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# Remarks on Vertex-Distinguishing IE-Total Coloring of Complete Bipartite Graphs $K_{4,n}$ and $K_{n,n}$

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**Abstract** Let $G$ be a simple graph. An IE-total coloring $f$ of $G$ refers to a coloring of the vertices and edges of $G$ so that no two adjacent vertices receive the same color. Let $C(u)$ be the set of colors of vertex $u$ and edges incident to $u$ under $f$. For an IE-total coloring $f$ of $G$ using $k$ colors, if $C(u) \neq C(v)$ for any two different vertices $u$ and $v$ of $V(G)$, then $f$ is called a $k$-vertex-distinguishing IE-total-coloring of $G$, or a $k$-VDIET coloring of $G$ for short. The minimum number of colors required for a VDIET coloring of $G$ is denoted by $\chi_{vt}^{ie}(G)$, and it is called the VDIET chromatic number of $G$. We will give VDIET chromatic numbers for complete bipartite graph $K_{4,n}$ $(n \geq 4)$, $K_{n,n}$ $(5 \leq n \leq 21)$ in this article.

**Keywords** graphs; IE-total coloring; vertex-distinguishing IE-total coloring; vertex-distinguishing IE-total chromatic number; complete bipartite graph.

**MR(2010) Subject Classification** 05C15

## 1. Introduction and preliminaries

The vertex distinguishing proper edge coloring and point distinguishing general edge coloring were studied in [1-5, 8-9] and [7, 10-14], respectively.

For a total coloring (proper or not) $f$ of $G$ and a vertex $v$ of $G$, denote by $C_f(v)$, or simply $C(v)$ if no confusion arises, the set of colors used to color the vertex $v$ as well as the edges incident to $v$. Let $\overline{C}(v)$ be the complementary set of $C(v)$ in the set of all colors we used. Obviously, $|C(v)| \leq d_G(v) + 1$ and the equality holds if the total coloring is proper.

For a proper total coloring, if $C(u) \neq C(v)$, i.e., $\overline{C}(u) \neq \overline{C}(v)$ for any two distinct vertices $u$ and $v$, then the coloring is called vertex-distinguishing (proper) total coloring and the minimum number of colors required for a vertex-distinguishing (proper) total coloring is denoted by $\chi_{vt}(G)$. This concept has been considered in [6, 15]. The following conjecture was given in [15].

**Conjecture 1** *Suppose $G$ is a simple graph and $n_d$ is the number of vertices of degree $d$,*

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*$\delta \le d \le \Delta$. Let $k$ be the minimum positive integer such that $\binom{k}{d+1} \ge n_d$ for all $d$ such that $\delta \le d \le \Delta$. Then $\chi_{vt}(G) = k$ or $k+1$.*

From [15] we know that the above conjecture is valid for complete graph, complete bipartite graph, path and cycle, etc.

The total coloring of a graph $G$ such that no two adjacent vertices receive the same color is called an IE- total coloring of a graph $G$. If $f$ is an IE- total coloring of graph $G$ using $k$ colors and $\forall u, v \in V(G)$, $u \ne v$, we have $C(u) \ne C(v)$, then $f$ is called $k$-vertex-distinguishing IE-total coloring, or $k$-VDIET coloring. The minimum number $k$ for which $G$ has a vertex-distinguishing IE-total coloring using $k$ colors is denoted by $\chi_{vt}^{ie}(G)$ and called the vertex-distinguishing IE-total chromatic number of graph $G$. The following proposition is obviously true.

**Proposition 1** $\chi_{vt}^{ie}(G) \le \chi_{vt}(G)$.

For a graph $G$, let $n_i$ denote the number of the vertices of degree $i$, $\delta \le i \le \Delta$. Let

$$\xi(G) = \min\{k | \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{s} + \binom{k}{s+1} \ge n_\delta + n_{\delta+1} + \cdots + n_s, \delta \le s \le \Delta\}.$$

Obviously, we have $\chi_{vt}^{ie}(G) \ge \xi(G)$. We will consider the VDIET colorings of complete bipartite graph $K_{4,n}$ ($n \ge 4$) and $K_{n,n}$ ($5 \le n \le 21$) in this paper.

## 2. Vertex distinguishing IE-total chromatic numbers of $K_{4,n}$

**Lemma 1** *For $4 \le n \le 7$, $K_{4,n}$ has a 4-VDIET coloring.*

**Proof** We give a VDIET coloring of $K_{4,n}$ with colors 1, 2, 3, 4 as follows. Let $u_1, u_2, u_3, u_4$ receive color 1. Let $\mathcal{S}_1 = (\{4\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\})$. Let $C(v_j)$ be the $j$-th term of $\mathcal{S}_1$, $j = 1, 2, \ldots, n$. Assign 4 to $v_1$ and all its incident edges. Assign 2 to $v_2$, $u_1v_2$, and $u_3v_2$ and assign 1 to $u_2v_2$, $u_4v_2$; Assign 3 to $v_3$ and 1 to all incident edges of $v_3$; Assign 2 to $v_4$ and 4 to all incident edges of $v_4$; Assign 3 to $v_5$ and 4 to all incident edges of $v_5$ (when $n \ge 5$). Color $u_1v_6, u_2v_6, u_3v_6, u_4v_6, v_6$ by $3, 1, 2, 1, 3$, respectively (when $n \ge 6$). Color $u_1v_7, u_2v_7, u_3v_7, u_4v_7, v_7$ by $2, 3, 2, 4, 2$, respectively (if $n = 7$). For the resulting coloring, $C(u_1) = \{1,2,3,4\}$, $C(u_2) = \{1,3,4\}$, $C(u_3) = \{1,2,4\}$ and $C(u_4) = \{1,4\}$. So the resulting coloring is 4-VDIET coloring of $K_{4,n}$. $\square$

**Lemma 2** *$K_{4,n}$ has a 5-VDIET coloring for $8 \le n \le 23$.*

**Proof** Arrange all the subsets of $\{1,2,3,4,5\}$, except for $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{2,3\}$, $\{1,2,3,4,5\}$, $\{1,3,4,5\}$, $\{1,2,4,5\}$, $\{1,4,5\}$, as follows.

