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Maximum Hexagon Packing of $K_v - F$ Where F is a Spanning Forest

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Abstract In this paper, we extend the result of packing the complete graph K_v with 6-cycles (hexagons). Mainly, the maximum packing of $K_v - F$ is obtained where the leave is an odd spanning forest.

Keywords 6-cycle (Hexagon); leave; complete graph; forest; packing.

MR(2010) Subject Classification 05C38

1. Introduction

An H-decomposition of the graph G is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. In the case where H is an m-cycle, such a decomposition is referred to as an m-cycle system of G. An m-cycle system of G will be formally described as an ordered pair (V, B), where V is the vertex set of G and B is the set of m-cycles.

A packing of a graph G with m-cycles is an m-cycle system of a subgraph P of G. The remainder graph of this packing, also known as the leave, is the subgraph $G - P$ formed from G by removing the edges in P. If the remainder graph is empty, we have an m-cycle system of the graph G. If the remainder graph is minimum in size (that is, has the least number of edges among all possible leaves of G), then the packing is called a maximum packing. All packings we consider in this paper are hexagon packings unless otherwise noted.

Hanani [3] showed the remainder graphs P for any maximum packing of K_v with triangles are in Table1:

$v \pmod 6$	0	1	2	3	4	5
P	F	$\emptyset$	F	$\emptyset$	F_1	C_4

Table 1 Relation between P and v

F is a 1-factor, F_1 is an odd spanning forest with $\frac{v}{2}+1$ edges (tripole), and C_4 is a cycle of length four.

Research on H-decomposition of a graph G dates back to the nineteenth century [5], and has received a lot of attention over the past 40 years. There have been many results found on

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H-decompositions of G for various graphs H and G, usually with $G = K_v$. One particularly enticing but difficult problem is to solve the case when H is an m-cycle (see [6,9] for surveys of results). This can alternatively be viewed as a partial m-cycle system of G in which the set of edges not in any m-cycles is either $\emptyset$ or induces a subgraph of G respectively.

Kennedy solved maximum packings of K_v with hexagons [4] and Ashe, Fu and Rodger [1] extended the results in [2,4] by finding necessary and sufficient conditions for the existence of a 6-cycle system of $K_v - E(F)$ where v is even and the leave F is an odd spanning forest (a forest where each vertex has odd degree). Pu and Chai extended the result of [2] by finding necessary and sufficient conditions for the existence of maximum hexagon Packing of $K_v - L$ where L is a 2-regular Subgraph [7]. The necessary and sufficient conditions for the existence of a 4-cycle system of $K_v - E(F)$ were also obtained [2].

In this paper, we extend the results of Ashe, Rodger and Fu [1]. We shall consider the maximum hexagon packing of $K_v - F$ where F is an odd spanning forest.

2. The small cases

In order to consider the necessary and sufficient conditions for the maximum packing of $K_v - E(F)$ for any spanning forest F, we need Lemma 2.1.

Lemma 2.1 *Let v be even and let F be a spanning forest of the complete graph K_v with $c(F)$ connected components. $|E(K_v - F)| \equiv i \pmod 6$ if and only if v and $c(F)$ are related as in Table 2.*

Proof

v	$12k$	$12k+2$	$12k+4$	$12k+6$	$12k+8$	$12k+10$
$\lvert E(K_v - F)\rvert \equiv 1 \pmod 6$						
$c(F)$	1	2	5	4	5	2
$\lvert E(K_v - F)\rvert \equiv 2 \pmod 6$						
$c(F)$	2	3	0	5	0	3
$\lvert E(K_v - F)\rvert \equiv 3 \pmod 6$						
$c(F)$	3	4	1	0	1	4
$\lvert E(K_v - F)\rvert \equiv 4 \pmod 6$						
$c(F)$	4	5	2	1	2	5
$\lvert E(K_v - F)\rvert \equiv 5 \pmod 6$						
$c(F)$	5	0	3	2	3	0

Table 2 The number of components required in F for $|E(K_v - F)| \equiv i \pmod 6$ when v is even

Clearly, $c(F) = v - |E(F)|$. So, if $|E(K_v - F)| \equiv i \pmod 6$, then $c(F) \pmod 6 \equiv (v - |E(F)|) \pmod 6 \equiv (v - \frac{v^2-v}{2} + i) \pmod 6$.

Also, if $c(F)$ and v are related as in Table 2, then $c(F) \pmod 6 \equiv (v - \frac{v^2-v}{2} + i) \pmod 6$. $\Box$

A cycle of length l is denoted by $C_l = (x_1, x_2, \ldots, x_l)$. Let G^c denote the complement of a graph G and $G \vee H$ denote the join of two vertex disjoint graphs G and H (so $E(G \vee H) = E(G) \cup E(H) \cup \{\{u,v\} : u \in V(G), v \in V(H)\}$). Let $G+H$ denote a graph with $E(G+H) = E(G) \cup E(H)$ and $V(G+H) = V(G) \cup V(H)$. In order to prove our main results, we need to solve the following small cases.

Lemma 2.2 *Let G_1 be the graph $K_6 \vee K_3^c$ with $V(K_6) = \{x_i | i \in Z_6\}$, $V(K_3^c) = \{y_i | i = 1,2,3\}$. Let G_2 be the graph $K_6 \vee K_4^c - K_1 \vee K_6^c + C_7$ with $V(K_6) = \{x_i | i \in Z_6\}$, $V(K_4^c) = \{y_0, y_1, y_5, y_6\}$, $V(K_1) = \{y_0\}$ and $C_7 = (y_0, y_1, y_2, y_3, y_4, y_5, y_6)$. Then there exists a 6-cycle system for G_1 and G_2 with leaves (y_2, x_2, x_5) and (x_4, y_5, x_5, x_2), respectively.*

Proof By direct construction, we have $G_1 = \{(x_1, y_1, x_2, y_3, x_5, x_4), (x_3, x_2, x_1, y_3, x_0, x_4), (y_1, x_0, y_2, x_3, x_1, x_5), (y_2, x_1, x_0, x_3, y_1, x_4), (x_0, x_2, x_4, y_3, x_3, x_5)\} \cup \{(y_2, x_2, x_5)\}$ and $G_2 = \{(y_1, y_2, y_3, y_4, y_5, x_0), (y_5, y_6, y_0, y_1, x_1, x_2), (x_0, y_6, x_1, y_5, x_3, x_1), (x_3, y_1, x_2, y_6, x_4, x_5), (x_4, y_1, x_5, y_6, x_3, x_0), (x_4, x_1, x_5, x_0, x_2, x_3)\} \cup \{(x_4, y_5, x_5, x_2)\}$. □

Lemma 2.3 *Let G_1 be the graph $K_6 \vee K_4^c - \{\{y_0, x_0\}, \{x_1, y_0\}, \{y_1, x_2\}, \{y_1, x_3\}, \{y_2, x_4\}, \{y_2, x_5\}\}$ with $V(K_6) = \{x_i | i \in Z_6\}$ and $V(K_4^c) = \{y_i | i \in Z_4\}$. Let $G_2 = G_1 + C_7$ with $C_7 = (y_0, y_1, y_2, y_3, y_4, y_5, y_6)$. Then there exists a 6-cycle system for G_1 and G_2 with leaves (x_2, x_3, x_5) and (x_3, y_2, x_0, y_3), respectively.*

Proof By direct construction, we have $G_1 = \{(x_2, y_0, x_3, y_2, x_1, x_4), (x_4, y_0, x_5, y_1, x_0, x_3), (x_2, y_2, x_0, y_3, x_5, x_1), (x_4, y_1, x_1, y_3, x_2, x_0), (x_3, y_3, x_4, x_5, x_0, x_1)\} \cup \{(x_2, x_3, x_5)\}$ and

$G_2 = \{(y_3, y_4, y_5, y_6, y_0, x_2), (y_0, y_1, y_2, y_3, x_1, x_3), (x_4, y_0, x_5, y_1, x_0, x_3), (x_1, y_1, x_4, y_3, x_5, x_2), (x_5, x_4, x_1, x_0, x_2, x_3), (x_1, x_5, x_0, x_4, x_2, y_2)\} \cup \{(x_3, y_2, x_0, y_3)\}$. □

Lemma 2.4 *Let G_0 be the graph $K_6 \vee K_4^c + \{\{y_0, y_1\}\} - \{\{y_0, x_0\}, \{y_1, x_1\}, \{y_1, x_2\}, \{y_1, x_3\}, \{y_1, x_4\}, \{y_1, x_5\}\}$ with $V(K_6) = \{x_i | i \in Z_6\}$ and $V(K_4^c) = \{y_i | i \in Z_4\}$. Let $G_1 = G_0 + C_3$, $G_2 = G_0 + C_4$ and $G_3 = G_0 + C_7$ with $C_3 = (y_0, y_2, y_3)$, $C_4 = (y_0, y_2, y_1, y_3)$, and $C_7 = (y_0, y_2, y_1, y_3, y_4, y_5, y_6)$ respectively. Then there exists a 6-cycle system for G_0, G_1, G_2, and G_3 with leaves (x_4, x_0, x_5, y_3), $(x_1, y_2, x_0, y_3, x_4, x_3, x_2)$, $(x_4, y_0, x_5, x_0, x_1) \cup (x_4, x_5, x_3)$, and $(x_3, x_0, x_1, x_5, x_2)$, respectively.*

