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Compact Operators on Weighted Bergman Spaces of the Unit Ball

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Abstract In this paper, we study the compact operators on weighted Bergman spaces of the unit ball. Extending Miao and Zheng'result in 2004, we obtain the necessary and sufficient conditions for the operator to be compact on weighted Bergman spaces of the unit ball under some integrable conditions.

Keywords compact operator; unit ball; weighted Bergman space.

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1. Preliminaries

Throughout let n be a fixed integer $n \geq 2$. Let $\mathbb{C}^n$ denote the Euclidean space of complex dimension n. For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we define

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,$$

where $\bar{w}_k$ is the complex conjugate of w_k.

For a multi-index $m = (m_1, \ldots, m_n)$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we also write

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \quad z^m = z_1^{m_1} \cdots z_n^{m_n}.$$

Denote the unit ball in $\mathbb{C}^n$ by $\mathbb{B}_n$. Let v denote the Lebesgue volume measure on $\mathbb{B}_n$, normalized so that $v(\mathbb{B}_n) = 1$. For $-1 < \alpha < \infty$, we denote by v_α the weighted Lebesgue measure on $\mathbb{B}_n$ defined by $\mathrm{d}v_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \mathrm{d}v(z)$, where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant so that dv_α is a probability measure on $\mathbb{B}_n$. For $\alpha > -1$ and $p \geq 1$ the weighted Bergman space A^p_α consists of holomorphic functions f in $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$, that is,

$$A^p_\alpha = L^p(\mathbb{B}_n, \mathrm{d}v_\alpha) \bigcap H(\mathbb{B}_n).$$

It is clear that A^p_α is a linear subspace of $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

When a function $f : \mathbb{B}_n \to \mathbb{C}$ is holomorphic, all higher order partial derivatives exist and are still holomorphic. For a multi-index $m = (m_1, \ldots, m_n)$ we will employ the notation

$$\partial^m f = \frac{\partial^m f}{\partial z^m} = \frac{\partial^{|m|} f}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}.$$

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For $z, w \in \mathbb{B}_n$, the reproducing kernel on $A^2_\alpha(\mathbb{B}_n)$ is given by

$$K_w^{(\alpha)}(z) = \frac{1}{(1 - \langle z, w\rangle)^{n+\alpha+1}}.$$

If $\langle \cdot, \cdot \rangle_\alpha$ denotes the inner product in $L^2(\mathbb{B}_n, \mathrm{d}v_\alpha)$, then $\langle h, K_w^{(\alpha)}\rangle_\alpha = h(w)$, for every $h \in A^2_\alpha(\mathbb{B}_n)$ and $w \in \mathbb{B}_n$. In this paper, we use $\| \cdot \|_{\alpha,p}$ to denote the norm in $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

Let P_α be the weighted Bergman orthogonal projection from $L^2(\mathbb{B}_n, \mathrm{d}v_\alpha)$ onto $A^2_\alpha(\mathbb{B}_n)$, which is given by

$$(P_\alpha g)(w) = \langle g, K_w^{(\alpha)}\rangle_\alpha = \int_{\mathbb{B}_n} g(z) \frac{1}{(1 - \langle w, z\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(z),$$

for $g \in L^2(\mathbb{B}_n, \mathrm{d}v_\alpha)$ and $w \in \mathbb{B}_n$. For $f \in L^1(\mathbb{B}_n, \mathrm{d}v_\alpha)$, the Toeplitz operator with symbol f is defined by

$$(T_f h)(w) = P_\alpha(fh)(w) = \int_{\mathbb{B}_n} f(z)h(z)K_z^{(\alpha)}(w)\mathrm{d}v_\alpha(z),$$

for $h \in H^\infty(\mathbb{B}_n)$. Clearly, T_f is densely defined on $A^p_\alpha(\mathbb{B}_n)$.

Using the reproducing property of $K_w^{(\alpha)}$, we have

$$\|K_w^{(\alpha)}\|^2_{\alpha,2} = \langle K_w^{(\alpha)}, K_w^{(\alpha)}\rangle_\alpha = K_w^{(\alpha)}(w) = \frac{1}{(1 - |w|^2)^{n+\alpha+1}}.$$

Thus, for $z, w \in \mathbb{B}_n$, the normalized reproducing kernel is given by

$$k_w^{(\alpha)}(z) = \frac{(1 - |w|^2)^{\frac{n+\alpha+1}{2}}}{(1 - \langle z, w\rangle)^{n+\alpha+1}}.$$

For $w \in \mathbb{B}_n$, let φ_w be the automorphism of $\mathbb{B}_n$ given by

$$\varphi_z(w) = \frac{z - P_z(w) - s_z Q_z(w)}{1 - \langle w, z\rangle}, \quad z \in \mathbb{B}_n,$$

where $s_z = \sqrt{1 - |z|^2}$, P_z is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[z]$ generated by z, and Q_z is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [z]$. It is clear that

$$P_z(w) = \frac{\langle w, z\rangle}{|z|^2} z, \ z \in \mathbb{C}^n, \quad Q_z(w) = w - \frac{\langle w, z\rangle}{|z|^2} z, \ z \in \mathbb{B}_n.$$

We can see that $\varphi_w(0) = w$ and $\varphi_w^{-1} = \varphi_w$. The function φ_w has the real Jacobian equal to

$$|\varphi'_w(z)|^2 = \frac{(1 - |w|^2)^{n+1}}{|1 - \langle z, w\rangle|^{2n+2}}.$$

Thus we have the change-of-variable formula

$$\int_{\mathbb{B}_n} h(\varphi_w(z))|k_w^{(\alpha)}(z)|^2 \mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n} h(u)\mathrm{d}v_\alpha(u),$$

for every $h \in L^1(\mathbb{B}_n, \mathrm{d}v_\alpha)$. It follows from above that the mapping $U_w^{(\alpha)} h = (h \circ \varphi_w) k_w^{(\alpha)}$ is an isometry on $A^2_\alpha(\mathbb{B}_n)$:

$$\|U_w^{(\alpha)} h\|^2_{\alpha,2} = \int_{\mathbb{B}_n} |h(\varphi_w(z))|^2 |k_w^{(\alpha)}(z)|^2 \mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n} |h(u)|^2 \mathrm{d}v_\alpha(u) = \|h\|^2_{\alpha,2},$$

for all $h \in A^2_\alpha(\mathbb{B}_n)$. Using the identity

$$k_w^{(\alpha)}(\varphi_w(z)) = \frac{1}{k_w^{(\alpha)}(z)},$$

we have, for any $f, g \in A^2_\alpha(\mathbb{B}_n)$,

$$\begin{aligned}\langle f, U^{(\alpha)}_w g\rangle_\alpha &= \int_{\mathbb{B}_n} f(z)\overline{(g\circ\varphi_w)(z)k^{(\alpha)}_w(z)}\mathrm{d}v_\alpha(z)\\ &= \int_{\mathbb{B}_n} (f\circ\varphi_w)(\lambda)k^{(\alpha)}_w(\lambda)\overline{g(\lambda)}\mathrm{d}v_\alpha(\lambda) = \langle U^{(\alpha)}_w f, g\rangle_\alpha.\end{aligned}$$

So $U^{(\alpha)}_w$ is an adjoint operator.