$\mathcal{S}_2 = (\{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{2,3,4,5\})$.

We give a 5-VDIET coloring as follows. Let $u_1, u_2, u_3, u_4$ receive color 1. Let $C(u_1) = \{1,2,3,4,5\}$, $C(u_2) = \{1,3,4,5\}$, $C(u_3) = \{1,2,4,5\}$, $C(u_4) = \{1,4,5\}$. Let $C(v_j)$ be the $j$-th term of $\mathcal{S}_2$, $j = 1, 2, \ldots, n$. Obviously, $C(u_i) \cap C(v_j) \ne \emptyset$, $1 \le i \le 4$, $1 \le j \le n$. Assign 4 to $v_1$ and all its incident edges and assign 5 to $v_2$ and all its incident edges. Color $u_1v_3, u_2v_3$,

$u_3v_3, u_4v_3$ and $v_3$ by $2, 1, 2, 1$ and $2$, respectively. Color $u_1v_4, u_2v_4, u_3v_4, u_4v_4$ and $v_4$ by $3, 3, 1, 1$ and $3$, respectively. For $j \geq 5$, if $C(v_j) = \{1, b\}$, then color $v_j$ by $b$ and $u_1v_j, u_2v_j, u_3v_j, u_4v_j$ by 1. If $C(v_j) = \{2, b\}$, $b = 4$ or $5$, then color $v_j$ by 2 and $u_1v_j, u_2v_j, u_3v_j, u_4v_j$ by $b$. If $C(v_j) = \{a, b\}$, $2 < a < b$, then color $v_j, u_1v_j, u_2v_j$ by $a$ and $u_3v_j, u_4v_j$ by $b$. If $C(v_j) = \{a, b, c\} \neq \{1, 2, 3\}$, $1 \leq a < b < c \leq 5$, then color $v_j$ by $b$, $u_1v_j$ by $a$, $u_2v_j, u_3v_j, u_4v_j$ by $c$. If $C(v_j) = \{1, 2, 3\}$, then color $v_j$ by 2, color $u_1v_j$ by 3, and color $u_2v_j, u_3v_j, u_4v_j$ by 1. If $C(v_j) = \{a, b, c, d\}$, $a < b < c < d$, then let $u_1v_j, u_2v_j, u_3v_j, u_4v_j$ and $v_j$ receive $b, c, a, d$ and $d$.

It is easy to verify that the resulting coloring is the required coloring. □

**Lemma 3** *If $24 \leq n \leq 55$, then $K_{4,n}$ has 6-VDIET coloring.*

**Proof** We give a sequence of all subsets of $\{1, 2, 3, 4, 5, 6\}$, except for $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{2, 3\}$, $\{1, 2, 3, 4, 5, 6\}$, $\{1, 3, 4, 5, 6\}$, $\{1, 2, 4, 5, 6\}$, $\{1, 4, 5, 6\}$, as follows.

$\mathcal{S}_3 = (\{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{2, 3, 4, 5, 6\})$.

The first 5 subsets of $\mathcal{S}_3$ are $\{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}$, respectively. Obviously, $\mathcal{S}_3$ has 55 terms (each term is a subset). Now we give a 6-VDIET coloring of $K_{4,n}$ as follows.

Let $u_1, u_2, u_3, u_4$ receive color 1. Let $C(u_1) = \{1, 2, 3, 4, 5, 6\}$, $C(u_2) = \{1, 3, 4, 5, 6\}$, $C(u_3) = \{1, 2, 4, 5, 6\}$, $C(u_4) = \{1, 4, 5, 6\}$. Let $C(v_j)$ be the $j$-th term of $\mathcal{S}_3$, $j = 1, 2, \ldots, n$. We color $v_j$ and its incident edges by $j + 3$, $j = 1, 2, 3$. We color $u_1v_4, u_2v_4, u_3v_4, u_4v_4, v_4$ by $2, 1, 2, 1$ and $2$, respectively. We color $u_1v_5, u_2v_5, u_3v_5, u_4v_5, v_5$ by $3, 3, 1, 1$ and $3$, respectively. For $j \geq 6$, if $C(v_j) = \{a, b\}$, $a < b$, then assign $a$ to $u_1v_j$, and $b$ to $u_2v_j, u_3v_j, u_4v_j$ and $v_j$.

If $C(v_j) = \{1, a, b\}$, $1 < a < b$, then assign 1 to $u_2v_j, u_3v_j, u_4v_j$, and assign $a$ and $b$ to $u_1v_j$ and $v_j$, respectively. If $C(v_j) = \{a, b, c\}$, $2 \leq a < b < c$, then assign $c$ to $u_2v_j, u_3v_j, u_4v_j$, and assign $a$ and $b$ to $u_1v_j$ and $v_j$, respectively. If $C(v_j) = \{1, 2, a, b\}$, $3 \leq a < b$, then assign $a, 1, 2, 1, b$ to $u_1v_j, u_2v_j, u_3v_j, u_4v_j$, and $v_j$, respectively. If $C(v_j) = \{a, b, c, d\}$, $a < b < c < d$, $a > 1$ or $a = 1, b > 2$, then assign $a, b, c, d, d$ to $u_1v_j, u_2v_j, u_3v_j, u_4v_j$, and $v_j$, respectively. If $C(v_j) = \{1, 2, 3, a, b\}$, then assign $1, 3, 2, a, b$ to $u_1v_j, u_2v_j, u_3v_j, u_4v_j$, and $v_j$, respectively. If $C(v_j) = \{2, 3, 4, 5, 6\}$, then assign $2, 3, 4, 5, 6$ to $u_1v_j, u_2v_j, u_3v_j, u_4v_j$, and $v_j$, respectively.