Proof By direct constructions, we have

$G_0 = \{(x_1, y_0, x_2, y_2, x_3, y_3), (x_3, y_0, x_4, y_2, x_5, x_2), (x_5, y_0, y_1, x_0, y_2, x_1), (x_2, y_3, x_0, x_1, x_3, x_4), (x_2, x_0, x_3, x_5, x_4, x_1)\} \cup \{(x_4, x_0, x_5, y_3)\}$,

$G_1 = \{(y_2, y_0, y_1, x_0, x_1, y_3), (x_2, y_0, x_3, y_2, x_5, y_3), (x_5, x_2, x_4, x_0, x_3, x_1), (x_5, x_0, x_2, y_2, x_4, y_0), (x_4, x_5, x_3, y_3, y_0, x_1)\} \cup \{(x_1, y_2, x_0, y_3, x_4, x_3, x_2)\}$,

$G_2 = \{(x_2, y_0, x_3, y_2, x_5, y_3), (x_5, x_2, x_4, x_0, x_3, x_1), (y_1, y_0, y_2, x_1, y_3, x_0), (y_2, y_1, y_3, x_3, x_2, x_0), (y_3, y_0, x_1, x_2, y_2, x_4)\} \cup \{(x_4, y_0, x_5, x_0, x_1) \cup (x_4, x_5, x_3)\}$,

$G_3 = \{(y_3, y_4, y_5, y_6, y_0, x_1), (y_0, y_2, y_1, y_3, x_0, x_2), (y_0, y_1, x_0, y_2, x_1, x_4), (x_3, x_1, x_2, x_4, x_5, y_3), (x_5, y_0, x_3, y_2, x_4, x_0), (x_5, x_3, x_4, y_3, x_2, y_2)\} \cup \{(x_3, x_0, x_1, x_5, x_2)\}$. □

Lemma 2.5 *Let G_1 be the graph $K_6 \vee K_4^c + \{\{y_0, y_1\}\} - \{\{y_0, x_0\}, \{y_2, x_2\}, \{y_2, x_3\},$*

$\{y_1,x_1\},\{y_3,x_4\},\{y_3,x_5\}\}$ with $V(K_6)=\{x_i|i\in Z_6\}$ and $V(K_4^c)=\{y_i|i\in Z_4\}$. Let $G_2=G_1+C_7$ with $C_7=(y_0,y_2,y_3,y_1,y_4,y_5,y_6)$. Then there exists a 6-cycle system for G_1 and G_2 with leaves (x_5,y_0,x_4,y_2) and (x_3,x_4,x_2,x_0,x_5), respectively.

Proof By direct constructions, we have

$G_1=\{(y_1,x_0,y_2,x_1,y_3,x_2),(x_0,y_3,x_3,x_2,x_4,x_1),(x_1,y_0,x_2,x_5,x_4,x_3),(y_1,y_0,x_3,x_5,x_0,x_4),(x_3,x_0,x_2,x_1,x_5,y_1)\}\cup\{(x_5,y_0,x_4,y_2)\}$,

$G_2=\{(y_1,y_4,y_5,y_6,y_0,x_5),(y_1,y_0,x_4,y_2,x_5,x_2),(x_0,y_1,x_3,y_0,x_2,x_1),(y_0,y_2,y_3,y_1,x_4,x_1),(x_1,y_2,x_0,y_3,x_2,x_3),(x_1,y_3,x_3,x_0,x_4,x_5)\}\cup\{(x_3,x_4,x_2,x_0,x_5)\}$. □

Lemma 2.6 *Let G_0 be the graph $K_6\vee K_4^c+\{\{y_1,y_2\},\{y_0,y_2\}\}-\{\{y_0,x_0\},\{y_1,x_1\},\{y_2,x_2\},\{y_2,x_3\},\{y_2,x_4\},\{y_2,x_5\}\}$ with $V(K_6)=\{x_i|i\in Z_6\}$ and $V(K_4^c)=\{y_i|i\in Z_4\}$. Let $G_1=G_0+C_3$ and $G_2=G_0+C_4$ with $C_3=(y_0,y_1,y_3)$ and $C_4=(y_0,y_1,y_3,y_4)$. Then there exists a 6-cycle system for G_0, G_1, and G_2 with leaves (x_5,x_4,x_3,x_0,x_1), $(y_3,y_0,x_1,x_0,x_3)\cup(x_4,y_1,y_3)$, and (x_4,y_1,y_3), respectively.*

Proof By direct construction, we have

$G_0=\{(y_1,y_2,y_0,x_1,y_3,x_2),(x_3,y_0,x_2,x_0,y_1,x_5),(x_5,y_0,x_4,y_1,x_3,y_3),(x_4,y_3,x_0,y_2,x_1,x_2),(x_5,x_0,x_4,x_1,x_3,x_2)\}\cup\{(x_5,x_4,x_3,x_0,x_1)\}$,

$G_1=\{(y_1,x_0,y_2,y_0,x_4,x_2),(x_0,y_3,x_1,y_2,y_1,x_5),(x_5,y_0,x_3,x_2,x_1,x_4),(x_5,x_3,x_4,x_0,x_2,y_3),(y_0,y_1,x_3,x_1,x_5,x_2)\}\cup\{(y_3,y_0,x_1,x_0,x_3)\cup(x_4,y_1,y_3)\}$,

$G_2=\{(y_1,x_0,y_2,y_0,x_4,x_2),(x_0,y_3,x_1,y_2,y_1,x_5),(x_5,y_0,x_3,x_2,x_1,x_4),(x_5,x_3,x_4,x_0,x_2,y_3),(y_0,y_1,x_3,x_1,x_5,x_2),(y_0,x_1,x_0,x_3,y_3,y_4)\}\cup\{(x_4,y_1,y_3)\}$. □

Lemma 2.7 *Let G_0 be the graph $K_6\vee K_4^c+\{\{y_1,y_2\},\{y_0,y_2\}\}-\{\{y_0,x_0\},\{y_1,x_1\},\{y_2,x_2\},\{y_2,x_3\},\{y_3,x_4\},\{y_3,x_5\}\}$ with $V(K_6)=\{x_i|i\in Z_6\}$ and $V(K_4^c)=\{y_i|i\in Z_4\}$. Let $G_1=G_0+C_3$,$G_2=G_0+C_4$, $C_3=(y_0,y_1,y_3)$ and $C_4=(y_0,y_1,y_3,y_4)$. Then there exists a 6-cycle system for G_0, G_1, and G_2 with leaves (x_1,y_0,x_2,x_3,x_4), $(x_1,y_0,x_2,x_5)\cup(x_3,x_4,y_1,y_3)$, and (x_3,x_4,y_1), respectively.*

Proof By direct constructions, we have

$G_0=\{(y_1,x_0,y_2,y_0,x_4,x_2),(x_0,y_3,x_1,y_2,y_1,x_5),(x_5,x_3,x_1,x_0,x_4,y_2),(x_2,x_0,x_3,y_1,x_4,x_5),(x_3,y_3,x_2,x_1,x_5,y_0)\}\cup\{(x_1,y_0,x_2,x_3,x_4)\}$,

$G_1=\{(y_1,x_0,y_2,y_0,x_4,x_2),(x_0,y_3,x_1,y_2,y_1,x_5),(x_5,y_0,x_3,x_2,x_1,x_4),(x_5,x_3,x_1,x_0,x_4,y_2),(y_3,y_0,y_1,x_3,x_0,x_2)\}\cup\{(x_1,y_0,x_2,x_5)\cup(x_3,x_4,y_1,y_3)\}$,

$G_2=\{(y_1,x_0,y_2,y_0,x_4,x_2),(x_0,y_3,x_1,y_2,y_1,x_5),(x_5,y_0,x_3,x_2,x_1,x_4),(x_5,x_3,x_1,x_0,x_4,y_2),(y_0,y_1,y_3,x_3,x_0,x_2),(y_3,y_4,y_0,x_1,x_5,x_2)\}\cup\{(x_3,x_4,y_1)\}$. □

Lemma 2.8 *Let G be the graph $K_6-\{\{x_0,x_1\},\{x_2,x_3\},\{x_4,x_5\}\}$ with $V(K_6)=\{x_i|i\in Z_6\}$. Then there exists a 6-cycle system for G.*

Proof By direct construction, we have $G=\{(x_2,x_0,x_3,x_4,x_1,x_5),(x_5,x_3,x_1,x_2,x_4,x_0)\}$. □

Lemma 2.9 Let G be the graph $K_6 \vee K_{10}^c + \{\{y_0, y_3\}, \{y_1, y_4\}, \{y_2, y_4\}\} - \{\{y_0, x_0\}, \{y_1, x_1\}, \{y_2, x_2\}, \{y_3, x_3\}, \{y_4, x_4\}, \{y_4, x_5\}\}$ where $V(K_6) = \{x_i | i \in Z_6\}$ and $V(K_{10}) = \{y_i | i \in Z_{10}\}$. Then there exists a 6-cycle system for G.

