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# Some Properties of Cauchy-Type Singular Integrals in Clifford Analysis

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**Abstract** First, we give a module estimation of the singular integral with a differential element. Then by proving the existences of Cauchy principal value we obtain the transformation formula of the Cauchy-type singular integrals with a parameter.

**Keywords** Clifford analysis; Cauchy principal value; Cauchy-type singular integral with a parameter; transformation formula.

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## 1. Introduction

Cauchy-type integral is a kind of singular integrals which has become one of the basic tools to solve various boundary value problems. Due to its good property it has been widely applied to the theories of partial differential equations, singular integral equations and generalized functions. Especially it seems that the disposition of the singular integral equations and differential equations becomes quite simple and profound when Cauchy-type integral is applied [1].

Exchanging order of Cauchy-type integral plays an important role in the regularization and composition of the singular integral operators. With the exchanging order formula of Cauchy-type integral, we can solve various boundary value problems [2]. Thus, the exchanging order of Cauchy-type integral is the core problem in solving boundary value problems of many equations. In complex analysis and complex analysis in several variables, the exchanging order of Cauchy-type integral and the relevant problems have been solved thoroughly and it has been applied to elastic mechanics, fluid mechanics, multi-dimensional singular integral and integral equation [3–7].

Starting from the above facts it naturally occurs to us whether there exists the corresponding conclusion in Clifford algebraic space. Clifford algebra $\mathcal{A}_n(R)$ is an associative and incommutable

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real algebra structure. Since the 1970s Clifford analysis has been well developed in light of complex analysis [8, 9]. But for the incommutable property of Clifford algebra, many conclusions in complex analysis are not right in Clifford analysis. Cauchy-type integral also plays a vital role in Clifford analysis. However, Clifford algebra's incommutability brings trouble in the exchanging order, composition and regularization of Cauchy-type integral. Huang Sha proved P-B (Poincaré-Bertrand) formula for singular integrals in Clifford analysis in 1998 (see [8]). But the definitions are not perfect. On the basis of above conclusions, this paper will modify the definitions and prove that the exchanging order formula of Cauchy-type integral in Clifford analysis still holds true.

## 2. Preliminaries

### 2.1 Clifford algebra $\mathcal{A}_n(R)$

Let $\mathcal{A}_n(R)$ be a real Clifford Algebra over an $n$-dimensional Euclidean space $R^n$ with orthogonal basis $e := \{e_1, e_2, \ldots, e_n\}$. Then $\mathcal{A}_n(R)$ has its basis $e_1,\ e_2, \ldots, e_n;\ e_2e_3, \ldots, e_{n-1}e_n; \ldots; e_2\cdots e_n$. Hence an arbitrary element of the basis may be written as $e_A = e_{\alpha_1}\cdots e_{\alpha_h}$, here

$$A = \{\alpha_1, \ldots, \alpha_h\} \subseteq \{2, \ldots, n\},\ 2 \le \alpha_1 < \alpha_2 < \cdots < \alpha_h \le n,$$

and when $A = \emptyset$, $e_A = e_1$. So the real Clifford algebra is composed of elements having the type $a = \sum_A x_A e_A$, where $x_A (\in R)$ are real numbers. We define

$$\begin{cases} e_1e_i = e_ie_1 = e_i,\ \ i = 1, 2, 3, \ldots, n, \\ e_i^2 = -1,\ \ i = 2, 3, \ldots, n, \\ e_ie_j = -e_je_i,\ \ 2 \le i < j \le n, (i \ne j) \\ e_{h1}e_{h2}\cdots e_{hr} = e_{h_1h_2\cdots h_r},\ \ 1 \le h_1 < \cdots < h_n \le n. \end{cases}$$

The norm for an element $a = \sum_A x_A e_A \in \mathcal{A}_n(R)$ is defined as $|a| = \sqrt{(a, a)} = (\sum_A x_A^2)^{\frac{1}{2}}$.

### 2.2 Outer algebra

A differential space with basis $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ can be denoted by $V_n$. A Grassman algebra defined in $V_n$ with basis $\{\mathrm{d}x^A, A \in PN\}$ can be denoted by $G(V_n)$. The outer multiplication can be defined as

$$\begin{cases} \mathrm{d}x^A \bigwedge \mathrm{d}x^B = (-1)^{P(A,B)}\mathrm{d}x^{A\cup B}, & A, B \in PN,\ A \bigcap B = \emptyset, \\ \mathrm{d}x^A \bigwedge \mathrm{d}x^B = 0, & A, B \in PN,\ A \bigcap B \ne \emptyset, \\ \eta \bigwedge \nu = \sum\limits_A \sum\limits_B \eta^A \nu^B \mathrm{d}x^A \bigwedge \mathrm{d}x^B, & \eta = \sum\limits_A \eta^A \mathrm{d}x^A,\ \nu = \sum\limits_B \nu^B \mathrm{d}x^B. \end{cases}$$

