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A Class of Strong Limit Theorems and Moment Generating Function Method

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Abstract In virtue of the notion of likelihood ratio and moment generating function, the limit properties of the sequences of absolutely continuous random variables are studied, and a class of strong limit theorems represented by inequalities with random bounds are obtained.

Keywords likelihood ratio; strong limit theorem; moment generating function.

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1. Introduction

Strong limit theory is one of the most important problems in probability theory, and has received extensive attention in the literature. Liu [1] obtained some strong limit theorems for a multivariate function sequence of discrete random variables by using the concept of the conditional moment generating function. Yang [2] proved two strong limit theorems for arbitrary stochastic sequences. Li, Chen and Zhang [3] studied the strong limit theorems of arbitrary dependent continuous random variables by using the analytic technique and the Laplace transform approach. Yang and Yang [4] established a strong limit theorem of the Dubins-Freedman type for arbitrary stochastic sequences. Furthermore, many comprehensive works can be found in [5] and references therein. The purpose of this paper is to establish a kind of strong limit theorems represented by inequalities with random bounds, and to extend the analytic technique proposed by Liu [5]. In the proof, the approach of applying the tool of moment generating function to the study of strong limit theorem of the sequences of continuous random variables is proposed.

Let $\{X_n, n \geq 1\}$ be a sequence of absolutely continuous random variables on the probability space $(\Omega, \mathcal{F}, P)$ with joint distribution density function $f_n(x_1, \ldots, x_n)$. Let $f(x_k),\ k = 1, 2, \ldots,$

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stand for the marginal density function of X_k, and call $\prod_{k=1}^{n} f(x_k)$ the reference product density function. Let

$$r_n(\omega) = \ln \Big[f_n(X_1, \ldots, X_n) / \prod_{k=1}^{n} f(X_k) \Big], \tag{1}$$

where ω is a sample point. In statistical terms, $r_n(\omega)$ is called the log-likelihood ratio. Let

$$r(\omega) = \limsup_{n\to\infty} \frac{1}{n} r_n(\omega) \tag{2}$$

with $\ln 0 = -\infty$. $r(\omega)$ is called asymptotic log-likelihood ratio.

Let $f(x) > 0$ stand for the density function of the random variable Y, and let the mathematical expectation and moment generating function of the random variable Y be

$$\int_{-\infty}^{\infty} x f(x) \mathrm{d}x = m = E(Y), \tag{3}$$

and

$$M(t) = \int_{-\infty}^{\infty} \mathrm{e}^{tx} f(x) \mathrm{d}x = E(\mathrm{e}^{tY}), \tag{4}$$

respectively. In this paper, we assume that there exists $t_0 \in (0, +\infty)$, such that $M(t) < \infty$, $t \in [-t_0, t_0]$.

In order to prove our main results, we first give a lemma, which will play a central role in the proof of Theorem 3.

Lemma 1 ([6, P. 54]) *There exists a random variable Y such that its mathematical expectation and moment generating function are defined by (3) and (4), respectively. If $EY = m = 0$, then there exist constants $g > EY^2/2$ and $a_0 > 0$, such that*

$$M(t) \le e^{gt^2}, \quad t \in (-a_0, a_0). \tag{5}$$

2. Main results

Our main existence results are the following:

Theorem 1 *Let $\{X_n, n \ge 1\}$ be a sequence of absolutely continuous random variables on the probability space $(\Omega, \mathcal{F}, P)$, $r(\omega)$, $M(t)$ be given as above, and $M(t)$ be defined in $[-t_0, t_0]$. Then there exists a constant $c > 0$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - m] \ge \alpha(c), \quad \text{a.e.}, \ \ \omega \in D(c), \tag{6}$$

where

$$D(c) = \{\omega : r(\omega) \le c\}, \tag{7}$$

$$\alpha(c) = \sup\{\varphi(t), -t_0 \le t < 0\}, \tag{8}$$

$$\varphi(t) = [\ln M(t) + c]/t - m, \quad -t_0 \le t < 0, \tag{9}$$

and then

$$\alpha(c) \le 0, \ -t_0 \le t < 0, \tag{10}$$

$$\lim_{c\to 0^+}\alpha(c)=\alpha(0)=0. \tag{11}$$

Proof For arbitrary $t\in[-t_0,t_0]$, let

$$g(t,x)=\mathrm{e}^{tx}f(x)/M(t). \tag{12}$$

Then

$$\int_{-\infty}^{\infty}g(t,x)\mathrm{d}x=1. \tag{13}$$

Let

$$q_n(t;x_1,\dots,x_n)=\prod_{k=1}^{n}g(t,x_k)=1/[M(t)]^n\exp(t\sum_{k=1}^{n}x_k)\cdot\prod_{k=1}^{n}f(x_k). \tag{14}$$