Proof By direct constructions, we have

$G = \{(y_2, y_4, y_1, x_3, x_4, x_5), (x_5, y_3, x_4, x_0, x_1, y_0), (y_3, y_0, x_4, y_7, x_3, x_2), (y_4, x_2, y_1, x_0, x_5, x_3), (y_5, x_5, y_9, x_1, y_2, x_4), (y_5, x_0, y_6, x_4, x_2, x_1), (y_6, x_2, y_7, x_0, y_8, x_3), (y_3, x_0, x_2, x_5, y_7, x_1), (y_4, x_0, x_3, y_9, x_4, x_1), (y_5, x_2, y_9, x_0, y_2, x_3), (y_6, x_1, y_8, x_4, y_1, x_5), (y_0, x_2, y_8, x_5, x_1, x_3)\}$. $\square$

Lemma 2.10 *Let G_1 be the graph $K_6 \vee K_{10}^c + \{\{y_0, y_3\}, \{y_1, y_3\}, \{y_2, y_3\}\} - \{\{y_0, x_0\}, \{y_1, x_1\}, \{y_2, x_2\}, \{y_3, x_3\}, \{y_3, x_4\}, \{y_3, x_5\}\}$ where $V(K_6) = \{x_i | i \in Z_6\}$ and $V(K_{10}) = \{y_i | i \in Z_{10}\}$. Then there exists a 6-cycle system for G.*

Proof By direct construction, we have

$G = \{(y_2, y_3, y_1, x_3, x_4, x_5), (x_5, y_4, x_4, x_0, x_1, y_0), (y_3, y_0, x_4, y_7, x_3, x_2), (y_4, x_2, y_1, x_0, x_5, x_3), (y_5, x_5, y_9, x_1, y_2, x_4), (y_5, x_0, y_6, x_4, x_2, x_1), (y_6, x_2, y_7, x_0, y_8, x_3), (y_3, x_0, x_2, x_5, y_7, x_1), (y_4, x_0, x_3, y_9, x_4, x_1), (y_5, x_2, y_9, x_0, y_2, x_3), (y_6, x_1, y_8, x_4, y_1, x_5), (y_0, x_2, y_8, x_5, x_1, x_3)\}$. $\square$

Lemma 2.11 *If F is a spanning forest of K_8 in which each vertex has odd degree and $|E(K_8 - F)| \equiv i \pmod 6$, then $K_8 - F$ can be packed with leave C_i for $i = 3, 4, 5$.*

Proof There are eight possibilities for F. For $1 \le i \le 8$, a 6-cycle system (Z_8, B) of $K_8 - E(F_i)$ is given below, where F_i is the forest induced by the edges in no hexagons in B.

$F_1 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_0, x_4\}, \{x_0, x_5\}, \{x_0, x_6\}, \{x_0, x_7\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_6, x_7, x_1, x_3, x_5, x_2), (x_4, x_6, x_3, x_7, x_5, x_1)\}$ with leave $C_3 = (x_2, x_4, x_7)$.

$F_2 = \{\{x_0, x_4\}, \{x_0, x_5\}, \{x_0, x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_3, x_6\}, \{x_3, x_7\}\}$: $B = \{(x_6, x_7, x_0, x_3, x_5, x_2), (x_1, x_6, x_0, x_2, x_4, x_7), (x_4, x_1, x_5, x_7, x_2, x_3)\}$ with leave $C_3 = (x_4, x_5, x_6)$.

$F_3 = \{\{x_0, x_1\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_2, x_6\}, \{x_2, x_7\}, \{x_1, x_2\}\}$: $B = \{(x_6, x_7, x_0, x_2, x_3, x_4), (x_3, x_5, x_7, x_1, x_6, x_0), (x_6, x_3, x_7, x_4, x_0, x_5)\}$ with the leave $C_3 = (x_5, x_1, x_4)$.

$F_4 = \{\{x_0, x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_5, x_6\}, \{x_5, x_7\}\}$: $B = \{(x_6, x_7, x_3, x_0, x_4, x_2), (x_5, x_2, x_7, x_4, x_6, x_0), (x_0, x_2, x_3, x_6, x_1, x_7)\}$ with leave $C_3 = (x_4, x_5, x_1)$.

$F_5 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}\}$: $B = \{(x_7, x_2, x_1, x_5, x_0, x_6), (x_6, x_1, x_4, x_0, x_7, x_3), (x_2, x_5, x_7, x_1, x_3, x_4)\}$ with leave $C_4 = (x_2, x_3, x_5, x_6)$.

$F_6 = \{\{x_1, x_4\}, \{x_1, x_5\}, \{x_1, x_0\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_6, x_7\}\}$: $B = \{(x_1, x_3, x_5, x_0, x_6, x_2), (x_6, x_1, x_7, x_0, x_4, x_5), (x_4, x_6, x_3, x_2, x_5, x_7)\}$ with leave $C_4 = (x_2, x_4, x_3, x_7)$.

$F_7 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_0, x_4\}, \{x_0, x_5\}, \{x_6, x_7\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_1, x_4, x_2, x_6, x_0, x_7), (x_5, x_1, x_3, x_6, x_4, x_7)\}$ with leave $C_4 = (x_3, x_5, x_2, x_7)$.

$F_8 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_4, x_5\}, \{x_6, x_7\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_6, x_5), (x_1, x_4, x_2, x_7, x_0, x_6), (x_1, x_3, x_6, x_2, x_5, x_7)\}$ with leave $C_5 = (x_5, x_0, x_4, x_7, x_3)$. $\square$

Lemma 2.12 *If F is a spanning forest of K_{10} in which each vertex has odd degree and $|E(K_{10} - F)| \equiv i \pmod 6$, then $K_{10} - F$ can be packed with leave L_i for $i = 1, 2, 3, 4$.*

Proof There are seven possibilities for F. For $1 \le i \le 7$, a 6-cycle system (Z_{10}, B) of $K_{10} - E(F_i)$ is given below, where F_i is the forest induced by the edges in no hexagons in B.

$F_1 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_9\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_3, x_6\}, \{x_6, x_7\}, \{x_6, x_8\}\}$: $B = \{(x_1, x_2, x_4, x_7, x_3, x_8), (x_2, x_3, x_1, x_4, x_8, x_5), (x_4, x_5, x_7, x_0, x_6, x_9), (x_5, x_6, x_4, x_0, x_3, x_9), (x_7, x_8, x_0, x_5, x_1, x_9)\}$ with leave $L_1 = (x_2, x_6, x_1, x_7) \cup (x_9, x_2, x_8)$.

$F_2 = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_2, x_7\}, \{x_2, x_4\}, \{x_7, x_8\}, \{x_7, x_9\}, \{x_4, x_6\}, \{x_4, x_5\}\}$: $B = \{(x_0, x_3, x_1, x_4, x_7, x_5), (x_1, x_2, x_5, x_3, x_7, x_6), (x_6, x_3, x_4, x_0, x_2, x_8), (x_1, x_5, x_6, x_2, x_9, x_8), (x_8, x_4, x_9, x_1, x_7, x_0)\}$ with leave $L_1 = (x_9, x_0, x_6) \cup (x_3, x_8, x_5, x_9)$.

$F_3 = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_7\}, \{x_7, x_9\}, \{x_7, x_8\}, \{x_8, x_6\}, \{x_8, x_5\}\}$: $B = \{(x_1, x_2, x_5, x_4, x_6, x_3), (x_0, x_4, x_8, x_2, x_6, x_9), (x_1, x_5, x_0, x_2, x_9, x_4), (x_4, x_3, x_5, x_6, x_0, x_7), (x_5, x_7, x_6, x_1, x_8, x_9)\}$ with leave $L_1 = (x_8, x_0, x_3) \cup (x_9, x_3, x_7, x_1)$.

$F_4 = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_7, x_9\}, \{x_7, x_8\}\}$: $B = \{(x_1, x_2, x_0, x_3, x_6, x_9), (x_0, x_4, x_8, x_3, x_7, x_5), (x_2, x_6, x_0, x_8, x_9, x_4), (x_5, x_6, x_7, x_1, x_3, x_9), (x_4, x_1, x_6, x_8, x_5, x_3)\}$ with leave $L_2 = (x_2, x_8, x_1, x_5) \cup (x_9, x_2, x_7, x_0)$.

$F_5 = \{\{x_0, x_1\}, \{x_2, x_5\}, \{x_2, x_4\}, \{x_2, x_3\}, \{x_6, x_7\}, \{x_6, x_8\}, \{x_6, x_9\}\}$: $B = \{(x_3, x_4, x_5, x_6, x_1, x_9), (x_7, x_8, x_9, x_0, x_3, x_5), (x_1, x_3, x_6, x_2, x_0, x_7), (x_7, x_4, x_6, x_0, x_5, x_9), (x_8, x_0, x_4, x_9, x_2, x_1)\}$ with leave $L_2 = (x_8, x_4, x_1, x_5) \cup (x_8, x_2, x_7, x_3)$.