We define $\mathrm{d}\widehat{x}_i = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n, i = 1, 2, \ldots, n$. And let $\mathrm{d}\sigma = \sum_{i=1}^n (-1)^{i+1} e_i \mathrm{d}\widehat{x}_i$. If $\mathrm{d}s$ stands for the classical surface element and $\overrightarrow{m} = \sum_{i=1}^n e_i n_i$, where $n_i$ is the $i$-th component of the unit outward normal vector, then $\mathrm{d}\sigma$ can be written as $\mathrm{d}\sigma = \overrightarrow{m}\mathrm{d}S$. Furthermore, the volume-element $\mathrm{d}x^n = \mathrm{d}x_1 \bigwedge \cdots \bigwedge \mathrm{d}x_n$ is used. Next, let $\Omega \subset R^n$ be a nonempty open connected set and the boundary $\partial\Omega$ be a Liapunov surface which is differentiable, oriented and compact [1]. Let $N_0 \in \partial\Omega$ be a fixed point. We establish a polar coordinate system with the origin at $N_0$ and the outward normal direction of $\partial\Omega$ at $N_0$ as the direction of the positive $\xi_n$ axis. Then the surface $\partial\Omega$ can be written as $\xi_n = \xi_n(\xi_1, \ldots, \xi_{n-1})$. And $\xi_n$ has partial derivatives about $\xi_i$ $(i = 1, \ldots, n-1)$. Now we establish a polar coordinate at $N_0$: $\xi_{n-1} = \rho_0 \cos\varphi_1 \cos\varphi_2 \cdots \cos\varphi_{n-3} \cos\varphi_{n-2}$, $\xi_{n-2} = \rho_0 \cos\varphi_1 \cos\varphi_2 \cdots \cos\varphi_{n-3} \sin\varphi_{n-2}$, $\ldots, \xi_2 = \rho_0 \cos\varphi_1 \sin\varphi_2$, $\xi_1 = \rho_0 \sin\varphi_1$, where $\rho_0$ is the length of $|NN_0|$ and $\varphi_i$ satisfy the conditions: $|\varphi_i| \le \frac{\pi}{2}, i = 1, 2, \ldots, n-3, 0 \le \varphi_{n-2} < 2\pi$. By [1], we have

$$|\mathrm{d}\sigma_x| = |\mathrm{d}s_x| \le 2\left|\frac{D(\xi_1, \xi_2, \ldots, \xi_{n-1})}{D(\rho_0, \varphi_1, \ldots, \varphi_{n-2})}\right||\mathrm{d}\rho_0 \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{n-2}| \le M_0 \rho_0^{n-2} \mathrm{d}\rho_0, \tag{1}$$

where $M_0$ is a positive constant.

### 2.3 Cauchy type singular integrals with a parameter

Let $\Omega \subset R^n$ be as stated above. Denote $\xi$ and $\eta$ on $\partial\Omega$ by $\partial\Omega_\xi$ and $\partial\Omega_\eta$, respectively. $f = \sum_A f_A e_A$ is a function defined on $\Omega$ and valued on $\mathcal{A}_n(R)$, where $f_A$ are real functions with $n$ variables and $A \in PN$. $f$ is called a Hölder continuous function on $\partial\Omega$ with the order $\beta$ if all $f_A$ are Hölder continuous functions on $\partial\Omega$ with the order $\beta$, where $0 < \beta < 1$. Let $H(\partial\Omega \times \partial\Omega, \beta)$ be a set which includes all the Hölder continuous functions with the order $\beta$ defined on $\partial\Omega \times \partial\Omega$ and valued on $\mathcal{A}_n(R)$.

**Definition 1** *Let $\Gamma$ be a P-chain in $R^n$, $f(x) = \sum_A f_A(x) e_A$, $g(x) = \sum_B g_B(x) e_B$, $f(x), g(x) \in H(\Gamma, \beta)$, $A, B \in PN$. Then we define*

$$\int_\Gamma f(x) \mathrm{d}\sigma_x g(x) = \sum_A \sum_{i=1}^n \sum_B (-1)^{i+1} e_A e_i e_B \int_\Gamma f_A(x) g_B(x) \mathrm{d}\widehat{x_i},$$

*where $x = (x_1, x_2, \ldots, x_n)$.*

**Remark 1** By Definition 1, we know when $f(x), g(x) \in H(\Gamma, \beta)$, the above integral is well defined.

Let $E(\eta, \zeta) = \frac{\overline{\eta} - \overline{\zeta}}{\omega_n |\eta - \zeta|^n}$, $E(\xi, \eta) = \frac{\overline{\xi} - \overline{\eta}}{\omega_n |\xi - \eta|^n}$, where $\omega_n$ is the area of the unit sphere in $R^n$. And we know $E(\eta, \zeta), E(\xi, \eta)$ are Cauchy integral kernels of regular functions. Let

$$P_1 f = \int_{\partial\Omega} E(\xi, \eta) \mathrm{d}\sigma_\eta f(\eta, \xi) = \lim_{\lambda_1 \to 0} \int_{\partial\Omega - \delta_{\lambda_1}} E(\xi, \eta) \mathrm{d}\sigma_\eta f(\eta, \xi),$$

