\mathrm{d}x. \qquad (2.14)
\end{aligned}
$$

Choosing $\eta = \frac{l}{1-l}$ and $\alpha = \frac{l}{4C^2(k+\|a\|_\infty)^2}$ yields (2.9). □

Lemma 2.6 *Let u be the solution of problem (1.1) derived in Proposition (1.4). Then we have*

$$
\begin{aligned}
\phi'(t) \le & \delta[1+2(1-l)^2 + C^{2r+2}(\frac{2E(0)}{l})^r]\xi(t)\int_\Omega |\nabla u(t)|^2 \mathrm{d}x + k_\delta \xi(t)(g\circ\nabla u)(t) - \\
& \frac{g(0)}{4\delta}C^2\xi(t)(g'\circ\nabla u)(t) + \Big[\delta(1+k+\|a\|_\infty) - \int_0^t g(s)\mathrm{d}s\Big]\xi(t)\int_\Omega |u_t|^2 \mathrm{d}x, \qquad (2.15)
\end{aligned}
$$

where

$$K_\delta = \frac{1-l}{4\delta} + (2\delta + \frac{1}{4\delta})(1-l) + (\|a\|_\infty + k + 1)\frac{C^2}{4\delta}(1-l). \qquad (2.16)$$

Proof It is obtained after direct computations that

$$
\begin{aligned}
\phi'(t) = & \xi(t)\int_\Omega \nabla u(t)\cdot\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big)\mathrm{d}x - \\
& \xi(t)\int_\Omega \Big(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\Big)\cdot\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big)\mathrm{d}x +
\end{aligned}
$$

$$\begin{aligned}
&\xi(t)\int_\Omega a(x)u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x+\\
&\xi(t)\int_\Omega |u|^r u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x-\\
&\xi(t)\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x-\xi(t)\int_0^t g(s)\mathrm{d}s\int_\Omega |u_t|^2\mathrm{d}x-\\
&\xi'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x. \qquad (2.17)
\end{aligned}$$

We now estimate the right-hand side terms of (2.17). Applying Young's inequality to the first term gives

$$\int_\Omega \nabla u(t)\cdot\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big)\mathrm{d}x\le\delta\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{1-l}{4\delta}(g\circ\nabla u)(t),\ \ \forall\delta>0. \quad (2.18)$$

Similarly, the second term can be estimated as follows:

$$\begin{aligned}
&\int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\Big)\cdot\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big)\mathrm{d}x\\
&\le\delta\int_\Omega\Big|\int_0^t g(t-s)|\nabla u(s)|\mathrm{d}s\Big|^2\mathrm{d}x+\frac{1}{4\delta}\int_\Omega\Big|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big|^2\mathrm{d}x\\
&\le\delta\int_\Omega\Big(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)\mathrm{d}s\Big)^2\mathrm{d}x+\\
&\quad\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big)^2\mathrm{d}x\\
&\le(2\delta+\frac{1}{4\delta})\int_\Omega\Big(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|\mathrm{d}s\Big)^2\mathrm{d}x+2\delta(1-l)^2\int_\Omega|\nabla u|^2\mathrm{d}x\\
&\le(2\delta+\frac{1}{4\delta})(1-l)(g\circ\nabla u)(t)+2\delta(1-l)^2\int_\Omega|\nabla u|^2\mathrm{d}x. \qquad (2.19)
\end{aligned}$$

As for the third term, we have

$$\int_\Omega a(x)u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\le\delta\|a\|_\infty\int_\Omega|u_t|^2\mathrm{d}x+\|a\|_\infty\frac{C^2}{4\delta}(1-l)(g\circ\nabla u)(t). \quad (2.20)$$

The fourth term

$$\begin{aligned}
&\int_\Omega |u|^r u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\
&\le\delta\int_\Omega|u|^{2r+2}\mathrm{d}x+\frac{C^2}{4\delta}(1-l)(g\circ\nabla u)(t)\\
&\le\delta C^{2r+2}\|\nabla u\|_2^{2r+2}+\frac{C^2}{4\delta}(1-l)(g\circ\nabla u)(t)\\
&\le\delta C^{2r+2}(\frac{2E(0)}{l})^r\|\nabla u\|_2^2+\frac{C^2}{4\delta}(1-l)(g\circ\nabla u)(t). \qquad (2.21)
\end{aligned}$$

The fifth term

$$-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\le\delta\int_\Omega|u_t|^2\,dx-\frac{g(0)}{4\delta}(g'\circ\nabla u)(t). \quad (2.22)$$

Combining (2.18)–(2.22) gives Lemma 2.6. □

3. Decay of solutions

Now we prove our main result Theorem 1.4.