Since $\varphi_w \circ \varphi_w = \mathrm{id}$, we see that

$$(U^{(\alpha)}_w(U^{(\alpha)}_w h))(z) = (U^{(\alpha)}_w h)(\varphi_w(z))k^{(\alpha)}_w(z) = h(z)k^{(\alpha)}_w(\varphi_w(z))k^{(\alpha)}_w(z) = h(z),$$

for all $z \in \mathbb{B}_n$ and $h \in A^2_\alpha(\mathbb{B}_n)$. Thus $(U^{(\alpha)}_w)^{-1} = U^{(\alpha)}_w$, and hence $U^{(\alpha)}_w$ is unitary on $A^2_\alpha(\mathbb{B}_n)$. Clearly, $U^{(\alpha)}_w$ is a bounded operator on $A^p_\alpha(\mathbb{B}_n)$.

For S a bounded operator on $A^p_\alpha(\mathbb{B}_n)$ for $1 < p < \infty$, we define S_z by $S_z = U^{(\alpha)}_z S U^{(\alpha)}_z$. The Berezin transform of S is the function $\widetilde{S}$ on $\mathbb{B}_n$ defined by

$$\widetilde{S}(z) = \langle S k^{(\alpha)}_z, k^{(\alpha)}_z\rangle_\alpha,$$

where $\langle f, g\rangle = \int_{\mathbb{B}_n} f\bar{g}\mathrm{d}v_\alpha(z)$, whenever $f\bar{g} \in L^1(\mathbb{B}_n, \mathrm{d}v_\alpha)$. Let $\widetilde{f}$ denote $\widetilde{T}_f$ and let

$$BT = \{f \in L^1 : \|f\|_{BT} = \sup_{z\in\mathbb{B}_n} \widetilde{|f|} < \infty\}.$$

In [1], Miao and Zheng proved that a operator S on the Bergman space of the unit disk is compact if and only if the Berezin transform of S vanishes on the boundary of the unit disk if S satisfies some integrable conditions. In this paper we extend the result to the general setting of the weighted Bergman space of the unit ball.

Throughout the paper we use p' to denote the conjugate of p, i.e., $(1/p) + (1/p') = 1$, for $1 < p < \infty$, and use p_1 to denote $\min\{p, p'\}$. The main results of the paper are stated as follows.

Theorem 1.1 *Suppose $\alpha > -1$, $1 < p < \infty$ and S is a bounded operator on $A^p_\alpha(\mathbb{B}_n)$ such that*

$$\sup_{z\in\mathbb{B}_n} \|S_z 1\|_{\alpha,m} < \infty, \quad \sup_{z\in\mathbb{B}_n} \|S^*_z 1\|_{\alpha,m} < \infty$$

for $m > \max\{3, 3(n+\alpha+1)/2(\alpha+1)(p_1-1)\}$. Then S is compact on $A^p_\alpha(\mathbb{B}_n)$ if and only if $\widetilde{S}(z) \longrightarrow 0$ as $z \to \partial\mathbb{B}_n$.

Theorem 1.2 *Suppose $\alpha > -1$ and S is a bounded operator on $A^2_\alpha(\mathbb{B}_n)$ such that*

$$\sup_{z\in\mathbb{B}_n} \|S_z 1\|_{\alpha,m} < \infty, \quad \sup_{z\in\mathbb{B}_n} \|S^*_z 1\|_{\alpha,m} < \infty$$

for $m > \max\{3, 3(n+\alpha+1)/2(\alpha+1)\}$. Then S is compact on $A^2_\alpha(\mathbb{B}_n)$ if and only if $\widetilde{S}(z) \longrightarrow 0$ as $z \to \partial\mathbb{B}_n$.

Theorem 1.3 *Suppose $\alpha > -1$, $1 < p < \infty$ and S is a finite sum of operators of the form $T_{f_1}\cdots T_{f_n}$, where each $f_j \in BT$. Then S is compact on $A^p_\alpha(\mathbb{B}_n)$ if and only if $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{B}_n$.*

2. The Berezin transform

The Berezin transform of a bounded operator on Bergman space $A^2_\alpha(\mathbb{B}_n)$ plays an important role and is one of the most useful tools in the study of Toeplitz operators.

For $z, w \in \mathbb{B}_n$, the distance in the Bergman metric on the unit ball is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

Let $D(z,r)$ denote the Bergman metric disk with center z and radius r. Thus

$$D(z,r) = \{w \in \mathbb{B}_n : \beta(z,w) < r\}, \quad z \in \mathbb{B}_n,\ r > 0.$$

We recall a positive Borel measure μ on $\mathbb{B}_n$ is a Carleson measure if there exist a constant $C > 0$ and $p > 0$ such that, for all f in $A^p_\alpha(\mathbb{B}_n)$,

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \le C \int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}v_\alpha(z).$$

The following Lemma 2.1 implies that a Carleson measure is independent of p.

Lemma 2.1 *Suppose $p > 0$, $r > 0$, $\alpha > -1$, and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following conditions are equivalent:*

(a) For all f in $A^p_\alpha(\mathbb{B}_n)$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \le C \int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}v_\alpha(z).$$

(b) For all $a \in \mathbb{B}_n$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} |k_a^{(\alpha)}(z)|^2 \mathrm{d}\mu(z) \le C.$$

(c) For all $a \in \mathbb{B}_n$, there exists a constant $C > 0$ such that

$$\mu(D(a,r)) \le C(1-|a|^2)^{n+1+\alpha}.$$

Lemma 2.2 *Suppose $\alpha > -1$ and $f \in L^1(\mathbb{B}_n, \mathrm{d}v_\alpha)$. Then $f \in BT$ if and only if $\mu = |f|\mathrm{d}v_\alpha$ is a Carleson measure on $\mathbb{B}_n$.*