It can be easily verified that the above coloring is a 6-VDIET coloring of $K_{4,n}$, where $24 \leq n \leq 55$. □

**Lemma 4** *If $56 \leq n \leq 115$, then $K_{4,n}$ has a 7-VDIET coloring. If $\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{5} - 4 < n \leq \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{5} - 4$, where $k \geq 8$, then $K_{4,n}$ has a $k$-VDIET coloring.*

**Proof** We give an order for all 1-combinations, 2-combinations, 3-combinations, 4-combinations and 5-combinations of $\{1, 2, \ldots, k\}$, except for, $\{1\}$, $\{2\}$, $\{3\}$, $\{2, 3\}$ such that the first $k - 1$ terms are $\{4\}, \{5\}, \ldots, \{k\}, \{1, 2\}, \{1, 3\}$, respectively. Obviously the resulting sequence, denoted by $\mathcal{S}_4$, has $\binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\binom{k}{4}+\binom{k}{5}-4$ terms (each term is a subset).

We give a coloring of $K_{4,n}$ as follows.

Let $u_1,u_2,u_3,u_4$ receive color 1. Let $C(u_1)=\{1,2,\ldots,k\}$, $C(u_2)=C(u_1)-\{2\}$, $C(u_3)=C(u_1)-\{3\}$, $C(u_4)=C(u_1)-\{2,3\}$. Let $C(v_j)$ be the $j$-th term of $\mathcal{S}_4$, $j=1,2,\ldots,n$. Let $v_j$ and its incident edges receive $j+3$, $j=1,2,\ldots,k-3$. Let $u_1v_{k-2},u_2v_{k-2},u_3v_{k-2},u_4v_{k-2},v_{k-2}$ receive $2,1,2,1$ and $2$. Let $u_1v_{k-1},u_2v_{k-1},u_3v_{k-1},u_4v_{k-1},v_{k-1}$ receive $3,3,1,1,3$.

For $j\geq k$, if $C(v_j)=\{a,b\}$, then assign $a$ to $u_1v_j$ and $b$ to $u_2v_j,u_3v_j,u_4v_j$ and $v_j$. If $C(v_j)=\{1,a,b\}$, $1<a<b$, then assign 1 to $u_2v_j,u_3v_j,u_4v_j$ and assign $a$ and $b$ to $u_1v_j$ and $v_j$, respectively. If $C(v_j)=\{a,b,c\}$, $2\leq a<b<c$, then assign $c$ to $u_2v_j,u_3v_j,u_4v_j$ and assign $a$ and $b$ to $u_1v_j$ and $v_j$, respectively. If $C(v_j)=\{1,2,a,b\}$, $3\leq a<b$, then assign $a,1,2,1,b$ to $u_1v_j,u_2v_j,u_3v_j,u_4v_j$ and $v_j$, respectively. If $C(v_j)=\{a,b,c,d\}$, $a<b<c<d$, $a>1$ or $a=1,b>2$, then assign $a,b,c,d,d$ to $u_1v_j,u_2v_j,u_3v_j,u_4v_j$ and $v_j$, respectively. If $C(v_j)=\{1,2,a,b,c\}$, then assign $1,a,2,b,c$ to $u_1v_j,u_2v_j,u_3v_j,u_4v_j$ and $v_j$, respectively. If $C(v_j)=\{a,b,c,d,e\}$, $a<b<c<d<e$, $a>1$ or $a=1,b>2$, then assign $a,b,c,d,e$ to $u_1v_j,u_2v_j,u_3v_j,u_4v_j$ and $v_j$, respectively.

It is easy to verify that the resulting coloring is a $k$-VDIET coloring of $K_{4,n}$. $\Box$

**Lemma 5** *If $4\leq n\leq 11$, then $\xi(K_{4,n})=4$; If $12\leq n\leq 27$, then $\xi(K_{4,n})=5$; If $28\leq n\leq 59$, then $\xi(K_{4,n})=6$.*

**Proof** This lemma is obviously true. $\Box$

**Theorem 1** *For $4\leq n\leq 58$, we have*

$$\chi^{ie}_{vt}(K_{4,n})=\begin{cases}4, & 4\leq n\leq 7;\\ 5, & 8\leq n\leq 23;\\ 6, & 24\leq n\leq 55;\\ 7, & 56\leq n\leq 58.\end{cases}$$

**Proof** (a) When $4\leq n\leq 7$, $\chi^{ie}_{vt}(K_{4,n})\geq\xi(K_{4,n})=4$ by Lemma 5. By Lemma 1, we know that $\chi^{ie}_{vt}(K_{4,n})=4$.

(b) When $8\leq n\leq 11$, $\chi^{ie}_{vt}(K_{4,n})\geq\xi(K_{4,n})=4$ by Lemma 5. Assume that $K_{4,n}$ has a VDIET coloring $g$ with colors 1, 2, 3, 4. Obviously, $|C(u_i)|\geq 2$, $i=1,2,3,4$.