Proof Since g is positive, continuous and $g(0) > 0$, for any $t_0 \geq 0$, we have

$$\int_0^t g(s)\mathrm{d}s \geq \int_0^{t_0} g(s)\mathrm{d}s = g_0 > 0, \quad \forall\, t \geq t_0.$$

By using (2.2), (2.9) and (2.16), we obtain for $t \geq t_0$,

$$\begin{aligned} F'(t) \leq &- \big[\varepsilon_2\{g_0 - \delta(1+k+\|a\|_\infty)\} - \varepsilon_1(1+\frac{C^2(k+\|a\|_\infty)^2}{l})\big]\xi(t)\int_\Omega |u_t|^2\mathrm{d}x - \\ &\big[\varepsilon_1\frac{1}{4l} - \varepsilon_2\delta\{1+2(1-l)^2 + C^{2r+2}(\frac{2E(0)}{l})^r\}\big]\xi(t)\int_\Omega |\nabla u|^2\mathrm{d}x + \\ &(\frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C^2M)(g'\circ\nabla u)(t) + (\varepsilon_1\frac{1-l}{2l} + \varepsilon_2 k_\delta)\xi(t)(g\circ\nabla u)(t) - \\ &\varepsilon_1\xi(t)\int_\Omega |u|^{r+2}\mathrm{d}x. \end{aligned} \tag{3.1}$$

At this point we choose δ so small that

$$g_0 - \delta(1+k) > \frac{1}{2}g_0,$$

$$\frac{4}{l}\delta[1+2(1-l)^2 + C^{2r+2}(\frac{2E(0)}{l})^r] < \frac{1}{4(1+\frac{C^2(k+\|a\|_\infty)^2}{l})}g_0, \tag{3.2}$$

where δ is fixed. The choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{g_0}{4(1+\frac{C^2(k+\|a\|_\infty)^2}{l})}\varepsilon_2 < \varepsilon_1 < \frac{g_0}{2(1+\frac{C^2(k+\|a\|_\infty)^2}{l})}\varepsilon_2 \tag{3.3}$$

will make

$$\begin{aligned} k_1 &= \varepsilon_2\{g_0 - \delta(1+k+|a|_\infty)\} - \varepsilon_1(1+\frac{C^2(k+\|a\|_\infty)^2}{l}) > 0, \\ k_2 &= \varepsilon_1\frac{l}{4} - \varepsilon_2\delta[1+2(1-l)^2 + C^{2r+2}(\frac{2E(0)}{l})^r] > 0. \end{aligned} \tag{3.4}$$

We then take ε_1 and ε_2 such that (2.6) and (3.3) remain valid. Further,

$$k_3 = (\frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C^2M) - (\varepsilon_1\frac{1-l}{2l} + \varepsilon_2 k_\delta) > 0. \tag{3.5}$$

Hence

$$(\frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C^2M)(g'\circ\nabla u)(t) + (\varepsilon_1\frac{1-l}{2l} + \varepsilon_2 k_\delta)\xi(t)(g\circ\nabla u)(t) \leq -k_3\xi(t)(g\circ\nabla u)(t). \tag{3.6}$$

Since ξ is nonincreasing, by using (2.6), (3.1) and (3.6) we arrive at

$$F'(t) \leq -\beta_1\xi(t)E(t) \leq -\beta_1\alpha_1\xi(t)F(t). \tag{3.7}$$

A simple integration of (3.7) leads to

$$F(t) \leq F(t_0)e^{-\beta_1\alpha_1\int_{t_0}^t \xi(s)\mathrm{d}s}, \quad \forall\, t \geq t_0. \tag{3.8}$$

Thus from (2.6) and (3.8) it follows

$$E(t) \leq \alpha_2 F(t_0)e^{-\beta_1\alpha_1\int_{t_0}^t \xi(s)\mathrm{d}s} = Ke^{-\lambda\int_{t_0}^t \xi(s)\mathrm{d}s}, \quad t \geq t_0, \tag{3.9}$$

where $K = \alpha_2 F(t_0)$, $\lambda = \beta_1 \alpha_1$. This completes the proof. □

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