Proof Since

$$\begin{aligned}\widetilde{|f|}(z) = \langle T_{|f|}k_z^{(\alpha)}, k_z^{(\alpha)}\rangle_\alpha &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} |f(u)|k_z^{(\alpha)}(u)k_u^{(\alpha)}(w)\mathrm{d}v_\alpha(u)\overline{k_z^{(\alpha)}(w)}\mathrm{d}v_\alpha(w)\\ &= \int_{\mathbb{B}_n} |f(u)||k_z^{(\alpha)}(u)|^2 \mathrm{d}v_\alpha(u),\end{aligned}$$

by Lemma 2.1 (b), the proof is completed. □

Lemma 2.3 *Suppose $\alpha > -1$, $1 < p < \infty$, $z \in \mathbb{B}_n$ and $f \in BT$. Then T_f is bounded on $A^p_\alpha(\mathbb{B}_n)$ and there is a constant C such that $\|T_f\|_{\alpha,p} \le C\|f\|_{BT}$.*

Proof It is well known that the dual of $A^p_\alpha(\mathbb{B}_n)$ is $A^{p'}_\alpha(\mathbb{B}_n)$. For $u, v \in H^\infty(\mathbb{B}_n)$, by Hölder's inequality and Fubini's theorem,

$$\langle T_f u, v\rangle_\alpha = \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{f(w)u(w)}{(1-\langle z,w\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(w)\overline{v(z)}\mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n} f(w)u(w)\overline{v(w)}\mathrm{d}v_\alpha(w).$$

Thus

$$\begin{aligned}|\langle T_f u, v\rangle_\alpha| &\le \int_{\mathbb{B}_n} |f||u||v|\mathrm{d}v_\alpha \le (\int_{\mathbb{B}_n} |u|^p|f|\mathrm{d}v_\alpha)^{1/p}(\int_{\mathbb{B}_n} |v|^{p'}|f|\mathrm{d}v_\alpha)^{1/p'}\\ &= (\int_{\mathbb{B}_n} |u|^p\mathrm{d}\mu)^{1/p}(\int_{\mathbb{B}_n} |v|^{p'}\mathrm{d}\mu)^{1/p'},\end{aligned}$$

where $\mathrm{d}\mu = |f|\mathrm{d}v_\alpha$. By Lemmas 2.1 and 2.2, there exists a constant C such that

$$|\langle T_f u, v\rangle_\alpha| \le C\|f\|_{BT}\|u\|_{\alpha,p}\|v\|_{\alpha,p'}$$

for $u, v \in H^\infty(\mathbb{B}_n)$, which shows that T_f is bounded on $A^p_\alpha(\mathbb{B}_n)$ and $\|T_f\|_{\alpha,p} \le C\|f\|_{BT}$.

Lemma 2.4 *If S is a bounded linear operator on $A^p_\alpha(\mathbb{B}_n)$, $\alpha > -1$ and $1 < p < \infty$, then $\widetilde{S}(\varphi_z(w)) = \widetilde{S_z}(w)$, $z, w \in \mathbb{B}_n$.*

Proof From the definition of the Berezin transform, we have

$$\widetilde{S_z}(w) = \langle S_z k^{(\alpha)}_w, k^{(\alpha)}_w\rangle_\alpha = \langle SU^{(\alpha)}_z k^{(\alpha)}_w, U^{(\alpha)}_z k^{(\alpha)}_w\rangle_\alpha = \langle S((k^{(\alpha)}_w\circ\varphi_z)k^{(\alpha)}_z), (k^{(\alpha)}_w\circ\varphi_z)k^{(\alpha)}_z\rangle_\alpha.$$

Using the identity

$$k^{(\alpha)}_{\varphi_z(w)} = \frac{(1-\langle z,w\rangle)^{n+\alpha+1}}{|1-\langle z,w\rangle|^{n+\alpha+1}}(k^{(\alpha)}_w\circ\varphi_z)k^{(\alpha)}_z,$$

we obtain

$$\begin{aligned}\widetilde{S}(\varphi_z(w)) &= \langle Sk^{(\alpha)}_{\varphi_z(w)}, k^{(\alpha)}_{\varphi_z(w)}\rangle_\alpha = \Big|\frac{(1-\langle z,w\rangle)^{n+\alpha+1}}{|1-\langle z,w\rangle|^{n+\alpha+1}}\Big|^2\langle S((k^{(\alpha)}_w\circ\varphi_z)k^{(\alpha)}_z), (k^{(\alpha)}_w\circ\varphi_z)k^{(\alpha)}_z\rangle_\alpha\\ &= \widetilde{S_z}(w).\end{aligned}$$

Lemma 2.5 *For any $z \in \mathbb{B}_n$, if $f \in BT$ and $p > 1$, then on $A^p_\alpha(\mathbb{B}_n)$, $T_{f\circ\varphi_z}U^{(\alpha)}_z = U^{(\alpha)}_z T_f$.*

Proof Since for any $u \in A^p_\alpha(\mathbb{B}_n)$ and $v \in H^\infty(\mathbb{B}_n)$, $\langle u, U^{(\alpha)}_z v\rangle_\alpha = \langle U^{(\alpha)}_z u, v\rangle_\alpha$, we have for any $u, v \in H^\infty(\mathbb{B}_n)$, $\langle U^{(\alpha)}_z T_f u, U^{(\alpha)}_z v\rangle_\alpha = \langle T_f u, v\rangle_\alpha = \int_{\mathbb{B}_n} f(\varphi_z(w))u(\varphi_z(w))\overline{v(\varphi_z(w))}|k^{(\alpha)}_z(w)|^2\mathrm{d}v_\alpha(w)$ $= \langle T_{f\circ\varphi_z}U^{(\alpha)}_z u, U^{(\alpha)}_z v\rangle_\alpha$. Note that $H^\infty(\mathbb{B}_n)$ is dense in $A^p_\alpha(\mathbb{B}_n)$ and $U^{(\alpha)}_z$ is surjective on $H^\infty(\mathbb{B}_n)$. The proof is completed. □

Lemma 2.6 *Suppose $f \in L^1(\mathbb{B}_n, \mathrm{d}v_\alpha)$, $z \in \mathbb{B}_n$ and $\alpha > -1$, then $\widetilde{f\circ\varphi_z} = \widetilde{f}\circ\varphi_z$.*

Proof From Lemmas 2.4 and 2.5, we have $\widetilde{f}\circ\varphi_z = \widetilde{T_f}\circ\varphi_z = \widetilde{(T_f)_z} = \widetilde{U^{(\alpha)}_z T_f U^{(\alpha)}_z} = \widetilde{T_{f\circ\varphi_z}} = \widetilde{f\circ\varphi_z}$.