$$P_2 f = \int_{\partial\Omega} f(\eta, \xi) \mathrm{d}\sigma_\xi E(\xi, \eta) = \lim_{\lambda_2 \to 0} \int_{\partial\Omega - \delta_{\lambda_2}} f(\eta, \xi) \mathrm{d}\sigma_\xi E(\xi, \eta),$$

where $\xi$ and $\eta$ are points in $\partial\Omega$, $\delta_{\lambda_1} = \{\eta \mid |\eta - \xi| < \lambda_1\} \bigcap \partial\Omega, \delta_{\lambda_2} = \{\xi \mid |\xi - \eta| < \lambda_2\} \bigcap \partial\Omega$.

**Definition 2** *Let $\partial\Omega$ be as stated above and $E(\eta, \zeta) = \sum_{l=1}^n g_l(\eta, \zeta) e_l$, $E(\xi, \eta) = \sum_{k=1}^n \varphi_k(\xi, \eta) e_k$, $f(\eta, \xi) = \sum_C f_C(\eta, \xi) e_C \in H(\partial\Omega \times \partial\Omega, \beta)$, here $g_l, \varphi_k$ and $f_C$ are real value functions and $C \in PN$. Then we define*

$$I_1 = \int_{\partial\Omega_\eta} E(\eta, \zeta) \mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi} E(\xi, \eta) \mathrm{d}\sigma_\xi f(\eta, \xi)$$

$$=\sum_{l=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{C}(-1)^{i+j+2}e_le_ie_ke_je_C[\int_{\partial\Omega_\eta}g_l(\eta,\zeta)\mathrm{d}\widehat{\eta_i}(\int_{\partial\Omega_\xi}\varphi_k(\xi,\eta)f_C(\eta,\xi)\mathrm{d}\widehat{\xi_j})];$$

$$I_2=\int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi f(\eta,\xi)]$$

$$=\sum_{l=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{C}(-1)^{i+j+2}e_le_ie_ke_je_C\int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta}g_l(\eta,\zeta)\varphi_k(\xi,\eta)f_C(\eta,\xi)\mathrm{d}\widehat{\eta_i}]\mathrm{d}\widehat{\xi_j},$$

*here $\zeta\in\partial\Omega$ and the singular integrals are their Cauchy principal values.*

**Remark 2** The singular integrals in this article are their Cauchy principal values.

## 3. Some properties of Cauchy-type singular integrals in Clifford analysis

### 3.1 Some Lemmas

**Lemma 1** ([8]) *Let $\partial\Omega$ be as stated above and $\zeta\in\partial\Omega$. Then we have*

$$\int_{\partial\Omega_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta=\int_{\partial\Omega_\xi}E(\xi,\eta)\mathrm{d}\sigma_\xi=\frac{1}{2};\quad \int_{B(\zeta,\varepsilon)}\mathrm{d}\sigma_\eta=\frac{\omega_n\varepsilon^{n-1}}{n-1},\ \text{where}\,B(\zeta,\varepsilon)=\{\eta\mid|\eta-\zeta|<\varepsilon\}.$$

**Lemma 2** ([8]) *When $f(\eta,\xi)\in H(\partial\Omega\times\partial\Omega,\beta)$, the singular integral operators $P_1f$, $P_2f$ all exist and $P_1f$, $P_2f\in H(\partial\Omega,\beta)$.*

**Lemma 3** ([8]) *Let $\partial\Omega$ be as stated above and $\zeta\in\partial\Omega$. Then we have*

$$\int_{\partial\Omega_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)=0.$$

**Lemma 4** ([8]) *Let $\partial\Omega$ be as stated above, $\zeta\in\partial\Omega$ and $\delta_\lambda(\zeta,\eta)=\{\eta\mid|\eta-\zeta|<\lambda,\ \eta\in\partial\Omega\}$. Then we have $\lim_{\lambda\to0}\int_{\delta_\lambda(\zeta,\eta)}E(\eta,\zeta)\mathrm{d}\sigma_\eta=0$.*

### 3.2 The main results

**Theorem 1** *Let $\Gamma$ be a differentiable, oriented, compact Liapunov surface in $R^n$, $\varphi(\eta,\xi)\in H(\Gamma\times\Gamma,\beta)$. Then for any points $\zeta\neq\xi\in\Gamma$, we have*

$$|\int_{\Gamma_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|\leq\frac{M_{13}}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|.$$

**Proof** Suppose $I=|\int_{\Gamma_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|$. And let $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$, where $\Gamma_1=\Gamma-B(\xi,\delta)$, $\Gamma_2=\Gamma\cap B(\xi,\frac{\delta}{4})$, $\Gamma_3=\Gamma-\Gamma_1-\Gamma_2$, and $\delta=2|\xi-\zeta|$. Then we can obtain

$$\begin{aligned}I\leq&|\int_{\Gamma_1}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|+\\&|\int_{\Gamma_2}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|+\\&|\int_{\Gamma_3}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|=L_1+L_2+L_3.\end{aligned}$$