$F_6 = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_6, x_8\}, \{x_6, x_9\}, \{x_6, x_7\}\}$: $B = \{(x_1, x_5, x_9, x_4, x_2, x_8), (x_1, x_2, x_7, x_9, x_8, x_4), (x_3, x_5, x_2, x_9, x_1, x_7), (x_6, x_5, x_7, x_8, x_0, x_3), (x_4, x_0, x_9, x_3, x_1, x_6), (x_7, x_0, x_5, x_8, x_3, x_4)\}$ with leave $L_3 = (x_6, x_0, x_2)$.

$F_7 = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_8, x_9\}, \{x_6, x_7\}\}$: $B = \{(x_8, x_0, x_3, x_7, x_2, x_5), (x_5, x_6, x_8, x_2, x_4, x_7), (x_3, x_1, x_2, x_6, x_0, x_5), (x_3, x_4, x_1, x_5, x_9, x_6), (x_9, x_7, x_8, x_1, x_6, x_4), (x_4, x_0, x_2, x_9, x_3, x_8)\}$ with leave $L_4 = (x_9, x_0, x_7, x_1)$. $\Box$

Lemma 2.13 *If F is a spanning forest of K_{12} in which each vertex has odd degree and $|E(K_{12} - F)| \equiv i \pmod 6$, then $K_{12} - F$ can be packed with leave L_i for $i = 1, 2, 3, 4, 5$.*

Proof There are 14 possibilities for F. For $1 \le i \le 14$, a 6-cycle system (Z_{12}, B) of $K_{12} - E(F_i)$ is given below, where F_i is the forest induced by the edges in no hexagons in B.

$F_1 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_0, x_4\}, \{x_0, x_5\}, \{x_0, x_6\}, \{x_0, x_7\}, \{x_0, x_8\}, \{x_0, x_9\}, \{x_0, x_{10}\}, \{x_0, x_{11}\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_6, x_7, x_8, x_9, x_{10}, x_{11}), (x_2, x_4, x_7, x_3, x_{11}, x_8), (x_4, x_6, x_8, x_5, x_9, x_1), (x_5, x_{10}, x_3, x_6, x_9, x_7), (x_1, x_3, x_5, x_2, x_{10}, x_7), (x_{10}, x_8, x_3, x_9, x_{11}, x_4), (x_1, x_8, x_4, x_9, x_2, x_{11}\}$ with leave $L_1 = (x_1, x_{10}, x_6, x_2, x_7, x_{11}, x_5)$.

$F_2 = \{\{x_0, x_1\}, \{x_0, x_3\}, \{x_0, x_2\}, \{x_2, x_5\}, \{x_2, x_4\}, \{x_4, x_{10}\}, \{x_4, x_{11}\}, \{x_3, x_6\}, \{x_3, x_7\}, \{x_1, x_8\}, \{x_1, x_9\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_6, x_7, x_8, x_9, x_{10}, x_{11}), (x_{11}, x_0, x_4, x_7, x_9, x_5), (x_3, x_5, x_8, x_2, x_9, x_{11}), (x_1, x_3, x_8, x_{10}, x_0, x_5), (x_6, x_8, x_4, x_9, x_3, x_{10}), (x_1, x_4, x_6, x_9, x_0, x_7), (x_8, x_{11}, x_7, x_2, x_6, x_0)\}$ with leave $L_1 = (x_2, x_{11}, x_1, x_{10}) \cup (x_5, x_7, x_{10})$.

$F_3 = \{\{x_0, x_1\}, \{x_0, x_{10}\}, \{x_0, x_{11}\}, \{x_1, x_2\}, \{x_1, x_7\}, \{x_2, x_4\}, \{x_2, x_3\}, \{x_3, x_5\}, \{x_3, x_6\}, \{x_7, x_9\}, \{x_7, x_8\}\}$: $B = \{(x_3, x_4, x_5, x_6, x_7, x_{10}), (x_8, x_9, x_{10}, x_{11}, x_1, x_6), (x_2, x_7, x_0, x_5, x_{10}, x_8), (x_9, x_{11}, x_2, x_{10}, x_6, x_0), (x_6, x_9, x_3, x_0, x_8, x_4), (x_2, x_5, x_7, x_3, x_1, x_9), (x_8, x_{11}, x_4, x_9, x_5, x_1), (x_4, x_7, x_{11}, x_6, x_2, x_0)\}$ with leave $L_1 = (x_5, x_{11}, x_3, x_8) \cup (x_4, x_{10}, x_1)$.

$F_4 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_0, x_4\}, \{x_0, x_5\}, \{x_6, x_7\}, \{x_6, x_8\}, \{x_6, x_9\}, \{x_6, x_{10}\}, \{x_6, x_{11}\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_7, x_8, x_9, x_{10}, x_{11}, x_0), (x_2, x_4, x_6, x_3, x_7, x_{11}), (x_2, x_5, x_8, x_{11}, x_3, x_9), (x_{10}, x_2, x_6, x_0, x_8, x_3), (x_1, x_3, x_5, x_7, x_9, x_{11}), (x_8, x_{10}, x_0, x_9, x_1, x_4), (x_4, x_7, x_{10}, x_1, x_5, x_9)\}$ with leave $L_1 = (x_2, x_7, x_1, x_8) \cup (x_5, x_{10}, x_4, x_{11})$.

$F_5 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_6, x_7\}, \{x_6, x_8\}, \{x_6, x_9\}, \{x_6, x_{10}\}, \{x_6, x_{11}\}, \{x_7, x_5\}, \{x_7, x_4\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_7, x_8, x_9, x_{10}, x_{11}, x_0), (x_1, x_3, x_5, x_8, x_{11}, x_4), (x_2, x_4, x_6, x_3, x_7, x_{11}), (x_7, x_9, x_{11}, x_1, x_{10}, x_2), (x_8, x_{10}, x_0, x_4, x_9, x_3), (x_9, x_2, x_8, x_4, x_{10}, x_5), (x_{10}, x_3, x_{11}, x_5, x_1, x_7)\}$ with leave $L_8 = (x_0, x_6, x_2, x_5) \cup (x_1, x_8, x_0, x_9)$.

$F_6 = \{\{x_0, x_1\}, \{x_6, x_7\}, \{x_6, x_8\}, \{x_6, x_{11}\}, \{x_7, x_5\}, \{x_7, x_4\}, \{x_8, x_{10}\}, \{x_8, x_9\}, \{x_9, x_2\}, \{x_9, x_3\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_7, x_8, x_5, x_9, x_1, x_3), (x_2, x_4, x_6, x_9, x_0, x_5), (x_9, x_{10}, x_{11}, x_0, x_2, x_7), (x_3, x_{10}, x_2, x_8, x_1, x_5), (x_5, x_{10}, x_4, x_0, x_7, x_{11}), (x_{11}, x_4, x_8, x_0, x_6, x_3), (x_{11}, x_2, x_6, x_{10}, x_7, x_1)\}$ with leave $L_2 = (x_{10}, x_1, x_4, x_9, x_{11}, x_8, x_3, x_0)$.

$F_7 = \{\{x_0, x_1\}, \{x_{11}, x_2\}, \{x_{11}, x_3\}, \{x_{11}, x_4\}, \{x_{11}, x_5\}, \{x_{11}, x_6\}, \{x_{11}, x_7\}, \{x_{11}, x_8\}, \{x_{11}, x_9\}, \{x_{11}, x_{10}\}\}$: $B = \{(x_1, x_2, x_3, x_4, x_5, x_6), (x_{11}, x_0, x_2, x_4, x_7, x_1), (x_1, x_3, x_5, x_7, x_9, x_4), (x_6, x_8, x_{10}, x_0, x_3, x_9), (x_3, x_6, x_2, x_7, x_0, x_8), (x_6, x_{10}, x_5, x_1, x_9, x_0), (x_0, x_4, x_8, x_2, x_9, x_5)\}$ with leave $L_2 = (x_{10}, x_1, x_8, x_5, x_2) \cup (x_{10}, x_3, x_7)$.

$F_8 = \{\{x_0, x_1\}, \{x_0, x_2\}, \{x_0, x_3\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_8, x_9\}, \{x_8, x_{10}\}, \{x_8, x_{11}\}\}$: $B = \{(x_1, x_2, x_3, x_5, x_{11}, x_9), (x_9, x_{10}, x_{11}, x_0, x_5, x_7), (x_6, x_9, x_0, x_7, x_1, x_{10}), (x_7, x_3, x_9, x_2, x_5, x_8), (x_1, x_6, x_7, x_2, x_4, x_3), (x_6, x_8, x_0, x_4, x_9, x_5), (x_7, x_{10}, x_0, x_6, x_2, x_{11}), (x_{11}, x_1, x_5, x_{10}, x_3, x_6), (x_4, x_{11}, x_3, x_8, x_2, x_{10})\}$ with leave $L_3 = (x_8, x_1, x_4)$.