Lemma 2.7 *Suppose $\alpha > -1$, $1 < p < \infty$ and $f \in BT$, $z \in \mathbb{B}_n$, then $T_{f\circ\varphi_z}$ is bounded on $A^p_\alpha(\mathbb{B}_n)$ and there is a constant C independent of z such that $\|T_{f\circ\varphi_z}\|_{\alpha,p} \le C\|f\|_{BT}$.*

Proof By Lemma 2.6, $\|f\circ\varphi_z\|_{BT} = \sup_{w\in\mathbb{B}_n}\widetilde{|f\circ\varphi_z|}(w) = \sup_{w\in\mathbb{B}_n}\widetilde{|f|}(\varphi_z(w)) = \|f\|_{BT}$, thus $f\circ\varphi_z \in BT$. According to Lemma 2.3, the proof of the lemma is completed. □

Lemma 2.8 *Let $\alpha > -1$, and S be a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in BT$. Then for every $p \in (1, \infty)$,*

$$\sup_{z \in \mathbb{B}_n} \|S_z 1\|_{\alpha,p} < \infty, \quad \sup_{z \in \mathbb{B}_n} \|S_z^* 1\|_{\alpha,p} < \infty.$$

Proof Without loss of generality we may assume that $S = T_{f_1} \cdots T_{f_n}$. Then $S_z = U_z^{(\alpha)} S U_z^{(\alpha)} = U_z^{(\alpha)} T_{f_1} U_z^{(\alpha)} U_z^{(\alpha)} T_{f_2} U_z^{(\alpha)} \cdots U_z^{(\alpha)} T_{f_n} U_z^{(\alpha)} = T_{f_1 \circ \varphi_z} T_{f_2 \circ \varphi_z} \cdots T_{f_n \circ \varphi_z}$. By Lemma 2.7, $\|S_z 1\|_{\alpha,p} = \|T_{f_1 \circ \varphi_z} T_{f_2 \circ \varphi_z} \cdots T_{f_n \circ \varphi_z} 1\|_{\alpha,p} \le C \|f_1\|_{BT} \cdots \|f_n\|_{BT}$. Clearly, each $\bar{f}_j \in BT$ and $\|\bar{f}_j\|_{BT} = \|f_j\|_{BT}$. Thus $\|S_z^* 1\|_{\alpha,p} = \|T_{\bar{f}_n \circ \varphi_z} T_{\bar{f}_{n-1} \circ \varphi_z} \cdots T_{\bar{f}_1 \circ \varphi_z} 1\|_{\alpha,p} \le C \|f_1\|_{BT} \cdots \|f_n\|_{BT}$.

3. Some lemmas

This section takes up some basic lemmas needed for the main theorem. The following lemma is the Theorem 1.12 in [2].

Lemma 3.1 *Suppose c is real and $t > -1$. Then the integral*

$$J_{c,t}(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t \mathrm{d}v(w)}{|1 - \langle z, w \rangle|^{n+1+t+c}}, \quad z \in \mathbb{B}_n,$$

has the following asymptotic properties.

(i) If $c < 0$, then $J_{c,t}$ is bounded in $\mathbb{B}_n$.

(ii) If $c = 0$, then

$$J_{c,t}(z) \sim \log \frac{1}{1 - |z|^2}, \quad |z| \to 1^-.$$

(iii) If $c > 0$, then $J_{c,t}(z) \sim (1 - |z|^2)^{-c}$, $|z| \to 1^-$.

Lemma 3.2 *Suppose $\alpha > -1$, $1 < p < \infty$, $0 < a < \alpha + 1$ and $1 < s < \min\{(\alpha+1)/a, (n+1+\alpha)/(n+\alpha+1-a)\}$. Then there exists a constant C such that if S is a bounded operator on $A_\alpha^p(\mathbb{B}_n)$, then*

$$\int_{\mathbb{B}_n} \frac{|(SK_z^{(\alpha)})(w)|}{(1 - |w|^2)^a} \mathrm{d}v_\alpha(w) \le \frac{C \|S_z 1\|_{\alpha,s'}}{(1 - |z|^2)^a} \tag{3.1}$$

for all $z \in \mathbb{B}_n$ and

$$\int_{\mathbb{B}_n} \frac{|(SK_z^{(\alpha)})(w)|}{(1 - |z|^2)^a} \mathrm{d}v_\alpha(z) \le \frac{C \|S_w^* 1\|_{\alpha,s'}}{(1 - |w|^2)^a} \tag{3.2}$$

for all $w \in \mathbb{B}_n$.

Proof To prove (3.1), fix $z \in \mathbb{B}_n$. From the definition of S_z and $U_z^{(\alpha)}$, we have

$$SK_z^{(\alpha)} = \frac{SU_z^{(\alpha)} 1}{(1 - |z|^2)^{(n+\alpha+1)/2}} = \frac{U_z^{(\alpha)} S_z 1}{(1 - |z|^2)^{(n+\alpha+1)/2}} = \frac{((S_z 1) \circ \varphi_z) k_z^{(\alpha)}}{(1 - |z|^2)^{(n+\alpha+1)/2}}.$$

Thus

$$\int_{\mathbb{B}_n} \frac{|(SK_z^{(\alpha)})(w)|}{(1 - |w|^2)^a} \mathrm{d}v_\alpha(w) = \frac{1}{(1 - |z|^2)^{(n+\alpha+1)/2}} \int_{\mathbb{B}_n} \frac{|(S_z 1)(\varphi_z(w))| |k_z^{(\alpha)}(w)|}{(1 - |w|^2)^a} \mathrm{d}v_\alpha(w).$$

In the last integral, make the change of variable $w=\varphi_z(\lambda)$ to obtain

$$\int_{\mathbb{B}_n}\frac{|(SK_z^{(\alpha)})(w)|}{(1-|w|^2)^a}\mathrm{d}v_\alpha(w)=\frac{1}{(1-|z|^2)^{(n+\alpha+1)/2}}\int_{\mathbb{B}_n}\frac{|(S_z1)(\lambda)|\cdot|k_z^{(\alpha)}(\varphi_z(\lambda))|}{(1-|\varphi_z(\lambda)|^2)^a}\cdot|k_z^{(\alpha)}(\lambda)|^2\mathrm{d}v_\alpha(\lambda).$$