By (14), it is easy to see that $q_n(t;x_1,\dots,x_n)$ is an n multivariate probability density function. Let

$$t_n(t,\omega)=\frac{q_n(t,X_1,\dots,X_n)}{f_n(X_1,\dots,X_n)}. \tag{15}$$

By [6], we can see that $t_n(t,\omega)$ is a nonnegative supermartingale that converges a.e. Hence there exists $A(t)\in\mathcal{F}$, $P(A(t))=1$, such that

$$\lim_{n\to\infty}t_n(t,\omega)<\infty,\quad \omega\in A(t). \tag{16}$$

This implies that

$$\limsup_{n\to\infty}\frac{1}{n}\ln t_n(t,\omega)\le 0,\quad \omega\in A(t). \tag{17}$$

By (17), (14) and (3), we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\Big\{-n\ln M(t)+t\sum_{k=1}^{n}X_k-\ln r_n(\omega)\Big\}\le 0,\quad \omega\in A(t). \tag{18}$$

Let $t=0$. We have

$$\liminf_{n\to\infty}\frac{1}{n}\ln r_n(\omega)\ge 0,\ \ \omega\in A(0), \tag{19}$$

that is

$$r(\omega)\ge 0,\ \ \omega\in A(0). \tag{20}$$

Let $-t_0\le t<0$. By (7) and (18), we have

$$t\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X_k\le\limsup_{n\to\infty}\Big[\frac{1}{n}\ln r_n(\omega)+\ln M(t)\Big]=r(\omega)+\ln M(t),\ \ \omega\in A(t)\cap D(c). \tag{21}$$

By the property of the inferior limit

$$\liminf_{n\to\infty}(a_n-b_n)\ge d\Rightarrow\liminf_{n\to\infty}(a_n-c_n)\ge\liminf_{n\to\infty}(b_n-c_n)+d, \tag{22}$$

and dividing the two sides of (21) by t, we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[X_k-m]\ge\frac{1}{t}r(\omega)+\frac{1}{t}\ln M(t)-m\ge[\ln M(t)+c]/t-m,\ \ \omega\in A(t)\cap D(c). \tag{23}$$

Let Q^- be the set of rational numbers in the interval $[-t_0, 0)$ and let $A^* = \cap_{t \in Q^-} A(t)$. Then $P(A^*) = 1$. By (23), we have

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - m] \geq [\ln M(t) + c]/t - m, \quad \omega \in A^* \cap D(c), \quad \forall\, t \in Q^-. \tag{24}$$

By (9), (20) and (24), we obtain

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - m] \geq \varphi(t), \quad \omega \in A^* \cap D(c), \quad \forall\, t \in Q^-. \tag{25}$$

We know that $\varphi(t)$ is a continuous function with respect to t on the interval $[-t_0, 0)$ and $\lim_{c \to \infty} \varphi(t) = -\infty$. By (8) and (20), for each $\omega \in A^* \cap D(c)$, take $t_n(\omega) \in Q^-$ $(n = 1, 2, \ldots)$, such that

$$\lim_{n \to \infty} \varphi[t_n(\omega)] = \alpha(c). \tag{26}$$

By (25), we have

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - m] \geq \varphi[t_n(\omega)], \quad \omega \in A^* \cap D(c). \tag{27}$$

From (26) and (27), we have

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - m] \geq \alpha(c), \quad \omega \in A^* \cap D(c). \tag{28}$$

Since $P(A^*) = 1$, (6) holds by (28). By the Jensen inequality we obtain

$$M(t) = E(\mathrm{e}^{tY}) \geq \mathrm{e}^{tE(Y)} = \mathrm{e}^{tm}. \tag{29}$$

By (9) and (29), we have

$$\varphi(t) \leq c/t \leq 0, \quad -t_0 \leq t < 0, \tag{30}$$

then

$$\alpha(x) \leq 0. \tag{31}$$

Thus, (10) follows from (31). By L′hospital rule, we have

$$\lim_{t \to 0^-} \ln M(t)/t = \lim_{t \to 0^-} M'(t)/M(t) = m. \tag{32}$$

By (8) and (32), we have

$$\alpha(0) = 0. \tag{33}$$

By (8) and (9), we have

$$\alpha(c) \geq \varphi(\sqrt{c}) = [M(\sqrt{c}) + c]/\sqrt{c} - m. \tag{34}$$

It is easy to see that

$$\lim_{c \to 0^+} \varphi(\sqrt{c}) = 0. \tag{35}$$

By (31), (34) and (35), we can show that

$$\lim_{c \to 0^+} \alpha(c) = 0. \tag{36}$$

Then (11) follows from (31) and (36). □

Theorem 2 *Under the conditions of the theorem 1, there holds*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[X_k-m]\le\beta(c),\quad \text{a.e.},\ \ \omega\in D(c), \tag{37}$$

where

$$\beta(c)=\inf\{\psi(t),0<t\le t_0\}, \tag{38}$$

$$\psi(t)=[\ln M(t)+c]/t-m,\quad 0<t\le t_0, \tag{39}$$

and then

$$\beta(c)\ge 0,\quad 0<t\le t_0, \tag{40}$$

$$\lim_{c\to 0^+}\beta(c)=\beta(0)=0. \tag{41}$$

The proof of Theorem 2 is similar to that of Theorem 1 and hence is omitted here.