$F_9 = \{\{x_0, x_1\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_7, x_2\}, \{x_7, x_3\}, \{x_8, x_9\}, \{x_8, x_{10}\}, \{x_8, x_{11}\}\}$: $B = \{(x_1, x_2, x_3, x_5, x_{11}, x_9), (x_3, x_4, x_2, x_5, x_8, x_1), (x_5, x_6, x_7, x_8, x_4, x_9), (x_6, x_8, x_0, x_4, x_1, x_{10}), (x_9, x_{10}, x_{11}, x_0, x_5, x_7), (x_7, x_{10}, x_0, x_6, x_2, x_{11}), (x_{11}, x_1, x_5, x_{10}, x_3, x_6), (x_4, x_{11}, x_3, x_8, x_2, x_{10}), (x_6, x_9, x_2, x_0, x_7, x_1)\}$ with leave $L_3 = (x_0, x_3, x_9)$.

$F_{10} = \{\{x_0, x_1\}, \{x_8, x_{11}\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_7, x_2\}, \{x_7, x_3\}, \{x_3, x_9\}, \{x_3, x_{10}\}\}$: $B = \{(x_1, x_7, x_0, x_8, x_5, x_6), (x_2, x_8, x_3, x_{11}, x_9, x_{10}), (x_4, x_{10}, x_5, x_{11}, x_6, x_2), (x_1, x_{11}, x_2, x_3, x_4, x_8), (x_6, x_7, x_8, x_9, x_1, x_{10}), (x_5, x_3, x_6, x_0, x_9, x_2), (x_4, x_9, x_5, x_0, x_2, x_1), (x_5, x_7, x_{10}, x_0, x_3, x_1), (x_9, x_6, x_8, x_{10}, x_{11}, x_7)\}$ with leave $L_3 = (x_0, x_{11}, x_4)$.

$F_{11} = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_4, x_8\}, \{x_4, x_9\}, \{x_4, x_{10}\}, \{x_4, x_{11}\}\}$: $B = \{(x_1, x_2, x_4, x_3, x_5, x_7), (x_5, x_6, x_7, x_8, x_9, x_{10}), (x_{10}, x_{11}, x_0, x_2, x_5, x_8), (x_6, x_8, x_{11}, x_7, x_2, x_9), (x_6, x_0, x_4, x_1, x_5, x_{11}), (x_{10}, x_0, x_5, x_9, x_3, x_7), (x_3, x_6, x_2, x_{10}, x_1, x_8), (x_1, x_3, x_0, x_8, x_2, x_{11}), (x_9, x_{11}, x_3, x_{10}, x_6, x_1)\}$ with leave $L_3 = (x_9, x_0, x_7)$.

$F_{12} = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_8, x_9\}, \{x_8, x_{10}\}, \{x_8, x_{11}\}\}$: $B = \{(x_3, x_4, x_2, x_1, x_{11}, x_9), (x_5, x_6, x_7, x_8, x_0, x_3), (x_9, x_{10}, x_{11}, x_0, x_2, x_5), (x_1, x_3, x_{11}, x_4, x_9, x_6), (x_5, x_7, x_9, x_0, x_4, x_8), (x_3, x_6, x_8, x_1, x_5, x_{10}), (x_6, x_{10}, x_4, x_1, x_9, x_2), (x_1, x_7, x_{11}, x_5, x_0, x_{10}), (x_7, x_0, x_6, x_{11}, x_2, x_{10})\}$ with leave $L_4 = (x_8, x_2, x_7, x_3)$.

$F_{13} = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_8, x_{11}\}, \{x_8, x_9\}, \{x_8, x_{10}\}, \{x_{10}, x_6\}, \{x_{10}, x_7\}\}$: $B = \{(x_5, x_6, x_7, x_8, x_0, x_9), (x_9, x_{10}, x_{11}, x_0, x_2, x_4), (x_1, x_3, x_5, x_7, x_9, x_{11}), (x_3, x_4, x_6, x_8, x_5, x_{11}), (x_{10}, x_0, x_3, x_6, x_9, x_1), (x_1, x_4, x_7, x_3, x_{10}, x_5), (x_2, x_5, x_0, x_6, x_1, x_7), (x_2, x_{10}, x_4, x_0, x_7, x_{11}), (x_{11}, x_4, x_8, x_1, x_2, x_6)\}$ with leave $L_4 = (x_8, x_3, x_9, x_2)$.

$F_{14} = \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_6, x_7\}, \{x_8, x_{10}\}, \{x_8, x_9\}, \{x_{11}, x_8\}\}$: $B = \{(x_1, x_3, x_5, x_7, x_9, x_{11}), (x_1, x_2, x_4, x_3, x_6, x_9), (x_5, x_6, x_4, x_1, x_8, x_0), (x_9, x_{10}, x_{11}, x_0, x_2, x_5), (x_{10}, x_0, x_3, x_{11}, x_7, x_1), (x_4, x_9, x_3, x_7, x_2, x_8), (x_{10}, x_3, x_8, x_5, x_1, x_6), (x_{11}, x_2, x_9, x_0, x_7, x_4), (x_5, x_{11}, x_6, x_0, x_4, x_{10})\}$ with leave $L_5 = ((x_7, x_{10}, x_2, x_6, x_8)$. $\Box$

3. The main results

The following result obtained from a special case of Sotteau's Theorem [10] is essential to the proof of our main results.

Lemma 3.1 ([10]) *There exists a 6-cycle system of $K_{a,b}$ if and only if:*

(1) a and b are even;

(2) 6 divides a or b, and

(3) $\min\{a, b\} \geq 4$.

Also, we need the following result which was proved by Ashe et al [1].

Lemma 3.2 ([1]) *Let F be a spanning forest in the complete graph K_v with $|E(F)| \geq 1$. There exists a 6-cycle system of $K_v - E(F)$ if and only if*

(1) All vertices in F have odd degree;

(2) $|E(K_v - F)|$ is divisible by 6, and

(3) v is even.

With the above preparation, we are now in a position to prove our main result, Theorem 3.1. Let $G[W]$ denote the subgraph of G induced by W.

Theorem 3.1 *Let F be a forest in the complete graph K_v with $|E(F)| \geq 1$. For any integer v, $v > 6$, $G = K_v - E(F)$ can be packed by 6-cycles with leave L_i if and only if*

(1) All vertices of F have odd degree;

(2) v is even, and

(3) $|E(K_v - F)| \equiv i \pmod 6$. Here, $L_0 = \emptyset$, $L_1 = C_7$, or $C_3 \cup C_4$, $L_2 = C_8$, $C_3 \cup C_5$, or $C_4 \cup C_4$, and $L_i = C_i$ for $i = 3, 4, 5$, respectively.

Proof First, we give the proof of necessity. Suppose that there exists a 6-cycle system (V, B) of $G = K_v - E(F) - L_i$. Then for each $v \in V$, the 6-cycles in B and the edges in L_i partition the edges incident with v into pairs, so $d_G(v)$ (the degree of v in graph G) is even. Since $|E(F)| \geq 1$ and F is a forest, F contains at least one vertex, say w, with $d_F(w) = 1$, so $d_G(w) = v - 2$. Therefore, v is even. Also, for each $v \in V$, $d_F(v) = (v - 1) - d_G(v)$, so $d_F(v)$ is odd. Then clearly F spans K_v. Since the 6-cycles in B partition the edges of G with leave L_i, we have $|E(K_v - E(F))| \equiv i \pmod 6$.

In the following, we will prove sufficiency. For $v = 8, 10, 12$, the proof is given in Lemma 2.11. The remaining cases are proved by induction. Suppose that for each positive integer α with $2 \leq \alpha < v$ and for any forest F' in K_α, the following conditions are satisfied:

$(1')$ All vertices in F' have odd degree (so F' is spanning), (2′) $|E(K_\alpha - E(F'))| \equiv i \pmod 6$, for $i = 0, 1, 2, 3, 4, 5$, and

(3′) α is even,

then $K_\alpha - E(F')$ can be packed with leave L_i. We will give the proof of sufficiency by considering several cases in turn: $c(F) = 1, 2, 3$ and $c(F) \geq 4$. We regularly make use of Table 1, since it is easier to find the number of components $c(F')$ in F', than to check that condition (2′) is satisfied. In the following let vertices of $V(K_v)$ be $X_v = \{x_i | i \in Z_v\}$.

Case 1 $c(F) = 1$.

By checking Table 2, we know $|E(K_v - F)| \equiv 1, 3, 4 \pmod 6$. We give two subcases as follows.

Case 1.1 F is a star.

If F is a star centered at vertex, say, x_6, then it has at least six leaves, namely x_0, x_1, x_2, x_3, x_4, and x_5. Then $F = F' + K_{\{x_6\},\{x_i|i\in Z_6\}}$ where F' satisfies conditions (1′)−(3′), and $K_{\{x_6\},\{x_i|i\in Z_6\}}$ is a star with center x_6 and arms x_0, x_1, x_2, x_3, x_4, and x_5.

We have $K_v - F = (K_{X_v\setminus\{x_i|i\in Z_6\}} - F') + K_{\{x_i|i\in Z_6\},X_v\setminus\{x_i|i\in Z_{10}\}} + [K_{\{x_i|i\in Z_6\},\{x_i|i=6,7,8,9\}} + K_{\{x_i|i\in Z_6\}} - K_{\{x_6\},\{x_i|i\in Z_6\}}]$.