(1) The colors of $u_1,u_2,u_3,u_4$ are the same. We may suppose that $g(u_1)=g(u_2)=g(u_3)=g(u_4)=1$. $C(v_j)\neq\{1\}$ (for the vertex coloring is proper). There exist $l,t\in\{2,3,4\}$, such that $l<t$, $\{l\}\neq C(v_j)\neq\{t\}$, $j=1,2,\ldots,n$, for otherwise there will be two same sets among the sets $C(u_1),C(u_2),C(u_3),C(u_4)$. If $\{p\}$ is the color set of some $v_j$, $p\in\{1,2,3,4\}-\{1,t,l\}$, then $\{1,p\}\subseteq C(u_i)$, $1\leq i\leq 4$. So $\{C(u_1),C(u_2),C(u_3),C(u_4)\}=\{\{1,2,3,4\},\{1,2,3,4\}-\{l\},\{1,2,3,4\}-\{t\},\{1,2,3,4\}-\{l,t\}\}$. Thus $\{l,t\}$ is not the color set of any vertex. Thereby $\{1\}$, $\{l\}$, $\{t\}$, $\{l,t\}$ are not the color set of any vertex. The number of subsets of $\{1,2,3,4\}$, except for $\emptyset,\{1\}$, $\{l\}$, $\{t\}$, $\{l,t\}$, is 11, but the number of vertices of $K_{4,n}$ is $n+4\geq 12$. This is a contradiction. If $C(v_j)\neq\{p\}$, $1\leq j\leq n$, then $\emptyset,\{1\}$, $\{l\}$, $\{t\}$, $\{p\}$ are not the color set of any vertex. We also get a contradiction.

(2) There are just two distinct elements in $\{g(u_1), g(u_2), g(u_3), g(u_4)\}$, say $g(u_1) = 1$, $g(u_2) = 2, g(u_3)$, $g(u_4) \in \{1, 2\}$. Then $C(u_i) \neq \{1\}, \{2\}$, $i = 1, 2, 3, 4$; $C(v_j) \neq \{1\}, \{2\}, \{1, 2\}$, $j = 1, 2, \ldots, n$. Obviously, there exists $l \in \{3, 4\}$, such that $C(v_j) \neq \{l\}, j = 1, 2, \ldots, n$. If there exists exactly one $l \in \{3, 4\}$, such that $C(v_j) \neq \{l\}$, $j = 1, 2, \ldots, n$, say $l = 3$, then $4 \in C(u_i), i = 1, 2, 3, 4$. So $\emptyset, \{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$ are not the color set of any vertex. This is a contradiction. If $\{3\} \neq C(v_j) \neq \{4\}$, $1 \leq j \leq n$, then $\emptyset, \{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ are not the color set of any vertex. This is also a contradiction.

(3) There are just three distinct elements in $\{g(u_1), g(u_2), g(u_3), g(u_4)\}$, say $g(u_i) = i$, $i = 1, 2, 3$, $g(u_4) \in \{1, 2, 3\}$. If $C(v_j) \neq \{4\}, j = 1, 2, \ldots, n$, then $\emptyset, \{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ are not the color set of any vertex. This is a contradiction. If $C(v_{j_0}) = \{4\}$ for some $j_0 \in \{1, 2, \ldots, n\}$, then $\emptyset, \{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ are not the color set of any vertex. This is also a contradiction.

(4) The colors of vertex $u_1, u_2, u_3, u_4$ are distinct. Without loss of generality, we assume that $g(u_i) = i$, $i = 1, 2, 3, 4$, then $\emptyset, \{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ are not the color set of any vertex. This is a contradiction. Thus $K_{4,n}$ has no 4-VDIET coloring. So $\chi^{ie}_{vt}(K_{4,n}) \geq 5$. Combining this with Lemma 2, we know that $\chi^{ie}_{vt}(K_{4,n}) = 5$, if $n = 8, \ldots, 11$.

(c) When $12 \leq n \leq 23$, we know that $\chi^{ie}_{vt}(K_{4,n}) = 5$ by Lemmas 2 and 5.

(d) When $n = 24, 25, 26, 27$, we have $\chi^{ie}_{vt}(K_{4,n}) \geq 5$. Assume that $K_{4,n}$ has a VDIET coloring using colors 1, 2, 3, 4, 5. Completely similar to the proof of the result that $K_{4,n}$ has no 4-VDIET coloring if $8 \leq n \leq 11$ in (b), we can show that $K_{4,n}$ has no 5-VDIET coloring if $24 \leq n \leq 27$. So $\chi^{ie}_{vt}(K_{4,n}) \geq 6$, and combining this with Lemma 3 gives $\chi^{ie}_{vt}(K_{4,n}) = 6$.

(e) When $28 \leq n \leq 55$, we can prove that $\chi^{ie}_{vt}(K_{4,n}) = 6$ by Lemmas 3 and 5.

(f) Suppose $n = 56, 57, 58$. From Lemma 5 we know that $\chi^{ie}_{vt}(K_{4,n}) \geq 6$. Completely similar to the proof of the result that $K_{4,n}$ has no 4-VDIET coloring if $8 \leq n \leq 11$ in (b), we can show that $K_{4,n}$ has no 6-VDIET coloring if $n = 56, 57, 58$, so $\chi^{ie}_{vt}(K_{4,n}) \geq 7$. Combining this with Lemma 4, we know that $\chi^{ie}_{vt}(K_{4,n}) = 7$.

The proof is completed. □

**Theorem 2** *If* $\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{5} - 4 < n \leq \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{5} - 4, k \geq 7$, *then* $\chi^{ie}_{vt}(K_{4,n}) = k$.

**Proof** Assume that $K_{4,n}$ has a $(k-1)$-VDIET coloring $g$.

**Case 1** The colors of $u_1, u_2, u_3, u_4$ are the same. We may assume that $g(u_1) = g(u_2) = g(u_3) = g(u_4) = 1$. Obviously, we have that $C(v_j) \neq \{1\}$, $j = 1, 2, \ldots, n$. There exist two colors $l, t \in \{2, 3, \ldots, k-1\}$, such that $\{l\} \neq C(v_j) \neq \{t\}$, $j = 1, 2, \ldots, n$, for otherwise there exists a color in $\{2, 3, \ldots, k-1\}$, say 2, such that $C(u_i) \supseteq \{1, 3, 4, \ldots, k-1\}$, $i = 1, 2, 3, 4$. So $C(u_1), C(u_2)$, $C(u_3), C(u_4)$ are all equal to $\{1, 3, 4, \ldots, k-1\}$ or $\{1, 2, 3, 4, \ldots, k-1\}$. This is a contradiction.