Using the identities

$$k_w^{(\alpha)}(\varphi_w(z))=\frac{1}{k_w^{(\alpha)}(z)},\quad 1-|\varphi_z(\lambda)|^2=\frac{(1-|z|^2)(1-|\lambda|^2)}{|1-\langle\lambda,z\rangle|^2},$$

we have

$$\int_{\mathbb{B}_n}\frac{|(SK_z^{(\alpha)})(w)|}{(1-|w|^2)^a}\mathrm{d}v_\alpha(w)=\frac{1}{(1-|z|^2)^a}\int_{\mathbb{B}_n}\frac{|(S_z1)(\lambda)|\mathrm{d}v_\alpha(\lambda)}{(1-|\lambda|^2)^a|1-\langle\lambda,z\rangle|^{n+\alpha+1-2a}}.$$

Applying Hölder's inequality to the integral above, we get

$$\int_{\mathbb{B}_n}\frac{|(SK_z^{(\alpha)})(w)|}{(1-|w|^2)^a}\mathrm{d}v_\alpha(w)\le\frac{\|S_z1\|_{\alpha,s'}}{(1-|z|^2)^a}\Big(\int_{\mathbb{B}_n}\frac{\mathrm{d}v_\alpha(\lambda)}{(1-|\lambda|^2)^{as}|1-\langle\lambda,z\rangle|^{(n+\alpha+1)s-2as}}\Big)^{1/s}.$$

Thus (3.1) follows from Lemma 3.1. To prove (3.2), replace S by S^* in (3.1), interchange w and z in (3.1) and then use the equation

$$(S^*K_w^{(\alpha)})(z)=\langle S^*K_w^{(\alpha)},K_z^{(\alpha)}\rangle_\alpha=\langle K_w^{(\alpha)},SK_z^{(\alpha)}\rangle_\alpha=\overline{SK_z^{(\alpha)}(w)}. \tag{3.3}$$

The following Schur's test in [3] is well known.

Lemma 3.3 *Suppose (X,μ) is a measure space, $1<p<\infty$, and $1/p+1/p'=1$. For a measurable function $H(x,y)$ consider the integral operator*

$$Tf(x)=\int_X H(x,y)f(y)\mathrm{d}\mu(y).$$

If there exist a positive function h on X and a positive constant C such that

$$\int_X|H(x,y)|h(y)^{p'}\mathrm{d}\mu(y)\le C_1h(x)^{p'}$$

for almost all $x\in X$, and

$$\int_X|H(x,y)|h(x)^{p}\mathrm{d}\mu(x)\le C_2h(y)^{p}$$

for almost all $y\in X$, then the operator T is bounded on $L^p(X,\mu)$ with $\|T\|_p\le(C_1)^{1/p'}(C_2)^{1/p}$.

Lemma 3.4 *Let $\alpha>-1$ and $1<p<\infty$ and S be a bounded operator on $A_\alpha^p(\mathbb{B}_n)$. If*

$$C_1=\sup_{z\in\mathbb{B}_n}\|S_z1\|_{\alpha,m}<\infty, C_2=\sup_{z\in\mathbb{B}_n}\|S_z^*1\|_{\alpha,m}<\infty$$

for $m>\max\{3,3(n+\alpha+1)/2(\alpha+1)(p_1-1)\}$, then there is a constant C such that

$$\|S\|_{\alpha,p}\le C(C_1)^{1/p}(C_2)^{1/p'}.$$

Proof For $f\in A_\alpha^p(\mathbb{B}_n)$ and $w\in\mathbb{B}_n$, we have

$$(Sf)(w)=\langle Sf,K_w^{(\alpha)}\rangle_\alpha=\int_{\mathbb{B}_n}f(z)\overline{(S^*K_w^{(\alpha)})}(z)\mathrm{d}v_\alpha(z)=\int_{\mathbb{B}_n}f(z)(SK_z^{(\alpha)})(w)\mathrm{d}v_\alpha(z).$$

To finish the proof, we just need to find the right test function $h(z)$ and apply Lemma 3.3. Choose $h(z)=1/(1-|z|^2)^\beta$, where $\beta=2(\alpha+1)(p_1-1)/3p_1$ and $p_1=\min\{p,p'\}$. It is easy to see that $0 < \beta < \min\{(\alpha+1)/p, (\alpha+1)/p'\}$, then $0 < \max\{\beta p, \beta p'\} < \alpha + 1$. It also follows from a simple computation that

$$\min\{\frac{\alpha+1}{\beta p}, \frac{(n+\alpha+1)}{(n+\alpha+1-\beta p)}\} = \begin{cases} \min\{\frac{3(n+\alpha+1)}{3(n+\alpha+1)-2(\alpha+1)(p-1)}, \frac{3}{2}\}, & p < 2, \\ \min\{\frac{3(n+\alpha+1)}{3(n+\alpha+1)-2(\alpha+1)}, \frac{3}{2}\}, & p \geq 2. \end{cases}$$

$$\geq \min\{\frac{3(n+\alpha+1)}{3(n+\alpha+1)-2(\alpha+1)(p_1-1)}, \frac{3}{2}\}.$$

Similarly, we can show that

$$\min\{\frac{\alpha+1}{\beta p'}, \frac{(n+\alpha+1)}{(n+\alpha+1-\beta p')}\} \geq \min\{\frac{3(n+\alpha+1)}{3(n+\alpha+1)-2(\alpha+1)(p_1-1)}, \frac{3}{2}\}.$$

Let $s = m'$. Then $m = s'$. For $m > \max\{3, \frac{3(n+\alpha+1)}{2(\alpha+1)(p_1-1)}\}$, $1 < s < \min\{\frac{3}{2}, \frac{3(n+\alpha+1)}{3(n+\alpha+1)-2(\alpha+1)(p_1-1)}\}$. Thus from Lemma 3.2, we have

$$\int_{\mathbb{B}_n} |(SK_z^{(\alpha)}(w))| h(z)^{p'} \mathrm{d}v_\alpha(z) \leq C\|S_w^* 1\|_{s'} h(w)^{p'}$$

$$\int_{\mathbb{B}_n} |(SK_z^{(\alpha)}(w))| h(w)^{p} \mathrm{d}v_\alpha(w) \leq C\|S_z 1\|_{s'} h(z)^{p}.$$

The conclusion of the proposition now follows from Lemma 3.3.

4. Proof of main results

Lemma 4.1 *Suppose $\alpha > -1$ and $1 < p < \infty$, then*

(a) $\|K_z^{(\alpha)}\|_{\alpha,p}$ is equivalent to $(1-|z|^2)^{-(n+\alpha+1)/p'}$ for all $z \in \mathbb{B}_n$.