By Lemma 3.1, $K_{\{x_i|i\in Z_6\},X_v\setminus\{x_i|i\in Z_{10}\}}$ can be packed by 6-cycles.

Let $H = K_{X_v\setminus\{x_i|i\in Z_6\}} - F'$.

When $|E(K_v - F)| \equiv 1, 3 \pmod 6$ and $|E(H)| \equiv 4, 0 \pmod 6$, H can be packed with leave C_4 or $\emptyset$ by induction. By Lemma 2.2, $K_{\{x_i|i\in Z_6\},\{x_i|i=6,7,8,9\}} + K_{\{x_i|i\in Z_6\}} - K_{\{x_6\},\{x_i|i\in Z_6\}}$ can be packed with leave C_3. Thus, $K_v - F$ can be packed with leave $C_4 \cup C_3$ or C_3, respectively.

When $|E(K_v - F)| \equiv 4 \pmod 6$, $|E(H)| \equiv 1 \pmod 6$, H can be packed with leave C_7 by induction. $K_{\{x_i|i\in Z_6\},\{x_i|i=6,7,8,9\}} + K_{\{x_i|i\in Z_6\}} - K_{\{x_6\},\{x_i|i\in Z_6\}} + C_7$ can be packed with leave C_4 by Lemma 2.2. Thus, $K_v - F$ can be packed with leave C_4.

Case 1.2 F is not a star.

A leaf pair is a set Y of two vertices each of degree 1 in F that have a common neighbor, $N(Y)$. We call $N(Y)$ the center of Y. If F is not a star, there must be three leaf pairs, denoted by $\{x_{v-1}, x_{v-2}\}$ (neighbor x_0), $\{x_{v-3}, x_{v-4}\}$ (neighbor x_1), and $\{x_{v-5}, x_{v-6}\}$ (neighbor x_2) (see Figure 1). Let F' be formed from $F[X_{v-6}]$ and let $\alpha = v - 6$. It is easy to check that conditions $(1' - 3')$ are satisfied.

$d_{F'}(x_i) = d_F(x_i) - 2$ for $i = 0, 1, 2$ and $d_{F'}(x_i) = d_F(x_i)$ for $i \in X_v \setminus \{x_i, x_{v-1-j} | i \in Z_3, j \in Z_6\}$.

Let $F = F' + T_1$ where $T_1 = \{x_0, x_{v-1}\} + \{x_0, x_{v-2}\} + \{x_1, x_{v-3}\} + \{x_1, x_{v-4}\} + \{x_2, x_{v-5}\} + \{x_2, x_{v-6}\}$.

$K_v - F = (K_{v-6} - F') + K_{6,v-10} + K_{6,4} + (K_6 - T_1)$ where $K_{v-6} - F'$ is defined on $Z_v \setminus \{x_{v-1-i} | i \in Z_6\}$; $K_{6,v-10}$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup X_v \setminus \{x_i, x_{v-1-j} | i \in Z_3, j \in Z_6\}$; $K_{6,4}$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup \{x_i | i \in Z_4\}$ and $K_6 - T_1$ is defined on $\{x_{v-1-i} | i \in Z_6\}$. By Lemma 3.1, $K_{6,v-10}$ can be packed by hexagons.

When $|E(K_v - F)| \equiv 1, 3 \pmod 6$ and $|E(K_{v-6} - F')| \equiv 4, 0 \pmod 6$, $K_{v-6} - F'$ can be packed with leave C_4 or $\emptyset$ by induction. By Lemma 2.3, $K_{6,4} + (K_6 - T_1)$ can be packed with leave C_3. Thus, $K_v - F$ can be packed with leave $C_4 \cup C_3$ or C_3, respectively.

When $|E(K_v - F)| \equiv 4 \pmod 6$ and $|E(K_{v-6} - F')| \equiv 1 \pmod 6$, $K_{v-6} - F'$ can be packed with leave C_7 by induction. $C_7 + K_{\{x_{v-1-i}|i\in Z_6\},\{x_i|i\in Z_4\}} + K_{\{\{x_{v-1-i}|i\in Z_6\}} - T_1$ can be packed with leave C_4 by Lemma 2.3. Thus, $K_v - F$ can be packed with leave C_4.

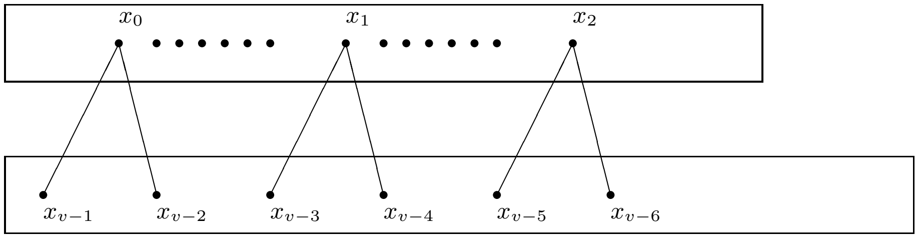

Figure 1 Case 1.2

Case 2 $c(F) = 2$.

By checking Table 1, we know $|E(K_v - F)| \equiv 1, 2, 4, 5 \pmod 6$.

Let C^0 and C^1 be two connected components in F. At least one of the connected components, say, C^1, is not K_2. Then we can proceed as follows.

Case 2.1 C^1 is not a star.

Let the second vertex in a maximum length path $P_i \in C^i$ be named x_i. Note that vertex x_i is adjacent to a vertex of degree 1 in F, namely the first vertex in P_i, denoted by x_{v-1-i} for $i = 0, 1$. There must be two leaf pairs in P_1, denoted by $\{x_{v-3}, x_{v-4}\}$ (with neighbor x_2) and $\{x_{v-5}, x_{v-6}\}$ (with neighbor x_3) (see Fig. 2). Let F' be formed from $F[X_{v-6}]$ and add edges $\{x_0, x_1\}$, and let $\alpha = v - 6$. We mainly check to see that condition $(1')$ is satisfied.

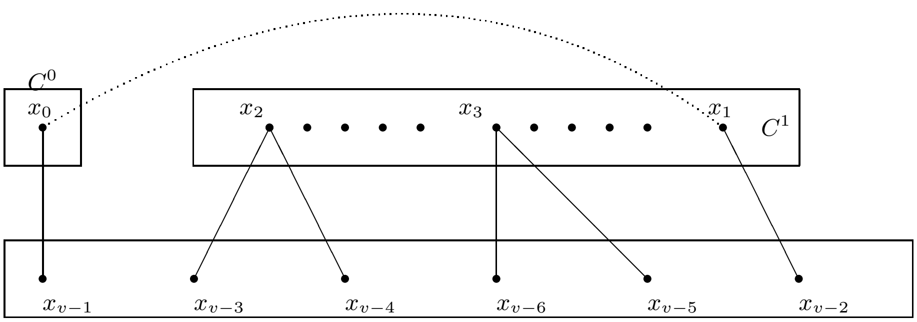

Figure 2 Case 2.1

$d_{F'}(x_i) = d_F(x_i)$ for $i = 0, 1$; $d_{F'}(x_i) = d_F(x_i) - 1$ for $i = 2, 3$, and $d_{F'}(x_i) = d_F(x_i)$ for $i \in X_v \setminus \{x_i, x_{v-1-j} | i \in Z_3, j \in Z_6\}$ (see Fig. 2).

Let $F = F' + T_2 - T_1$ where $T_2 = \{x_0, x_{v-1}\} + \{x_2, x_{v-3}\} + \{x_2, x_{v-4}\} + \{x_1, x_{v-2}\} + \{x_3, x_{v-5}\} + \{x_3, x_{v-6}\}$, and $T_1 = \{x_0, x_1\}$.

Then $K_v - F = (K_{v-6} - F') + K_{6,v-10} + K_{6,4} + (K_6 + T_1 - T_2)$ where $K_{v-6} - F'$ is defined on X_{v-6}, $K_{6,v-10}$ $(v \geq 14)$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup (X_v \setminus \{x_i, x_{v-1-j} | i \in Z_3, j \in Z_6\})$,

$K_{6,4}$ is defined on $\{x_{v-1-i}|i \in Z_6\} \cup \{x_i|i \in Z_4\}$, and K_6 is defined on $\{x_{v-1-i}|i \in Z_6\}$.

When $|E(K_v - F)| \equiv 1, 2, 4 \pmod 6$ and $|E(K_{v-6} - F')| \equiv 3, 4, 0 \pmod 6$, by induction, $K_{v-6} - F'$ can be packed with leave C_3, C_4, and $\emptyset$, respectively. $K_{6,v-10}$ $(v \geq 14)$ can be packed by Lemma 3.1. $K_{6,4} + (K_6 + T_1 - T_2)$ can be packed with leave C_4 by Lemma 2.5. Thus, $K_v - F$ can be packed with leave $C_4 \cup C_3$, $C_4 \cup C_4$, and C_4, respectively.