Without loss of generality, suppose $\{2\} \neq C(v_j) \neq \{3\}$.

(1) $\{4\}, \{5\}, \ldots, \{k-1\}$ are all the color sets of some $v_j$'s, $j = 1, 2, \ldots, n$. Then $C(u_i) \supseteq$

$\{1,4,5,\ldots,k-1\}$, and $\{C(u_1),C(u_2),C(u_3),C(u_4)\}=\{\{1,4,5,\ldots,k-1\},\{1,2,4,5,\ldots,k-1\},\{1,3,4,5,\ldots,k-1\},\{1,2,3,4,5,\ldots,k-1\}\}$. So $\{2,3\}$ is not the color set of any vertex $v_j$. Thus $\{1\}$, $\{2\}$, $\{3\}$, $\{2,3\}$ are not available for any $v_j$. And $\binom{k-1}{1}+\binom{k-1}{2}+\cdots+\binom{k-1}{5}-4(<n)$ subsets cannot distinguish $n$ vertices $v_1,v_2,\ldots,v_n$. This is a contradiction.

(2) $\exists r\in\{4,5,\ldots,k-1\}$, such that $C(v_j)\neq\{r\}$, $1\le j\le n$. Then $\{1\}$, $\{2\}$, $\{3\}$, $\{r\}$ are not available for any $v_j$. It is also a contradiction.

**Case 2** There are only two different colors among $g(u_1),g(u_2),g(u_3),g(u_4)$. Without loss of generality we assume that $g(u_1)=1,g(u_2)=2$, $g(u_3),g(u_4)\in\{1,2\}$. If for each $r\in\{3,4,\ldots,k-1\}$, $\{r\}$ is a color set of some vertex $v_j$, then $C(u_i)\supseteq\{3,4,\ldots,k-1\}$. Hence each $C(u_i)$, $i=1,2,3,4$, is equal to one of the following sets $\{1,3,4,\ldots,k-1\}$, $\{2,3,4,\ldots,k-1\}$, $\{1,2,3,4,\ldots,k-1\}$. Three subsets cannot distinguish 4 vertices $u_1,u_2,u_3,u_4$, this is a contradiction. If there exists $r\in\{3,4,\ldots,k-1\}$, such that $\{r\}$ is not a color set of any vertex $v_j$, then $\{1\}$, $\{2\}$, $\{1,2\}$, $\{r\}$ are not available for any vertex $v_j,j=1,2,\ldots,n$. The number of available subsets (for $v_j$) is at most $\binom{k-1}{1}+\binom{k-1}{2}+\cdots+\binom{k-1}{5}-4$. But we have $n$ vertices $v_1,v_2,\ldots,v_n$, leading to a contradiction.

**Case 3** In $\{g(u_1),g(u_2),g(u_3),g(u_4)\}$, there are at least three different colors. Without loss of generality we assume that $g(u_i)=i,i=1,2,3$. Then $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$ are not the color sets of any vertex $v_j,1\le j\le n$. So the number of available subsets (for $v_j$) is at most $\binom{k-1}{1}+\binom{k-1}{2}+\cdots+\binom{k-1}{5}-7<\binom{k-1}{1}+\binom{k-1}{2}+\cdots+\binom{k-1}{5}-4<n$. This is a contradiction.

So $K_{4,n}$ has no VDIET coloring using $(k-1)$ colors. i.e., $\chi^{ie}_{vt}(K_{4,n})\ge k$. From this result and Lemma 4, we know that $\chi^{ie}_{vt}(K_{4,n})=k$. The proof is completed. □

## 3. Vertex distinguishing IE-total chromatic numbers of $K_{n,n}$ with $5\le n\le 21$

**Theorem 3** *For complete graph $K_{5,5}$, we have $\chi^{ie}_{vt}(K_{5,5})=4$.*

**Proof** Obviously, $\chi^{ie}_{vt}(K_{5,5})\ge\xi(K_{5,5})=4$. In order to complete the proof of this theorem, we give a VDIET coloring using 4 colors 1, 2, 3, 4 as follows.

Let $u_1,u_2,u_3,u_4,u_5$ receive color 1. Let $C(u_1)=\{1,2,3,4\}$, $C(u_2)=\{1,2,4\}$, $C(u_3)=\{1,2,3\}$, $C(u_4)=\{1,3,4\}$, $C(u_5)=\{1,4\}$; $C(v_1)=\{1,2\}$, $C(v_2)=\{1,3\}$, $C(v_3)=\{3,4\}$, $C(v_4)=\{2,4\}$, $C(v_5)=\{2,3,4\}$.

For $i=1,2,3,4,5$, we assign color 2 to $u_iv_1$ if $2\in C(u_i)\cap C(v_1)$ and assign color 1 to $u_iv_1$ otherwise. We assign 3 to $u_iv_2$ if $3\in C(u_i)\cap C(v_2)$ and assign color 1 to $u_iv_2$ otherwise. We assign color 4 to $u_iv_3$ if $4\in C(u_i)\cap C(v_3)$ and assign color 3 to $u_iv_3$ otherwise. And then we color $u_1v_4,u_2v_4,u_3v_4,u_4v_4,u_5v_4$ by $2,4,2,4$ and 4, respectively and color $u_1v_5,u_2v_5,u_3v_5,u_4v_5,u_5v_5$ by $2,4,2,3$ and 4, respectively. We assign $2,3,4,2,3$ to vertices $v_1,v_2,v_3,v_4$ and $v_5$, respectively.

It is easy to verify that the resulting coloring is 4-VDIET coloring of $K_{5,5}$.