(b) $K_z^{(\alpha)}/\|K_z^{(\alpha)}\|_{\alpha,p} \to 0$ weakly in $A_\alpha^p(\mathbb{B}_n)$ as $z \to \partial\mathbb{B}_n$.

Proof (a) Note that

$$\|K_z^{(\alpha)}\|_{\alpha,p}^p = \int_{\mathbb{B}_n} |K_z^{(\alpha)}(w)|^p \mathrm{d}v_\alpha(w) = \int_{\mathbb{B}_n} |\frac{1}{(1-\langle w,z\rangle)^{n+\alpha+1}}|^p \mathrm{d}v_\alpha(w)$$
$$= \int_{\mathbb{B}_n} \frac{|1-|w|^2|^\alpha \mathrm{d}v(w)}{|1-\langle z,w\rangle|^{p(n+\alpha+1)}}.$$

Applying Lemma 3.1 gives

$$\|K_z^{(\alpha)}\|_{\alpha,p} \sim (1-|z|^2)^{\frac{-(n+\alpha+1)}{p'}}.$$

(b) Let $g \in H^\infty(\mathbb{B}_n)$. Then

$$|\langle g, \frac{K_z^{(\alpha)}}{\|K_z^{(\alpha)}\|_{\alpha,p}}\rangle_\alpha| \sim |g(z)| \cdot (1-|z|^2)^{\frac{n+\alpha+1}{p'}} \to 0,$$

as $|z| \to 1^-$. Since $H^\infty(\mathbb{B}_n)$ is dense in $A_\alpha^{p'}(\mathbb{B}_n)$, $K_z^{(\alpha)}/\|K_z^{(\alpha)}\|_{\alpha,p} \to 0$ weakly in $A_\alpha^p(\mathbb{B}_n)$ as $z \to \partial\mathbb{B}_n$.

See Ex. 7 on Page 181 of [3] for the following lemma.

Lemma 4.2 *Suppose $1 < p < \infty$ and $K(z,w)$ is a measurable function on $X \times X$ such that*

$$\int_X \Big(\int_X |K(z,w)|^p \mathrm{d}v(w)\Big)^{p'-1} \mathrm{d}v(z) < \infty.$$

Then the integral operator T defined by

$$Tf(w) = \int_X f(z)K(z,w)\mathrm{d}v(z)$$

is compact on L^p.

To write the Berezin transform $\tilde{S}(z)$ precisely, we need a power series formula for the Berezin transform of a bounded operator S on $A^2_\alpha(\mathbb{B}_n)$. From the definition of the reproducing kernel we get

$$k_z^{(\alpha)}(w) = \frac{(1-|z|^2)^{(n+\alpha+1)/2}}{(1-\langle w,z\rangle)^{n+\alpha+1}} = (1-|z|^2)^{(n+\alpha+1)/2} \cdot \sum_{k=0}^{\infty} C_{\alpha,k} \langle w,z\rangle^k,$$

where $C_{\alpha,k} = \frac{(n+\alpha+1)\cdots(n+\alpha+k)}{k!}$. Considering the multinomial formula

$$(z_1 + \cdots + z_n)^N = \sum_{|P|=N} \frac{N!}{P!} z^P,$$

we obtain

$$\begin{aligned}
\widetilde{S}(z) &= \langle Sk_z^{(\alpha)}, k_z^{(\alpha)}\rangle_\alpha = (1-|z|^2)^{n+\alpha+1}\langle SK_z^{(\alpha)}, K_z^{(\alpha)}\rangle_\alpha \\
&= (1-|z|^2)^{n+\alpha+1}\langle S\sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} w^\gamma \bar{z}^\gamma, \sum_{l=0}^{\infty} C_{\alpha,l} \sum_{|\beta|=l} \frac{l!}{\beta!} w^\beta \bar{z}^\beta\rangle_\alpha \\
&= (1-|z|^2)^{n+\alpha+1} \sum_{k,l=0}^{\infty} C_{\alpha,k}\overline{C_{\alpha,l}} \cdot \sum_{|\gamma|=k} \frac{k!}{\gamma!} \sum_{|\beta|=l} \frac{l!}{\beta!} \langle Sw^\gamma, w^\beta\rangle_\alpha \bar{z}^\gamma z^\beta.
\end{aligned}$$

The following lemma is key to the proof of Theorem 1.1. While the way of the proof of Lemma 4.3 is basically adapted from Lemma 14 of [1], substantial amount of extra work is necessary for the setting of weighted Bergman spaces of the unit ball.

Lemma 4.3 *Let $\alpha > -1$ and S be a bounded operator on $A^p_\alpha(\mathbb{B}_n)$ for some $p \in (1,\infty)$ such that*

$$\sup_{z\in\mathbb{B}_n} \|S_z 1\|_{\alpha,m} < \infty$$

for some $m > 1$. Then $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{B}_n$ if and only if for every $t \in [1,m)$, $\|S_z 1\|_{\alpha,t} \to 0$ as $z \to \partial\mathbb{B}_n$.

Proof Suppose that for every $t \in [1,m)$, $\|S_z 1\|_{\alpha,t} \to 0$ as $z \to \partial\mathbb{B}_n$. In particular, $\|S_z 1\|_{\alpha,1} \to 0$ as $z \to \partial\mathbb{B}_n$. Thus $|\widetilde{S}(z)| = |\langle Sk_z^{(\alpha)}, k_z^{(\alpha)}\rangle_\alpha| = |\langle SU_z^{(\alpha)}1, U_z^{(\alpha)}1\rangle_\alpha| = |\langle S_z 1, 1\rangle_\alpha| \le \|S_z 1\|_{\alpha,1} \to 0$ as $z \to \partial\mathbb{B}_n$. Suppose $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{B}_n$. Fix $t \in [1,m)$. We will show that $\|S_z 1\|_{\alpha,t} \to 0$ as $z \to \partial\mathbb{B}_n$. For $z \in \mathbb{B}_n$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $\gamma_i, \beta_i \in \mathrm{Z}_+$, we have

$$\begin{aligned}
|\langle S_z w^\gamma, w^\beta\rangle_\alpha| &= |\langle SU_z^{(\alpha)} w^\gamma, U_z^{(\alpha)} w^\beta\rangle_\alpha| = |(1-|z|^2)^{n+\alpha+1}\langle S(w^\gamma \circ \varphi_z K_z^{(\alpha)}), w^\beta \circ \varphi_z K_z^{(\alpha)}\rangle_\alpha| \\
&\le (1-|z|^2)^{n+\alpha+1}\|S\|_{\alpha,p}\|w^\gamma \circ \varphi_z K_z^{(\alpha)}\|_{\alpha,p}\|w^\beta \circ \varphi_z K_z^{(\alpha)}\|_{\alpha,p'}
\end{aligned}$$

$$\leq (1-|z|^2)^{n+\alpha+1}\|S\|_{\alpha,p}\|K_z^{(\alpha)}\|_{\alpha,p}\|K_z^{(\alpha)}\|_{\alpha,p'} \leq C\|S\|_{\alpha,p} = M,$$

where the last inequality comes from Lemma 4.1(a).