When $|E(K_v - F)| \equiv 5 \pmod 6$ and $|E(K_{v-6} - F')| \equiv 1 \pmod 6$, by induction, $K_{v-6} - F'$ can be packed by hexagons with leave C_7. By Lemma 3.1, $K_{6,v-10}$ $(v \geq 14)$ can be packed by hexagons. $C_7 + K_{6,4} + (K_6 + T_1 - T_2)$ can be packed with leave C_5. Thus, $K_v - F$ can be packed with leave C_5.

Case 2.2 C^1 is a star

If C^1 is a star centered at vertex, say, x_1, then it has at least five leaves, named as x_{v-2}, x_{v-3}, x_{v-4}, x_{v-5}, and x_{v-6}, respectively (see Fig. 3). Let the second vertex in a maximum length path $P_0 \in C^0$ be named as x_0. Then vertex x_0 is adjacent to a vertex of degree 1 in C^0, namely the first vertex in P_0, which we call x_{v-1} and add edges $\{x_0, x_1\}$.

Let $F = F' + T_2 - T_1$ where $T_2 = \{x_0, x_{v-1}\} + \{x_1, x_{v-2}\} + \{x_1, x_{v-3}\} + \{x_1, x_{v-4}\} + \{x_1, x_{v-5}\} + \{x_1, x_{v-6}\}$ and $T_1 = \{x_0, x_1\}$. Obviously, $d_{F'}(x_0) = d_F(x_0)$, $d_{F'}(x_1) = d_F(x_1) - 4$, and $d_{F'}(x_i) = d_F(x_i)$ for $i \in X_v \setminus \{x_i, x_{v-1-j}|i \in Z_2, j \in Z_6\}$.

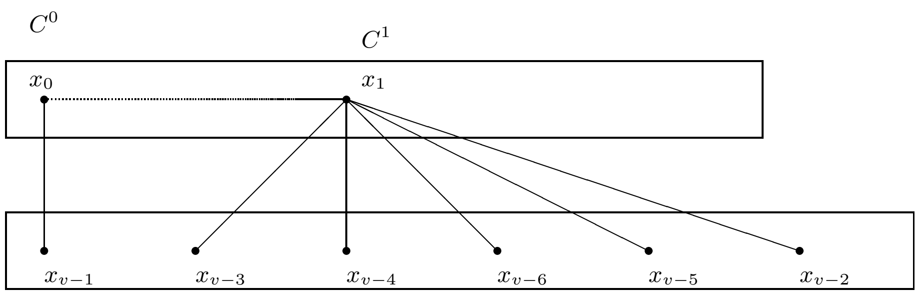

Figure 3 Case 2.2

$K_v - F = (K_{v-6} - F') + K_{6,v-10} + (K_{6,4} + K_6 - T_2 + T_1)$ where $K_{v-6} - F'$ is defined on $Z_v \setminus \{x_{v-1-i}|i \in Z_6\}$, $K_{6,v-10}$ is defined on $\{x_{v-1-i}|i \in Z_6\} \cup X_v \setminus \{x_i, x_{v-1-j}|i \in Z_4, j \in Z_6\}$, $K_{6,4}$ is defined on $\{x_{v-1-i}|i \in Z_6\} \cup \{x_i|i \in Z_4\}$, and K_6 is defined on $\{x_{v-1-i}|i \in Z_6\}$.

When $|E(K_v - F)| \equiv 1, 2, 4, 5 \pmod 6$ and $|E(K_{v-6} - F')| \equiv 3, 4, 0, 1 \pmod 6$, by induction, $K_{v-6} - F'$ can be packed with leave C_3, C_4, $\emptyset$, and C_7. $K_{6,v-10}(v \geq 14)$ can be packed by hexagons by Lemma 3.1. $C_3 + K_{6,4} + (K_6 + T_1 - T_2)$, $C_4 + K_{6,4} + (K_6 + T_1 - T_2)$, $K_{6,4} + (K_6 + T_1 - T_2)$ and $C_7 + K_{6,4} + (K_6 + T_1 - T_2)$ can be packed with leave C_7, $C_5 \cup C_3$, C_4, and C_5 respectively by Lemma 2.4. Thus, $K_v - F$ can be packed with leave C_7, $C_5 \cup C_3$, C_4, and C_5, respectively.

Case 3 $c(F) = 3$.

By checking Table 2, $|E(K_v - F)| \equiv 2, 3, 5 \pmod 6$. Let C^0, C^1 and C^2 be three connected components in F. We know that at least one of the components $C^2 \neq K_2$. Let P_i be a maximum path in C^i. Let x_{v-i-1} be the first vertex in P_i and x_i be the second vertex in P_i for $i = 0, 1$.

We consider the following subcases.

Case 3.1 C^2 is a star.

If C^2 is a star centered at vertex, say, x_2, then it has at least 5 vertices. So we choose any four and call them x_{v-3}, x_{v-4}, x_{v-5}, and x_{v-6} (see Fig. 4), respectively. Add edges $\{x_0, x_2\}$ and $\{x_1, x_2\}$.

Let $F = F' + T_2 - T_1$ where $T_2 = \{x_0, x_{v-1}\} + \{x_1, x_{v-2}\} + \{x_2, x_{v-3}\} + \{x_2, x_{v-4}\} + \{x_2, x_{v-5}\} + \{x_2, x_{v-6}\}$ and $T_1 = \{x_0, x_2\} + \{x_1, x_2\}$. Clearly F' satisfies condition ($1'$) and ($3'$).

Then $K_v - F = (K_{v-6} - F') + K_{6,v-10} + K_{6,4} + (K_6 - T_2 + T_1)$ where $K_{v-6} - F'$ is defined on $X_v \setminus \{x_{v-1-i} | i \in Z_6\}$, $K_{6,v-10} (v \geq 14)$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup X_v \setminus \{x_i, x_{v-1-j} | i \in Z_4, j \in Z_6\}$, $K_{6,4}$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup \{x_i | i \in Z_4\}$, and $K_6 - T_2 + T_1$ is defined on $\{x_{v-1-i} | i \in Z_6\}$.

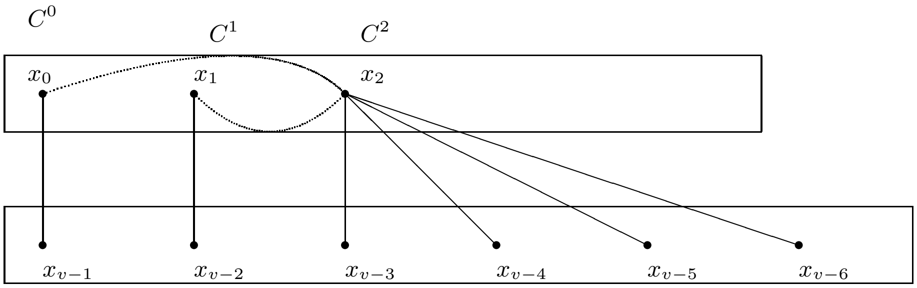

Figure 4 Case 3.1

By checking Table 2, $|E(K_v - F)| \equiv 2, 3, 5 \pmod 6$. Thus $|E(K_{v-6} - F')| \equiv 3, 4, 0 \pmod 6$. By induction, $K_{v-6} - F'$ can be packed with leave C_3, C_4, and $\emptyset$. By Lemma 3.1, $K_{6,v-10} (v \geq 14)$ can be packed by hexagons. By Lemma 2.6, $C_3 + K_{6,4} + (K_6 - T_2 + T_1)$, $C_4 + K_{6,4} + (K_6 - T_2 + T_1)$, and $K_{6,4} + (K_6 - T_2 + T_1)$ can be packed with leave C_8, C_3, and C_5, respectively.

Case 3.2 C^2 is not a star.

If C^2 is not a star, there must be two leaf pairs, call them $\{x_{v-1}, x_{v-2}\}$ (neighbor x_0), $\{x_{v-3}, x_{v-4}\}$ (neighbor x_2), and $\{x_{v-5}, x_{v-6}\}$ (neighbor x_3) (see Fig. 5). Let F' be formed from $F[X_{v-6}]$ and let $\alpha = v - 6$. We check to see that conditions ($1'$) is satisfied.

Now that we have selected 6 special vertices, namely x_{v-6}, x_{v-5}, x_{v-4}, x_{v-3}, x_{v-2}, and x_{v-1}, we proceed as follows. Let F' be formed from $F[X_{v-6}]$ by adding edges $\{x_0, x_2\}$ and $\{x_1, x_2\}$.

Clearly F' spans K_{v-6}. Then either (i) or (ii) holds as follows.

(i) $d_{F'}(x_i) = d_F(x_i)$ for $i = 0, 1, 2$ and $d_{F'}(x_3) = d_F(x_3) - 2$;

(ii) $d_{F'}(x_i) = d_F(x_i)$ for $i \in X_v \setminus \{x_i, x_{v-1-j} | i \in Z_4, j \in Z_6\}$ if C^2 is a star, and $i \in X_v \setminus \{x_i, x_{v-1-j} | i \in Z_4, j \in Z_6\}$ if C^2 is not a star.