Remark 1 In Theorems 1 and 2, a subset $D(c)$ of the sample space is determined by using the condition $r(\omega)\le c$, and on this subset the limit properties of $\frac{1}{n}\sum_{k=1}^{n}[X_k-m]$ as $n\to\infty$ are studied. $r(\omega)\le c$ is the key condition for determining $D(c)$, where the asymptotic log-likelihood ratio $r(\omega)$ can be regarded as a stochastic measure of the deviation between $\{X_n,n\ge 1\}$ and the sequence of random variables with product density function.

Remark 2 Roughly speaking, condition (7) can be regarded as a restriction on the deviation between $\{X_n,n\ge 1\}$ and the sequence of random variables with product density function $\prod_{k=1}^{n}f(x_k)$. From (6) and (37), it is easy to see that the inferior and superior limits depend on c. The smaller c is, the smaller the deviation is. For example, let $\{X_n,n\ge 1\}$ be a sequence of absolutely continuous random variables with joint distribution density function $f_n(x_1,\ldots,x_n)$, and $Y_k,k=1,\ldots,n$, are i.i.d. random variable sequence with exponential density function $f(x,\lambda)$, where $\lambda>0$ is a parameter. By the definition of log-likelihood ratio, we have

$$r_n(\omega)=\ln\Big[f_n(X_1,\ldots,X_n)/\prod_{k=1}^{n}f(X_k,\lambda)\Big]\approx\frac{f_n(X_1,\ldots,X_n)-\prod\limits_{k=1}^{n}f(X_k,\lambda)}{\prod\limits_{k=1}^{n}f(X_k,\lambda)}.$$

Let $D(c)=\{\omega:\limsup_{n\to\infty}\frac{1}{n}r_n(\omega)\le c\}$ denote a subset of sample space. It is easy to show that the subset $D(c)$ controls the deviation between $f_n(X_1,\ldots,X_n)$ and the product exponential density function $\prod_{k=1}^{n}f(X_k,\lambda)$, and the smaller c is, the smaller the deviation is. Obviously, $r(\omega)=0$ if and only if $c=0$, that is, $X_k(1\le k\le n)$ are independent exponential random variables with parameter λ.

Theorem 3 *Under the conditions of the theorem 1, let $m=E(Y)=0,EY^2<\infty$. Let $c\le ga_0^2$ and $a_0\le t_0$. Then*

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X_k\ge -2\sqrt{gc},\quad -a_0<t<0,\quad \text{a.e.},\ \ \omega\in D(c), \tag{42}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k \le 2\sqrt{gc}, \quad 0<t<a_0, \quad \text{a.e.}, \quad \omega \in D(c), \tag{43}$$

where g and a_0 are given by Lemma 1.

Proof Since the proof of (43) is similar to that of (42), we only prove (42) here. By Lemma 1, we have

$$\begin{aligned}\alpha(c) &= \sup\{[M(t)+c]/t, -a_0<t<0\} \le \sup\{[gt^2+c]/t, -a_0<t<0\} \\ &= -\inf\{(-gt)+(-c/t), \ -a_0<t<0\} = -2\sqrt{gc}, \quad -a_0<t<0. \end{aligned}\tag{44}$$

Hence, (42) follows from (44) directly. □

Corollary 1 *Let $\{X_n, n\ge 1\}$ be independent random variables. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(X_k - m) = 0, \quad \text{a.e.} \tag{45}$$

Proof In this case, $f_n(x_1,\ldots,x_n) = \prod_{k=1}^n f(x_k), n\ge 1$, and $r(\omega)=0$. Hence (45) follows directly from (11) and (41). □

Corollary 2 *Under the conditions of the theorem 1, if $f(x_k)$ is a Normal distribution with parameters μ and σ^2, where $E(X_k)=\mu$ and $\mathrm{Var}(X_k)=\sigma^2$, then*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}[X_k-\mu] \ge -\sigma\sqrt{2c}, \quad \text{a.e.}, \quad \omega\in D(c), \tag{46}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}[X_k-\mu] \le \sigma\sqrt{2c}, \quad \text{a.e.}, \quad \omega\in D(c). \tag{47}$$

Proof By (4), the moment generating function of the normal density function $f(x_k)$ is defined by

$$M(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}. \tag{48}$$

By (9) and (48), let $t<0$. We have

$$\varphi(t) = \frac{\mu t + \frac{1}{2}t^2\sigma^2 + c}{t} - \mu = \frac{1}{2}t\sigma^2 + \frac{c}{t}. \tag{49}$$

It is easy to see that if $t=-\sqrt{2c}/\sigma$, the function $\varphi(t)$ attains its largest value $\varphi(-\sqrt{2c}/\sigma) = -\sigma\sqrt{2c}$. Therefore, (46) follows from (6) and (49).

Let $t>0$. We have

$$\frac{1}{2}t\sigma^2 + \frac{c}{t} \ge \sigma\sqrt{2c}. \tag{50}$$

By (39) and (50), it is easy to see that if $t=\sqrt{2c/\sigma}$, the function $\psi(t)$ attains its smallest value $\psi(\sqrt{2c/\sigma}) = \sigma\sqrt{2c}$. Therefore, (47) follows from (37) and (50). □

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