When $\eta \in \Gamma_1$, $2|\eta-\xi| > |\eta-\xi| + |\zeta-\xi| \geq |\eta-\zeta|$. Then by equation (1), we have

$$\begin{aligned}
L_1 \leq& \frac{M_1M_2}{\omega_n^2}\int_{\Gamma_1}\frac{|\eta-\zeta|}{|\eta-\zeta|^n}|\mathrm{d}\sigma_\eta|\frac{|\xi-\eta|}{|\xi-\eta|^n}|\mathrm{d}\sigma_\xi||\eta-\zeta|^\beta\\
=& \frac{M_1M_2}{\omega_n^2}\int_{\Gamma_1}\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta|\frac{1}{|\eta-\xi|^{n-1}}|\mathrm{d}\sigma_\xi| \leq \frac{2^{n-1}M_1M_2}{\omega_n^2}\int_{\Gamma_1}\frac{1}{|\eta-\zeta|^{2n-2-\beta}}|\mathrm{d}\sigma_\eta||\mathrm{d}\sigma_\xi|\\
\leq& \frac{2^{n-1}M_2M_1M_0}{\omega_n^2}\int_{\frac{\delta}{2}}^{L}\frac{1}{\rho_1^{2n-2-\beta-n+2}}\mathrm{d}\rho_1|\mathrm{d}\sigma_\xi| = \frac{2^{n-1}M_2M_1M_0}{\omega_n^2}\int_{\frac{\delta}{2}}^{L}\frac{1}{\rho_1^{n-\beta}}\mathrm{d}\rho_1|\mathrm{d}\sigma_\xi|\\
=& \frac{2^{n-1}M_2M_1M_0}{\omega_n^2(n-\beta-1)}[(\frac{\delta}{2})^{\beta-n+1} - L^{\beta-n+1}]|\mathrm{d}\sigma_\xi| \leq M_3\delta^{\beta-n+1}|\mathrm{d}\sigma_\xi| = M_4\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|,
\end{aligned}$$

where $\rho_1 = |\eta-\zeta|$ and $L = \max_{\eta\in\Gamma}(|\eta-\zeta|)$.

When $\eta \in \Gamma_3$, $|\eta-\xi| \geq \frac{\delta}{4}$. Then from equation (1), we have

$$\begin{aligned}
L_3 \leq& \frac{M_1M_2}{\omega_n^2}\int_{\Gamma_3}\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta|\frac{1}{|\eta-\xi|^{n-1}}|\mathrm{d}\sigma_\xi|\\
\leq& 4^{n-1}M_1M_2\frac{1}{\delta^{n-1}}\int_{\Gamma_3}\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta||\mathrm{d}\sigma_\xi| \leq 4^{n-1}M_1M_2M_0\frac{1}{\delta^{n-1}}\int_0^{\frac{3}{2}\delta}\frac{\rho_1^{n-2}}{\rho_1^{n-1-\beta}}\mathrm{d}\rho_1|\mathrm{d}\sigma_\xi|\\
=& 4^{n-1}M_1M_2M_0\frac{1}{\delta^{n-1}\beta}(\frac{3}{2}\delta)^\beta|\mathrm{d}\sigma_\xi| = \frac{M_5}{\delta^{n-1-\beta}}|\mathrm{d}\sigma_\xi| = \frac{M_6}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|,
\end{aligned}$$

where $\rho_1 = |\eta-\zeta|$.

When $\eta \in \Gamma_2$, $|\eta-\xi| < \frac{\delta}{4}$, $|\eta-\zeta| \geq |\xi-\zeta| - |\eta-\xi| > \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4}$. Then

$$\begin{aligned}
L_2 \leq& \frac{1}{\omega_n^2}|\int_{\Gamma_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\eta-\xi|^n}\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\xi,\xi)]|+\\
&\frac{1}{\omega_n^2}|\int_{\Gamma_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\eta-\xi|^n}\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\zeta,\zeta)]| = v_1 + v_2.\\
v_1 \leq& \frac{M_1M_2}{\omega_n^2}\int_{\Gamma_2}\frac{1}{|\eta-\xi|^{n-1-\beta}}\frac{1}{|\eta-\zeta|^{n-1}}|\mathrm{d}\sigma_\eta||\mathrm{d}\sigma_\xi|\\
\leq& \frac{M_1M_24^{n-1}}{\omega_n^2}\frac{1}{\delta^{n-1}}\int_{\Gamma_2}\frac{1}{|\eta-\xi|^{n-1-\beta}}|\mathrm{d}\sigma_\eta||\mathrm{d}\sigma_\xi|\\
\leq& \frac{M_0M_1M_24^{n-1}}{\omega_n^2}\frac{1}{\delta^{n-1}}\int_0^{\frac{\delta}{4}}\frac{1}{\rho_2^{1-\beta}}\mathrm{d}\rho_2|\mathrm{d}\sigma_\xi| \leq M_7\frac{1}{\delta^{n-1-\beta}}|\mathrm{d}\sigma_\xi| = M_8\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|,
\end{aligned}$$

where $\rho_2 = |\eta-\xi|$.