First we show that $\langle S_z 1, w^\beta\rangle_\alpha \to 0$ as $z \to \partial\mathbb{B}_n$ for every multi-index β. If this is not true, then there is a sequence $z_k \in \mathbb{B}_n$ such that $\langle S_{z_k} 1, w^{\beta_0}\rangle_\alpha \to a_{0\beta_0}$ as $|z_k| \to 1$ for some nonzero constant $a_{0\beta_0}$ and some $\beta_0 = (\beta_{01}, \beta_{02}, \dots, \beta_{0n})$, $\beta_{0i} \in \mathrm{Z}_+$. We have showed that $|\langle S_z w^\gamma, w^\beta\rangle_\alpha|$ is uniformly bounded for $z \in \mathbb{B}_n$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, $\gamma_i, \beta_i \in \mathrm{Z}_+$.

Without loss of generality we may assume that for each γ, β and some constant $a_{\gamma\beta}$, $\langle S_{z_k} w^\gamma, w^\beta\rangle_\alpha \to a_{\gamma\beta}$. For $z, \lambda \in \mathbb{B}_n$, by Lemma 2.4, we have

$$\widetilde{S}_z(\lambda) = \widetilde{S}(\varphi_z(\lambda)) = (1-|\lambda|^2)^{n+\alpha+1}\sum_{k,l=0}^{\infty} C_{\alpha,k}\overline{C_{\alpha,l}} \cdot \sum_{|\gamma|=k}\frac{k!}{\gamma!}\sum_{|\beta|=l}\frac{l!}{\beta!}\langle S_z w^\gamma, w^\beta\rangle_\alpha \bar{\lambda}^\gamma \lambda^\beta. \tag{4.1}$$

For each $\lambda \in \mathbb{B}_n$, it is easy to see that $\varphi_{z_k}(\lambda) \to \partial\mathbb{B}_n$ as $z_k \to \partial\mathbb{B}_n$. Thus $\widetilde{S}(\varphi_{z_k}(\lambda)) \to 0$ as $z_k \to \partial\mathbb{B}_n$ for each $\lambda \in \mathbb{B}_n$. Since

$$\begin{aligned}
&\sum_{k,l=0}^{\infty}\sum_{|\gamma|=k}\sum_{|\beta|=l}|C_{\alpha,k}\overline{C_{\alpha,l}}\frac{k!}{\gamma!}\frac{l!}{\beta!}\lambda^\gamma\lambda^\beta \cdot \langle S_z w^\gamma, w^\beta\rangle_\alpha| \\
&\quad \leq \sum_{k,l=0}^{\infty}\sum_{|\gamma|=k}\sum_{|\beta|=l}|C_{\alpha,k}\overline{C_{\alpha,l}}\frac{k!}{\gamma!}\frac{l!}{\beta!}\lambda^\gamma\lambda^\beta| \cdot |\langle S_z w^\gamma, w^\beta\rangle_\alpha| \\
&\quad \leq M\sum_{k,l=0}^{\infty}\sum_{|\gamma|=k}\sum_{|\beta|=l}C_{\alpha,k}\overline{C_{\alpha,l}}\frac{k!}{\gamma!}\frac{l!}{\beta!}|\lambda|^\gamma|\lambda|^\beta = M[1-(|\lambda_1|+\cdots+|\lambda_n|)]^{-2(n+\alpha+1)},
\end{aligned}$$

the power series above converges absolutely. Replacing z by z_k in (4.1) and taking the limit as $z_k \to \partial\mathbb{B}_n$, we have

$$(1-|\lambda|^2)^{n+\alpha+1}\sum_{k,l=0}^{\infty} C_{\alpha,k}\overline{C_{\alpha,l}} \cdot \sum_{|\gamma|=k}\frac{k!}{\gamma!}\sum_{|\beta|=l}\frac{l!}{\beta!}a_{\gamma\beta}\bar{\lambda}^\gamma\lambda^\beta = 0$$

for each $\lambda \in \mathbb{B}_n$. Let

$$f(\lambda) = \sum_{k=0}^{\infty} C_{\alpha,k}\overline{C_{\alpha,l}} \cdot \sum_{|\gamma|=k}\frac{k!}{\gamma!}\sum_{|\beta|=l}\frac{l!}{\beta!}a_{\gamma\beta}\bar{\lambda}^\gamma\lambda^\beta.$$

Then $f(\lambda) = 0$ for all $\lambda \in \mathbb{B}_n$. This gives

$$\frac{\partial^{|\beta|+|\gamma|}f}{\partial\lambda^\beta\partial\bar{\lambda}^\gamma}(0) = 0$$

for each γ, β. On the other hand, we have for each γ, β

$$\frac{\partial^{|\beta|+|\gamma|}f}{\partial\lambda^\beta\partial\bar{\lambda}^\gamma}(0) = C_{\alpha,k}\overline{C_{\alpha,l}}\frac{k!}{\gamma!}\frac{l!}{\beta!}a_{\gamma\beta}.$$

In particular, $a_{0\beta_0} = 0$. This is a contradiction. Hence we obtain $\lim_{z\to\partial\mathbb{B}_n}\langle S_z 1, w^\beta\rangle_\alpha = 0$.