The proof is completed. □

**Lemma 6** *$K_{6,6}$ and $K_{7,7}$ has no 4-VDIET coloring.*

**Proof** Assume that $K_{n,n}$ has a 4-VDIET coloring $g$, $n = 6, 7$.

**Case 1** The colors of $u_1, u_2, \ldots, u_n$ are the same. Without loss of generality, we assume $g(u_i) = 1$, $i = 1, 2, \ldots, n$.

(1) $|C(u_i)| \geq 2$, $i = 1, 2, \ldots, n$. Otherwise the color set of each vertex contains 1, but the number of subsets of $\{1, 2, 3, 4\}$ which contain 1 is only 8. Eight subsets cannot distinguish 12 vertices or 14 vertices.

(2) $C(v_j) \neq \{1\}$, $j = 1, 2, \ldots, n$. This is obvious for no two adjacent vertices receive the same color.

(3) $|C(v_j)| \geq 2$, $j = 1, 2, \ldots, n$. Otherwise $C(v_{j_0}) = \{l\}$, $j_0 \in \{1, 2, \ldots, n\}, l \in \{2, 3, 4\}$. Then $\{1, l\} \subseteq C(u_i)$, $i = 1, 2, \ldots, n$. But the number of subsets of $\{1, 2, 3, 4\}$ which contain $\{1, l\}$ is 4. These 4 subsets cannot distinguish $u_1, u_2, \ldots, u_n$. So the number of subsets of $\{1, 2, 3, 4\}$ which contain at least 2 elements is 11 and the number of vertices of $K_{n,n}$ is $2n \geq 12$. This is a contradiction.

**Case 2** There are just two different elements in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$.

We may assume $g(u_i) = i$, $i = 1, 2$.

(1) $C(v_j) \neq \{1\}, \{2\}, \{1, 2\}$, $j = 1, 2, \ldots, n$.

(2) $|C(v_j)| \geq 2$, $j = 1, 2, \ldots, n$. Otherwise if some $|C(v_{j_0})| = 1$, say $C(v_{j_0}) = \{3\}$, then each $C(u_i)$, $i = 1, 2, \ldots, n$, is one of the following subsets: $\{1, 3\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$, $\{2, 3\}$, $\{2, 3, 4\}$. We immediately obtain a contradiction if $n = 7$. In the case $n = 6$, each $C(v_j)$ is not among the above 6 subsets or the complementary subsets of the above 6 subsets in $\{1, 2, 3, 4\}$. So each $C(v_j)$ is one of the following sets: $\{3\}, \{3, 4\}, \{1, 2, 4\}$. This is a contradiction.

(3) $C(u_i) \neq \{3\}, \{4\}$, $i = 1, 2, \ldots, n$.

(4) If some $C(u_{i_1}) = \{1\}$ and some $C(u_{i_2}) = \{2\}$, then $\{1, 2\} \subseteq C(v_j)$, $1 \leq j \leq n$. The number of subsets of $\{1, 2, 3, 4\}$ which contain 1 and 2 is 4. Four subsets cannot distinguish $n$ vertices $v_1, v_2, \ldots, v_n (n = 6, 7)$. This is a contradiction. If $C(u_i) \neq \{1\}, \{2\}$, $i = 1, 2, \ldots, n$, then $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}$ are not the color set of any vertex. But the number of subsets of $\{1, 2, 3, 4\}$ which contain at least 2 elements is 11. Eleven subsets cannot distinguish $2n(\geq 12)$ vertices. This is a contradiction. If $C(u_i) \neq \{1\}$, $i = 1, 2, \ldots, n$, and some $C(u_{i_0}) = \{2\}$, then the color set of each vertex is not equal to $\emptyset, \{1\}, \{3\}, \{4\}, \{3, 4\}$. Note that in this case $2 \in C(v_j), 1 \leq j \leq n$. This is a contradiction. We can also get a contradiction in the case $C(u_i) \neq \{2\}, 1 \leq i \leq n$, and some $C(u_{i_0}) = \{1\}$.

**Case 3** There are just 3 different elements in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. In this case the colors of $v_1, v_2, \ldots, v_n$ are the same. So similarly to Case 1, we can obtain a contradiction (we only exchange $u$ and $v$ in Case 1).

**Case 4** There are just 4 different elements in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. In this case the color of $v_1$ must be in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. This is a contradiction. □

**Theorem 4** *If $6 \le n \le 11$, then $\chi^{ie}_{vt}(K_{n,n}) = 5$.*

**Proof** From Lemma 6 we know $\chi^{ie}_{vt}(K_{n,n}) \ge 5$, if $n = 6, 7$. And $\chi^{ie}_{vt}(K_{n,n}) \ge \xi(K_{n,n}) = 5$, $n = 8, 9, 10, 11$. So we only need to give a 5-VDIET coloring of $K_{n,n}$, when $6 \le n \le 11$.

**Case 1** $6 \le n \le 10$. Let

$\mathscr{A} = (\{1,2,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}, \{1,2,3,4\}, \{1,3,4,5\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\})$,

$\mathscr{B} = (\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,2,3\}, \{3,4,5\}, \{2,4,5\}, \{2,3,5\}, \{2,3,4\}, \{4,5\})$.

Let $C(u_i)$ be the $i$-th term of $\mathscr{A}$, and $C(v_i)$ be the $i$-th term of $\mathscr{B}$, $i = 1, 2, \dots, n$. Let $u_1, u_2, \dots, u_n$ receive color 1, and vertex $v_j$ receive a color in $C(v_j) - \{1\}$, $j = 1, 2, \dots, n$. For $1 \le i \le n$, $1 \le j \le 4$, if $j + 1 \in C(u_i) \cap C(v_j)$, then assign $j + 1$ to $u_iv_j$; if $j + 1 \overline{\in} C(u_i) \cap C(v_j)$, then assign 1 to $u_iv_j$. For $1 \le i \le 5$, $5 \le j \le n$, if $i \in C(u_i) \cap C(v_j)$, then assign $i$ to $u_iv_j$; if $i \overline{\in} C(u_i) \cap C(v_j)$, then assign a color in $C(u_i) \cap C(v_j)$ to $u_iv_j$. For $6 \le i \le n$, $5 \le j \le n$, we assign a color in $C(u_i) \cap C(v_j)$ to $u_iv_j$.