$$v_2 \leq \frac{M_1M_2}{\omega_n^2}|\xi-\zeta|^\beta|\int_{\Gamma_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}||\mathrm{d}\sigma_\xi|.$$

Let $v_2^* = |\int_{\Gamma_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}|$, $\delta_\lambda = \{\eta \mid |\eta-\xi| < \lambda < \frac{\delta}{4}, \eta\in\Gamma_2\}$, $L_1 = \{\eta \mid |\eta-\xi| = \lambda,\ \eta\in\Gamma_2^+\}$, $L_2 = \{\eta \mid |\eta-\xi| = \frac{\delta}{4},\ \eta\in\Gamma_2^+\}$, where $\Gamma_2^+$ is the domain outside of $\Gamma$ and $L_1$, $L_2$ have inverse orientations. Then

$$\begin{aligned}
v_2^* =& \lim_{\lambda\to 0}|\int_{\Gamma_2-\delta_\lambda}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}| \leq \lim_{\lambda\to 0}|\int_{\Gamma_2-\delta_\lambda+L_1+L_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}|+\\
&\lim_{\lambda\to 0}|\int_{L_1}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}| + \lim_{\lambda\to 0}|\int_{L_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}| = \tau_1+\tau_2+\tau_3.
\end{aligned}$$

Suppose $\Omega_1$ is the domain closed by $\Gamma_2-\delta_\lambda+L_1+L_2$. Then $\zeta,\xi\notin\Omega_1$ and

$$(\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n})\partial_\eta=0;\quad \partial_\eta(\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n})=0.$$

By Stokes formula [8] we know

$$\begin{aligned}\tau_1&=\lim_{\lambda\to0}|\int_{\Gamma_2-\delta_\lambda+L_1+L_2}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}|\\&=\lim_{\lambda\to0}|\int_{\Omega_1}[(\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n})\partial_\eta]\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}+\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}[\partial_\eta(\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n})]\mathrm{d}\sigma_\eta|=0.\\\tau_2&=\lim_{\lambda\to0}|\int_{L_1}\frac{\overline{\eta}-\overline{\zeta}}{|\eta-\zeta|^n}\mathrm{d}\sigma_\eta\frac{\overline{\xi}-\overline{\eta}}{|\xi-\eta|^n}|\le\lim_{\lambda\to0}M_2\int_{L_1}\frac{1}{|\eta-\zeta|^{n-1}}|\mathrm{d}\sigma_\eta|\frac{1}{|\eta-\xi|^{n-1}}.\end{aligned}$$

When $\eta\in L_1$, we know $|\eta-\zeta|>\frac{\delta}{4}$ and $|\eta-\xi|=\lambda$. By Lemma 1 we have

$$\begin{aligned}\tau_2&\le\lim_{\lambda\to0}M_2\int_{L_1}\frac{4^{n-1}}{\delta^{n-1}}|\mathrm{d}\sigma_\eta|\frac{1}{\lambda^{n-1}}\le\lim_{\lambda\to0}\frac{4^{n-1}M_2}{\delta^{n-1}}\frac{1}{\lambda^{n-1}}\int_{L_1}|\mathrm{d}\sigma_\eta|\\&\le\lim_{\lambda\to0}\frac{4^{n-1}M_2}{\delta^{n-1}(n-1)}\frac{1}{\lambda^{n-1}}\omega_n\lambda^{n-1}=M_9\frac{1}{|\xi-\zeta|^{n-1}}.\end{aligned}$$

When $\eta\in L_2$, $|\eta-\zeta|\ge\frac{\delta}{4}$ and $|\eta-\xi|>\lambda$. Then by Lemma 1 we can obtain

$$\begin{aligned}\tau_3&\le\lim_{\lambda\to0}M_2\int_{L_2}\frac{4^{n-1}}{\delta^{n-1}}|\mathrm{d}\sigma_\eta|\frac{1}{|\eta-\zeta|^{n-1}}\le\lim_{\lambda\to0}\frac{4^{n-1}M_2}{\delta^{n-1}(\frac{\delta}{4})^{n-1}}\int_{L_2}|\mathrm{d}\sigma_\eta|\\&\le\lim_{\lambda\to0}\frac{4^{n-1}M_2}{\delta^{n-1}(\frac{\delta}{4})^{n-1}(n-1)}\omega_n(\frac{\delta}{4})^{n-1}=M_{10}\frac{1}{|\xi-\zeta|^{n-1}}.\end{aligned}$$

Hence $v_2^*\le M_{11}\frac{1}{|\xi-\zeta|^{n-1}}$ and $v_2\le\frac{M_1M_2}{\omega_n^2}v_2^*|\xi-\zeta|^\beta|\mathrm{d}\sigma_\xi|\le\frac{M_1M_2M_{11}}{\omega_n^2}\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|$. Then $L_2\le v_1+v_2\le M_{12}\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|$. Hence $I\le L_1+L_2+L_3\le M_{13}\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|$.