For $\lambda \in \mathbb{B}_n$, we have

$$(S_z 1)(\lambda) = \langle S_z 1, \frac{1}{(1-\langle w,\lambda\rangle)^{n+\alpha+1}}\rangle_\alpha = \langle S_z 1, \sum_{k=0}^{\infty} C_{\alpha,k}(\langle w,\lambda\rangle)^k\rangle_\alpha$$

$$= \langle S_z 1, \sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} w^\gamma \bar{\lambda}^\gamma \rangle_\alpha = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} \overline{C_{\alpha,k}} \frac{k!}{\gamma!} \langle S_z 1, w^\gamma \rangle_\alpha \lambda^\gamma.$$

Since

$$\sum_{k=0}^{\infty} \sum_{|\gamma|=k} |\overline{C_{\alpha,k}} \frac{k!}{\gamma!} \langle S_z 1, w^\gamma \rangle_\alpha \lambda^\gamma| \le \sum_{k=0}^{\infty} \sum_{|\gamma|=k} |\overline{C_{\alpha,k}} \frac{k!}{\gamma!} \lambda^\gamma| |\langle S_z 1, w^\gamma \rangle_\alpha|$$
$$\le M \sum_{k=0}^{\infty} \sum_{|\gamma|=k} \overline{C_{\alpha,k}} \frac{k!}{\gamma!} |\lambda|^\gamma = M[1 - (|\lambda_1| + \cdots + |\lambda_n|)]^{-(n+\alpha+1)}$$

for each fixed $\lambda \in \mathbb{B}_n$, we can see the power series above converges uniformly for $z \in \mathbb{B}_n$. This gives $\lim_{z\to\partial\mathbb{B}_n} (S_z 1)(\lambda) = 0$ for each $\lambda \in \mathbb{B}_n$. Thus $\lim_{z\to\partial\mathbb{B}_n} |(S_z 1)(\lambda)|^t = 0$ for each $\lambda \in \mathbb{B}_n$ and $t \in [1, m)$. Let $s = m/t$. Then $s > 1$. Thus

$$\int_{\mathbb{B}_n} [|(S_z 1)(\lambda)|^t]^s \mathrm{d}v_\alpha(\lambda) = \|S_z 1\|_{\alpha,m}^m \le \sup_{z\in\mathbb{B}_n} \|S_z 1\|_{\alpha,m}^m < \infty.$$

This implies that $\{|S_z 1|^t\}_{z\in\mathbb{B}_n}$ is uniformly integrable. By Ex.10 (Vitali's Theorem) on Page 133 of [4], $\lim_{z\to\partial\mathbb{B}_n} \|S_z 1\|_{\alpha,t} = 0$.

Proof of Theorem 1.1 If S is compact on $A^p_\alpha(\mathbb{B}_n)$, then by Lemma 4.1 (b), $\langle SK_z^{(\alpha)}/\|K_z^{(\alpha)}\|_{\alpha,p}, K_z^{(\alpha)}/\|K_z^{(\alpha)}\|_{\alpha,p'}\rangle_\alpha \to 0$ as $z \to \partial\mathbb{B}_n$. By Lemma 4.1(a), it is easy to see that $\widetilde{S}(z)$ is equivalent to $\langle SK_z^{(\alpha)}/\|K_z^{(\alpha)}\|_{\alpha,p}, K_z^{(\alpha)}/\|K_z^{(\alpha)}\|_{\alpha,p'}\rangle_\alpha$ for $z \in \mathbb{B}_n$. Thus $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{B}_n$.

Suppose that $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{B}_n$. By Lemma 4.3 we have that $\|S_z 1\|_{\alpha,t} \to 0$ as $z \to \partial\mathbb{B}_n$ for every $t \in [1, m)$. We will show that S is compact on $A^p_\alpha(\mathbb{B}_n)$. Fix t such that $\max\{3, 3(n+\alpha+1)/2(\alpha+1)(p_1-1)\} < t < m$ in the rest of the proof.

For $f \in A^p_\alpha(\mathbb{B}_n)$ and $w \in \mathbb{B}_n$, we have from the proof of Lemma 3.4,

$$(Sf)(w) = \int_{\mathbb{B}_n} f(z)(SK_z^{(\alpha)})(w)\mathrm{d}v_\alpha(z).$$

For $0 < r < 1$, define an operator S_r on $A^p_\alpha(\mathbb{B}_n)$ by

$$(S_r f)(w) = \int_{r\mathbb{B}_n} f(z)(SK_z^{(\alpha)})(w)\mathrm{d}v_\alpha(z).$$

In other words, S_r is the integral operator with kernel $(SK_z^\alpha)(w)\chi_{r\mathbb{B}_n}(z)$. We will use Lemma 4.2 to show that S_r is compact on A^p_α. Let

$$I_p(r) = \int_{\mathbb{B}_n} \Big(\int_{\mathbb{B}_n} |(SK_z^{(\alpha)})(w)\chi_{r\mathbb{B}_n}(z)|^p \mathrm{d}v_\alpha(w)\Big)^{(p'-1)} \mathrm{d}v_\alpha(z).$$

By Lemma 4.1(a),

$$I_p(r) = \int_{r\mathbb{B}_n} \Big(\int_{\mathbb{B}_n} |(SK_z^{(\alpha)})(w)|^p \mathrm{d}v_\alpha(w)\Big)^{(p'-1)} \mathrm{d}v_\alpha(z)$$
$$\le \|S\|_{\alpha,p}^{p'} \int_{r\mathbb{B}_n} \|K_z^{(\alpha)}\|_{\alpha,p}^{p'} \mathrm{d}v_\alpha(z) \le C\|S\|_{\alpha,p}^{p'} \int_{r\mathbb{B}_n} \frac{\mathrm{d}v_\alpha(z)}{(1-|z|^2)^{n+\alpha+1}} < \infty.$$

Thus from Lemma 4.2, S_r is compact on $A^p_\alpha(\mathbb{B}_n)$. Hence to prove that S is compact, we only need to show that $\|S - S_r\|_{\alpha,p} \to 0$ as $r \to 1^-$.

If $r \in (0,1)$, then $S - S_r$ is the integral operator with kernel $(SK_z)(w)\chi_{\mathbb{B}_n\setminus r\mathbb{B}_n}(z)$, as can be seen from above. The proof of Lemma 3.4 indicates that $\|S - S_r\|_{\alpha,p} \le C(C_1)^{1/p}(C_2)^{1/p'}$, where

$$C_1 = \sup\{\|S_z 1\|_{\alpha,t} : r \le |z| < 1\}, C_2 = \sup\{\|S_z^* 1\|_{\alpha,t} : z \in \mathbb{B}_n\}.$$

We have showed above that $C_1 \to 0$ as $r \to 1^-$. The hypothesis of the theorem gives that $C_2 < \infty$. Thus $\|S - S_r\|_{\alpha,p} \to 0$ as $r \to 1^-$, completing the proof. □

Proof of Theorem 1.3 Suppose S is a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in BT$. By Lemmas 2.3 and 2.8, we have that S is bounded on $A^p_\alpha(\mathbb{B}_n)$ for $1 < p < \infty$ and

$$\sup_{z\in\mathbb{B}_n} \|S_z 1\|_{\alpha,m} < \infty, \sup_{z\in\mathbb{B}_n} \|S_z^* 1\|_{\alpha,m} < \infty,$$

for all $1 < m < \infty$. Hence Theorem 1.3 follows from Theorem 1.1. □

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