We can verify that the above coloring is a 5-VDIET coloring of $K_{n,n}$, when $6 \le n \le 10$.

**Case 2** $n = 11$.

Let $C(u_1) = \{1,2,3,4,5\}$, $C(u_2) = \{1,2,3,4\}$, $C(u_3) = \{1,2,3,5\}$, $C(u_4) = \{1,2,4,5\}$, $C(u_5) = \{1,3,4,5\}$, $C(u_6) = \{1,3,4\}$, $C(u_7) = \{2,3,4,5\}$, $C(u_8) = \{2,3,5\}$, $C(u_9) = \{2,4,5\}$, $C(u_{10}) = \{1,2,5\}$, $C(u_{11}) = \{2,5\}$. Let $C(v_1) = \{1,5\}$, $C(v_2) = \{2,3\}$, $C(v_3) = \{3,5\}$, $C(v_4) = \{2,4\}$, $C(v_5) = \{4,5\}$, $C(v_6) = \{1,2,3\}$, $C(v_7) = \{1,2,4\}$, $C(v_8) = \{1,3,5\}$, $C(v_9) = \{1,4,5\}$, $C(v_{10}) = \{2,3,4\}$, $C(v_{11}) = \{3,4,5\}$.

We give a 5-VDIET coloring of $K_{11,11}$ according to the above color sets.

Let $u_1, u_2, u_3, u_4, u_5, u_6$ receive color 1 and $u_7, u_8, u_9, u_{10}, u_{11}$ receive color 2. For each $i = 1, 2, \dots, 11$, we color $u_iv_1$ by 1 if $1 \in C(u_i) \cap C(v_1)$ and color $u_iv_1$ by 5 otherwise; We color $u_iv_2$ by 2 if $2 \in C(u_i) \cap C(v_2)$ and color $u_iv_2$ by 3 otherwise; We color $u_iv_3$ by 3 if $3 \in C(u_i) \cap C(v_3)$ and color $u_iv_3$ by 5 otherwise; We color $u_iv_4$ by 4 if $4 \in C(u_i) \cap C(v_4)$ and color $u_iv_4$ by 2 otherwise; We color $u_iv_5$ by 5 if $5 \in C(u_i) \cap C(v_5)$ and color $u_iv_5$ by 4 otherwise. For each $i = 1, 2, 3, 4, 5$, $j = 6, 7, \dots, 11$, we assign $i$ to edge $u_iv_j$ if $i \in C(u_i) \cap C(v_j)$ and assign a color in $C(u_i) \cap C(v_j)$ to $u_iv_j$ if $i \overline{\in} C(u_i) \cap C(v_j)$. For each $i = 6, 7, \dots, 11$, $j = 6, 7, \dots, 11$, we assign a color in $C(u_i) \cap C(v_j)$ to edge $u_iv_j$. For each $j = 1, 2, \dots, 11$, we color $v_j$ by a color in $C(v_j) - \{1, 2\}$.

The above coloring is the required coloring. $\Box$

**Lemma 7** *$K_{n,n}$ has no 5-VDIET coloring if $n = 12, 13, 14, 15$.*

**Proof** Assume that $K_{n,n}$ has a 5-VDIET coloring $g$ with colors $1, 2, 3, 4, 5$ and $n \in \{12, 13, 14, 15\}$.

**Case 1** The colors of $u_1, u_2, \dots, u_n$ are the same. We may suppose $g(u_i) = 1$, $i = 1, 2, \dots, n$.

The complementary subset of each $C(u_i)$ in $\{1,2,3,4,5\}$ is not the color set of any vertex, $1 \le i \le n$. So we have at most $2^5 - n \le 2^5 - 12 = 20$ available subsets. This is a contradiction.

**Case 2** There are just two different elements in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. We may suppose $g(u_1) = 1, g(u_2) = 2$.

(1) $C(v_j) \neq \{1\}, \{2\}, \{1, 2\}$, $1 \leq j \leq n$, $C(u_i) \neq \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}$, $1 \leq i \leq n$.

(2) $C(v_j) \neq \{3\}, \{4\}, \{5\}$, $1 \leq j \leq n$. Otherwise if some $C(v_{j_0}) = \{5\}$, then each $C(u_i)$ is one of the following sets: $\{1, 5\}, \{2, 5\}$, $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$, $\{1, 2, 3, 4, 5\}$. We immediately get a contradiction when $n = 13, 14, 15$. When $n = 12$, any one of the complementary subsets of the above 12 subsets in $\{1, 2, 3, 4, 5\}$ is not the color set of any vertex. This is also a contradiction.

(3) $|C(u_i)| \geq 2$, $1 \leq i \leq n$. Otherwise if some $C(u_{i_0}) = \{1\}$, then any one of the 7 nonempty subsets of $\{3, 4, 5\}$ is not the color set of any vertex. In this case for $i = 1, 2, \ldots, n$, $C(u_i) \neq \{2\}$(For otherwise each $C(v_j) \supseteq \{1, 2\}$ and the number of subsets of $\{1, 2, 3, 4, 5\}$ which contain 1 and 2 is 8 and we have $n(\geq 12)$ vertices $v_1, v_2, \ldots, v_n$). Except for $\emptyset, \{2\}$ and nonempty subsets of $\{3, 4, 5\}$, we have 23 subsets of $\{1, 2, 3, 4, 5\}$ left. Such 23 subsets cannot distinguish $2n(\geq 24)$ vertices. This is also a contradiction.

Now we give two facts before further discussion.

Fact 1. For $1 \leq i, j \leq n$, we have $C(u_i) \cap C(v_j) \neq \emptyset$.