Then $K_v - F = (K_{v-6} - F') + K_{6,v-10} + K_{6,4} + (K_6 - T_2 + T_1)$ where $K_{v-6} - F'$ is defined on $X_v \setminus \{x_{v-1-i} | i \in Z_6\}$, $K_{6,v-10} (v \geq 14)$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup X_v \setminus \{x_i, x_{v-1-j} | i \in Z_4, j \in Z_6\}$, $K_{6,4}$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup \{x_i | i \in Z_4\}$, and $K_6 - T_2 + T_1$ is defined on $\{x_{v-1-i} | i \in Z_6\}$.

By checking Table 2, $|E(K_v - F)| \equiv 2, 3, 5 \pmod 6$, thus $|E(K_{v-6} - F')| \equiv 3, 4, 0 \pmod 6$. By induction, $K_{v-6} - F'$ can be packed by hexagons with leave C_3, C_4, and $\emptyset$. $C_3 + (K_{6,4} + K_6 - T_2 + T_1)$, $C_4 + (K_{6,4} + K_6 - T_2 + T_1)$, and $K_{6,4} + K_6 - T_2 + T_1$ can be packed by hexagons with leave $C_4 \cup C_4$, C_3, and C_5 by Lemma 2.7.

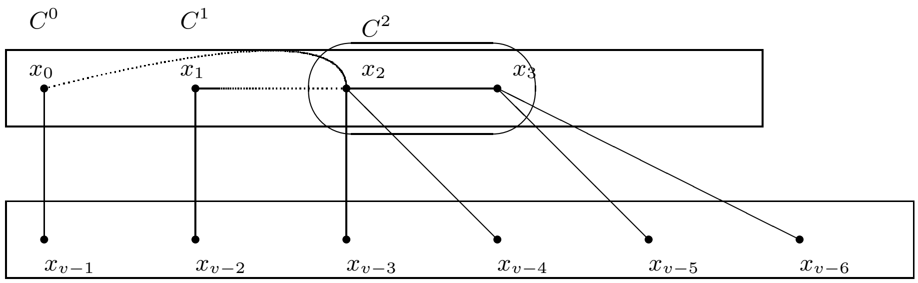

Figure 5 Case 3.2

Case 4 $c(F) \geq 4$.

Case 4.1 Suppose F has three components isomorphic to K_2.

Let the vertex sets of these three components be $\{x_{v-i}, x_{v-i-1}\}$, where $i = 1, 3, 5$. Let $F' = F[X_{v-6}]$ and let $\alpha = v - 6$, and $F = F' + T_1$ where $T_1 = \{x_{v-1}, x_{v-2}\} + \{x_{v-3}, x_{v-4}\} + \{x_{v-5}, x_{v-6}\}$.

We must check to see that F' and $\alpha = v - 6$ satisfy conditions $(1') - (3')$. Since F' is formed by removing the three components of F isomorphic to K_2, $d_{F'}(x_i) = d_F(x_i)$ for each $i \in Z_{v-6}$.

$K_v - F = (K_{v-6} - F') + K_{6,v-6} + (K_6 - T_1)$ where $K_{v-6} - F'$ is defined on $X_v \setminus \{x_{v-1-i} | i \in Z_6\}$, $K_{6,v-6}$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup X_v \setminus \{x_{v-1-i} | i \in Z_6\}$, and $K_6 - T_1$ is defined on $\{x_{v-1-i} | i \in Z_6\}$.

When $|E(K_v - F)| \equiv i \pmod 6$, $|E(K_{v-6} - F')| \equiv i \pmod 6$. By induction, $K_{v-6} - F'$ can be packed by hexagons and leave C_i for $i = 3, 4, 5$, $C_3 \cup C_4$, or C_7 for $i = 1$ and $C_3 \cup C_5$, $C_4 \cup C_4$, or C_8. By Lemma 2.8, $K_6 - T_1$ can be packed by hexagons. $K_{6,v-6} (v \geq 10)$ can be packed by hexagons by Lemma 3.1. Thus, $K_v - F$ can be packed by hexagons with leave C_i.

Case 4.2 Suppose F has three connected components not all isomorphic to K_2.

Let C^0, C^1, C^2, and C^3 be connected components in F. We also know that one of the connected components, say C^3, is not K_2. For $0 \leq i \leq 3$, let P_i be a maximum path in C^i, and let x_{v-i-1} and x_i be the first vertex and the second vertex in P_i, respectively. If C^3 is a star, then let x_{v-5} and x_{v-6} be two additional vertices of degree one adjacent to vertex 3. If C^3 is not a star, then let vertex 4 be the second to last vertex on P_3. Since P_3 is maximal, vertex x_4 is adjacent to at least two vertices of degree one, call them x_{v-5} and x_{v-6} (see Fig. 6).

Now that we have selected 6 special vertices, namely x_{v-6}, x_{v-5}, x_{v-4}, x_{v-3}, x_{v-2}, and x_{v-1}. We proceed as follows. Let F' be formed from $F[X_{v-6}]$ by adding edges

(i) $\{x_0, x_3\}$, $\{x_1, x_4\}$, and $\{x_2, x_4\}$ if C^3 is not a star, and

(ii) $\{x_0, x_3\}$, $\{x_1, x_3\}$, and $\{x_2, x_3\}$ if C^3 is a star.

Clearly, F' spans K_{v-6}. Since either

(i) $d_{F'}(x_i) = d_F(x_i) + 1 - 1$ for $0 \le i \le 3$ and $d_{F'}(x_i) = d_F(x_i) + 2 - 2$ for $i = 4$ or

(ii) $d_{F'}(x_i) = d_F(x_i) + 1 - 1$ for $0 \le i \le 2$ and $d_{F'}(x_i) = d_F(x_i) + 3 - 3$ for $i = 3$, all of the vertices in F' have odd degree, $(1')$ is satisfied.

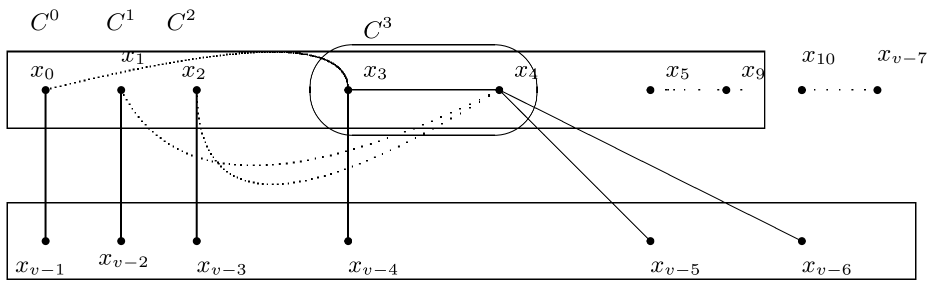

Figure 6 Case 4.2

Let $T_1 = \{x_0, x_3\} + \{x_1, x_4\} + \{x_2, x_4\}$ or $T_1' = \{x_0, x_3\} + \{x_1, x_3\} + \{x_2, x_3\}$, $T_2 = \{x_0, x_{v-1}\} + \{x_1, x_{v-2}\} + \{x_2, x_{v-3}\} + \{x_3, x_{v-4}\} + \{x_4, x_{v-5}\} + \{x_4, x_{v-6}\}$ and $T_2' = \{x_0, x_{v-1}\} + \{x_1, x_{v-2}\} + \{x_2, x_{v-3}\} + \{x_3, x_{v-4}\} + \{x_3, x_{v-5}\} + \{x_3, x_{v-6}\}$.

$F = F' + T_2 - T_1$ or $F = F' + T_2 - T_1'$.

Then $K_v - F = (K_{v-6} - F') + K_{6,v-16} + K_{6,10} + (K_6 - T_2 + T_1)$.

$K_v - F = (K_{v-6} - F') + K_{6,v-16} + K_{6,10} + (K_6 - T_2' + T_1')$ where $K_{v-6} - F'$ is defined on $X_v \setminus \{x_{v-1-i} | i \in Z_6\}$, $K_{6,v-16}$ is defined on $\{x_{v-1-i} | i \in Z_6\} \cup Z_v \setminus \{x_{v-1-i}, x_j | i \in Z_6, j \in Z_{10}\}$, and K_6 is defined on $\{x_{v-1-i} | i \in Z_6\}$.

When $|E(K_v - F)| \equiv i \pmod 6$, $|E(K_{v-6} - F')| \equiv i$ (mod 6). By induction, $K_{v-6} - F'$ can be packed with leave C_i for $i = 3, 4, 5$, $C_3 \cup C_4$, or C_7 for $i = 1$ and $C_3 \cup C_5$, $C_4 \cup C_4$, or C_8 for $i = 2$. $K_{6,v-16}$ can be packed by hexagons by Lemma 3.1. $K_6 - T_2 + T_1$ and $K_6 - T_2' + T_1'$ can be packed by hexagons by Lemmas 2.9 and 2.10. Thus, $K_v - F$ can be packed with leave C_i for $i = 3, 4, 5$, $C_3 \cup C_4$, or C_7 for $i = 1$ and $C_3 \cup C_5$, $C_4 \cup C_4$, or C_8 for $i = 2$. $\Box$

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