**Remark 3** Theorem 1 is used to prove Theorem 3, which has a special conclusion and gives a module estimation of the singular integral expression with a differential element. Because the differential elements in Clifford analysis are also vector-valued and they are incommutable with functions, they should be estimated together rather than separately.

**Theorem 2** *Let $\Gamma_1,\Gamma_2$ be differentiable, oriented, compact Liapunov surfaces in $R^n$, $\phi(\eta)\in H(\Gamma_1,\beta)$, $g(\eta,\xi)\in H(\Gamma_1\times\Gamma_2,\beta)$, $f(\eta,\xi)\in H(\Gamma_1\times\Gamma_2,\beta)$. We have*

$$\int_{\Gamma_1}\phi(\eta)\mathrm{d}\sigma_\eta[\int_{\Gamma_2}f(\eta,\xi)\mathrm{d}\sigma_\xi g(\eta,\xi)]=\int_{\Gamma_2}[\int_{\Gamma_1}\phi(\eta)\mathrm{d}\sigma_\eta f(\eta,\xi)\mathrm{d}\sigma_\xi g(\eta,\xi)],$$

*where $\xi=(\xi_1,\xi_2,\ldots,\xi_n)$, $\eta=(\eta_1,\eta_2,\ldots,\eta_n)$.*

**Proof** Let $\phi(\eta)=\sum_A\phi_A(\eta)e_A$, $g(\eta,\xi)=\sum_C g_C(\eta,\xi)e_C$, $f(\eta,\xi)=\sum_B f_B(\eta,\xi)e_B$. By Definitions 1, 2 and Fubini Theorem, we have

$$\begin{aligned}&\int_{\Gamma_1}\phi(\eta)\mathrm{d}\sigma_\eta[\int_{\Gamma_2}f(\eta,\xi)\mathrm{d}\sigma_\xi g(\eta,\xi)]\\&=\int_{\Gamma_1}\sum_A\phi_A(\eta)e_A\sum_{i=1}^n(-1)^{i+1}e_i\mathrm{d}\widehat{\eta}_i\int_{\Gamma_2}\sum_B f_B(\eta,\xi)e_B\sum_{j=1}^n(-1)^{j+1}e_j\mathrm{d}\widehat{\xi}_j\sum_C g_C(\eta,\xi)e_C\end{aligned}$$

$$= \sum_A \sum_{i=1}^n \sum_B \sum_{j=1}^n \sum_C (-1)^{i+j+2} e_A e_i e_B e_j e_C \int_{\Gamma_1} [\int_{\Gamma_2} \phi_A(\eta) f_B(\eta,\xi) g_C(\eta,\xi) \mathrm{d}\widehat{\xi_j}] \mathrm{d}\widehat{\eta_i}$$

$$= \sum_A \sum_{i=1}^n \sum_B \sum_{j=1}^n \sum_C (-1)^{i+j+2} e_A e_i e_B e_j e_C \int_{\Gamma_1 \times \Gamma_2} \phi_A(\eta) f_B(\eta,\xi) g_C(\eta,\xi) \mathrm{d}\widehat{\eta_i} \mathrm{d}\widehat{\xi_j}.$$

$$\int_{\Gamma_2} [\int_{\Gamma_1} \phi(\eta) \mathrm{d}\sigma_\eta f(\eta,\xi)] \mathrm{d}\sigma_\xi g(\eta,\xi)$$

$$= \int_{\Gamma_2} [\int_{\Gamma_1} \sum_A \phi_A(\eta) e_A \sum_{i=1}^n (-1)^{i+1} e_i \mathrm{d}\widehat{\eta_i} \sum_B f_B(\eta,\xi) e_B] \sum_{j=1}^n (-1)^{j+1} e_j \mathrm{d}\widehat{\xi_j} \sum_C g_C(\eta,\xi) e_C$$

$$= \sum_A \sum_{i=1}^n \sum_B \sum_{j=1}^n \sum_C (-1)^{i+j+2} e_A e_i e_B e_j e_C \int_{\Gamma_2} [\int_{\Gamma_1} \phi_A(\eta) f_B(\eta,\xi) g_C(\eta,\xi) \mathrm{d}\widehat{\eta_i}] \mathrm{d}\widehat{\xi_j}$$

$$= \sum_A \sum_{i=1}^n \sum_B \sum_{j=1}^n \sum_C (-1)^{i+j+2} e_A e_i e_B e_j e_C \int_{\Gamma_1 \times \Gamma_2} \phi_A(\eta) f_B(\eta,\xi) g_C(\eta,\xi) \mathrm{d}\widehat{\eta_i} \mathrm{d}\widehat{\xi_j}$$

$$= \int_{\Gamma_1} \phi(\eta) \mathrm{d}\sigma_\eta [\int_{\Gamma_2} f(\eta,\xi) \mathrm{d}\sigma_\xi g(\eta,\xi)].$$

**Remark 4** Theorem 2 shows that for normal integrals integral order can be commuted though the multiplication order of functions is not changed.