Fact 2. If $A \subseteq \{1, 2, 3, 4, 5\}$ is a color set of some $u_i$ (or $v_i$), then each subset $B$ of $\{1, 2, 3, 4, 5\} - A$ is also a color set of some $u_j$ (or $v_j$) when $B$ is a color set of some vertex.

(4) $C(u_i) \neq \{1, 2\}$, $1 \leq i \leq n$. Otherwise if some $C(u_{i_0}) = \{1, 2\}$, then from Fact 1 we know that $\{3, 4, 5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$ as well as 0-subsets, 1-subsets of $\{1, 2, 3, 4, 5\}$ are not the color set of any vertex. Thus the number of the remaining subsets is 22. This is a contradiction.

(5) From the foregoing discussion we know that each color set belongs to $2^{\{1,2,3,4,5\}} - \{\emptyset, \{1, 2\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. So we can obtain a contradiction if $n = 13, 14, 15$ and we can obtain the color sets of all vertices of $K_{12,12}$ by deleting exactly one set in $2^{\{1,2,3,4,5\}} - \{\emptyset, \{1, 2\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. Let $\mathcal{S} = \{\{1, 3\}, \{2, 4, 5\}, \{1, 4\}, \{2, 3, 5\}, \{1, 5\}, \{2, 3, 4\}, \{2, 3\}, \{1, 4, 5\}, \{2, 4\}, \{1, 3, 5\}, \{2, 5\}, \{1, 3, 4\}, \{3, 4\}, \{1, 2, 5\}, \{3, 5\}, \{1, 2, 4\}, \{4, 5\}, \{1, 2, 3\}\}$. There are 18 sets in $\mathcal{S}$. When $n = 12$, by Fact 2 we know that there are at least 16 subsets in $\mathcal{S}$ which are all the color sets of $v_j$'s. This is a contradiction.

**Case 3** There are just 3 or 4 different elements in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. Then there are 1 or 2 different elements in $\{g(v_1), g(v_2), \ldots, g(v_n)\}$. Similarly to Case 1 or Case 2, we can get a contradiction (we only exchange $u$ and $v$ in Case 1 or Case 2).

**Case 4** There are just 5 different elements in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. Then the color of $v_1$ must be in $\{g(u_1), g(u_2), \ldots, g(u_n)\}$. This is a contradiction.

The proof is completed. □

**Theorem 5** *If* $12 \leq n \leq 21$, *then* $\chi_{vt}^{ie}(K_{n,n}) = 6$.

**Proof** From Lemma 7, we know that $\chi_{vt}^{ie}(K_{n,n}) \geq 6$, if $n = 12, 13, 14, 15$. But $\chi_{vt}^{ie}(K_{n,n}) \geq 6$, if $n = 16, 17, \ldots, 21$. Thus we only need to show that $K_{n,n}$ has a 6-VDIET coloring with colors

1, 2, 3, 4, 5, 6, if $12 \le n \le 21$. Let $\mathscr{A} = (\{1,2,3,4,5,6\}, \{1,2,4,5,6\}, \{1,2,3,5,6\}, \{1,2,3,4,6\}$, $\{1,2,3,4,5\}$, $\{1,3,4,5,6\}$, $\{1,3,5\}$, $\{1,3,6\}$, $\{1,4,5\}$, $\{1,4,6\}$, $\{1,5,6\}$, $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,3,6\}$, $\{1,2,4,5\}$, $\{1,2,4,6\}$, $\{1,2,5,6\}$, $\{1,3,4,5\}$, $\{1,3,4,6\}$, $\{1,3,5,6\}$, $\{1,4,5,6\})$;

$\mathscr{B} = (\{1,2\}, \{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,2,6\}$, $\{1,3,4\}$, $\{3,4,5,6\}$, $\{2,4,5,6\}$, $\{2,3,5,6\}$, $\{2,3,4,6\}$, $\{2,3,4,5\}$, $\{4,5,6\}$, $\{3,5,6\}$, $\{3,4,6\}$, $\{3,4,5\}$, $\{2,5,6\}$, $\{2,3,4,5,6\})$.

We will color the vertices and edges of $K_{n,n}$ according to the given color sets of all vertices. Let $u_1, u_2, \ldots, u_n$ receive color 1. Let $C(u_1), C(u_2), \ldots, C(u_n)$ be the first $n$ terms of $\mathscr{A}$, respectively and $C(v_1), C(v_2), \ldots, C(v_n)$ be the first $n$ terms of $\mathscr{B}$, respectively. For $i = 1, 2, \ldots, n$, $j = 1, 2, 3, 4, 5$, if $j+1 \in C(u_i) \cap C(v_j)$, then assign $j+1$ to $u_iv_j$; if $j+1 \overline{\in} C(u_i) \cap C(v_j)$, then assign 1 to $u_iv_j$. For $i = 1, 2, 3, 4, 5, 6$, $6 \le j \le n$, if $i \in C(u_i) \cap C(v_j)$, then assign $i$ to $u_iv_j$; if $i \overline{\in} C(u_i) \cap C(v_j)$, then assign one color in $C(u_i) \cap C(v_j)$ to $u_iv_j$. For $i = 7, 8, \ldots, n$, $j = 6, 7, \ldots, n$, we assign one color in $C(u_i) \cap C(v_j)$ to $u_iv_j$. For $1 \le j \le n$, let $v_j$ receive a color in $C(v_j) - \{1\}$. Obviously, the above coloring is what we need. The proof is completed. $\Box$

By the method used in the proof of the above theorem, we can easily obtain the following proposition.

**Proposition 6** *Suppose $k \ge 4$, $n \ge 4$. If $\lceil \frac{2^{k-1}-2}{3} \rceil < n \le \lceil \frac{2^k-2}{3} \rceil$, then $\chi^{ie}_{vt}(K_{n,n}) \le k$.*

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