**Theorem 3** *Let $\Omega \subset R^n$ be as stated above. Suppose $\varphi(\eta,\xi) \in H(\partial\Omega \times \partial\Omega, \beta)$ and $\zeta \in \partial\Omega$. Then the following integrals all exist and we can obtain the following equations.*

$$\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi} E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\eta,\eta) - \varphi(\zeta,\zeta)]$$
$$= \int_{\partial\Omega_\xi} [\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\eta,\eta) - \varphi(\zeta,\zeta)]]; \tag{2}$$

$$\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi} E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\eta,\xi) - \varphi(\xi,\xi)]$$
$$= \int_{\partial\Omega_\xi} [\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\eta,\xi) - \varphi(\xi,\xi)]]; \tag{3}$$

$$\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi} E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\xi,\xi) - \varphi(\eta,\eta)]$$
$$= \int_{\partial\Omega_\xi} [\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\xi,\xi) - \varphi(\eta,\eta)]]. \tag{4}$$

**Proof** We only prove that the integrals exist and the first equation is right. The other equations can be proved similarly.

(i) First we prove that the integrals all exist. Let

$$I = \int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi} E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\eta,\eta) - \varphi(\zeta,\zeta)],$$

$$I' = \int_{\partial\Omega_\xi} [\int_{\partial\Omega_\eta} E(\eta,\zeta) \mathrm{d}\sigma_\eta E(\xi,\eta) \mathrm{d}\sigma_\xi [\varphi(\eta,\eta) - \varphi(\zeta,\zeta)]].$$

From Lemma 1 we have

$$
\begin{aligned}
|I| &= |\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]| \\
&= \frac{1}{2}|\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]| \le \frac{M_1M_2}{2\omega_n}\int_{\partial\Omega_\eta}\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta| \\
&\le \frac{M_1M_0M_2}{2\omega_n}\int_0^L \frac{1}{\rho_1^{1-\beta}}d\rho_1 = \frac{M_1M_0M_2}{2\omega_n\beta}L^\beta,
\end{aligned}
$$

here $\rho_1=|\eta-\zeta|$. Hence $I$ exists. By Theorem 1 we can obtain

$$
\begin{aligned}
|I'| &= |\int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]| \\
&\le M_2\int_{\partial\Omega_\xi}|\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]| \\
&\le M_{13}M_2\int_{\partial\Omega_\xi}\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi| \le M_{13}M_2M_0\int_0^L\frac{1}{\rho_2^{1-\beta}}\mathrm{d}\rho_2 = \frac{M_{13}M_0}{\beta}L^\beta,
\end{aligned}
$$

here $\rho_2=|\xi-\zeta|$. Hence $I'$ exists.

By Lemma 2, we know $\int_{\partial\Omega_\xi}E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\xi)-\varphi(\xi,\xi)]$ and $\int_{\partial\Omega_\xi}E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\eta,\eta)]$ are Hölder continuous functions about $\eta$. Then we can obtain that the left two integrals of equation (3) and equation (4) all exist. By Theorem 1 and the proof of $I'$ we can know the right two integrals of equation (3) and equation (4) all exist.

(ii) Next we prove $I=I'$.

$$
\begin{aligned}
I =& \int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi\setminus|\xi-\eta|<2\delta} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]+ \\
& \int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi\cap|\xi-\eta|<2\delta} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)] = I_1+I_2. \\
I' =& \int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta\setminus|\eta-\xi|<2\delta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]]+ \\
& \int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]] = I_1'+I_2'.
\end{aligned}
$$

We only need to prove $I_1=I_1'$ and $\lim_{\delta\to 0} I_2=\lim_{\delta\to 0} I_2'=0$.

$$
\begin{aligned}
I_1 =& \int_{\partial\Omega_\eta\setminus|\eta-\zeta|<\frac{\delta}{2}} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi\setminus|\xi-\eta|<2\delta} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]+ \\
& \int_{\partial\Omega_\eta\cap|\eta-\zeta|<\frac{\delta}{2}} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi\setminus|\xi-\eta|<2\delta} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)] = I_3+I_4. \\
I_1' =& \int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta\setminus(|\eta-\xi|<2\delta\cup|\eta-\zeta|<\frac{\delta}{2})} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]]+ \\
& \int_{\partial\Omega_\xi}[\int_{(\partial\Omega_\eta\setminus|\eta-\xi|<2\delta)\cap|\eta-\zeta|<\frac{\delta}{2}} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]] = I_3'+I_4'.
\end{aligned}
$$

Let $E(\eta,\zeta)=\sum_{l=1}^n f_l(\eta,\zeta)e_l$, $B(\eta,\xi)=\sum_{k=1}^n g_k(\eta,\xi)e_k, \varphi(\eta,\eta)-\varphi(\zeta,\zeta)=\sum_C\varphi_C(\eta,\zeta)e_C$. By

Definition 1 and Theorem 2 we have

$$
\begin{aligned}
I_3 =&\int_{\partial\Omega_\eta\backslash|\eta-\zeta|<\frac{\delta}{2}} E(\eta,\zeta)\mathrm{d}\sigma_\eta \int_{\partial\Omega_\xi\backslash|\xi-\eta|<2\delta} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]\\
=&\int_{\partial\Omega_\eta\backslash|\eta-\zeta|<\frac{\delta}{2}} \sum_{l=1}^{n} f_l(\eta,\zeta)e_l \sum_{i=1}^{n}(-1)^{i+1}e_i\mathrm{d}\widehat{\eta}_i\\
&\int_{\partial\Omega_\xi\backslash|\xi-\eta|<2\delta} \sum_{k=1}^{n} g_k(\eta,\xi)e_k \sum_{j=1}^{n}(-1)^{j+1}e_j\mathrm{d}\widehat{\xi}_j \sum_{C}\varphi_C(\eta,\zeta)e_C\\
=&\sum_{l=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{C}(-1)^{i+j+2}e_le_ie_ke_je_C\int\int_{\Sigma_1} f_lg_k\varphi_C\mathrm{d}\widehat{\eta}_i\mathrm{d}\widehat{\xi}_j.
\end{aligned}
$$

Here $\Sigma_1=\{(\eta,\xi)\mid \eta\in(\partial\Omega_\eta\backslash|\eta-\zeta|<\frac{\delta}{2}),\xi\in(\partial\Omega_\xi\backslash|\eta-\xi|<2\delta)\}$.

$$
\begin{aligned}
I_3' =& \int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta\backslash(|\eta-\xi|<2\delta\cup|\eta-\zeta|<\frac{\delta}{2})} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]]\\
=&\sum_{l=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{C}(-1)^{i+j+2}e_le_ie_ke_je_C\int\int_{\Sigma_2} f_lg_k\varphi_C\mathrm{d}\widehat{\eta}_i\mathrm{d}\widehat{\xi}_j.
\end{aligned}
$$

Here $\Sigma_2=\{(\eta,\xi)\mid \xi\in\partial\Omega_\xi,\eta\in(\partial\Omega_\eta\backslash(|\eta-\xi|<2\delta\cup|\eta-\zeta|<\frac{\delta}{2}))\}$. Let $\Sigma_3=\{(\eta,\xi)\mid\eta\in\partial\Omega,\xi\in\partial\Omega,|\eta-\zeta|>\frac{\delta}{2},|\eta-\xi|>2\delta\}$. Then $\Sigma_1=\Sigma_3=\Sigma_2$. Hence $I_3=I_3'$.

$$
\begin{aligned}
|I_4| =& |\int_{\partial\Omega_\eta\cap|\eta-\zeta|<\frac{\delta}{2}} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi\backslash|\eta-\xi|<2\delta} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|\\
\le& \frac{M_1M_2}{\omega_n^2}\int_{\partial\Omega_\eta\cap|\eta-\zeta|<\frac{\delta}{2}}\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta|\int_{\partial\Omega_\xi\backslash|\eta-\xi|<2\delta}\frac{1}{|\eta-\xi|^{n-1}}|\mathrm{d}\sigma_\xi|\\
\le& \frac{M_0^2M_1M_2}{\omega_n^2}\int_0^{\frac{\delta}{2}}\frac{1}{\rho_1^{1-\beta}}\mathrm{d}\rho_1\int_{2\delta}^{L}\frac{1}{\rho_2}\mathrm{d}\rho_2\le\frac{M_0^2M_1M_2}{\omega_n^2\beta}(\frac{\delta}{2})^\beta|\ln L|+|\ln 2\delta|)\\
\le& \frac{M_0^2M_1M_2}{\omega_n^2\beta}(\frac{\delta}{2})^\beta(|\ln L|+|2\delta|^{-\varepsilon})\le M_{14}\delta^{\beta-\varepsilon},
\end{aligned}
$$

where $0<\varepsilon<\beta$. Hence $\lim_{\delta\to0}|I_4|=0$. In integral $I_4'$, $\zeta$, $\xi$ and $\eta$ satisfy the following inequalities:

$$
|\xi-\zeta|\ge|\eta-\xi|-|\eta-\zeta|>2\delta-\frac{\delta}{2}>\delta;
$$

$$
|\eta-\xi|\ge|\xi-\zeta|-|\zeta-\eta|>|\xi-\zeta|-\frac{\delta}{2}>\frac{1}{2}|\xi-\zeta|.
$$

Then we have

$$
\begin{aligned}
|I_4'|\le& \frac{M_1M_2}{\omega_n^2}\int_{\partial\Omega_\xi}[\int_{(\partial\Omega_\eta\backslash|\eta-\xi|<2\delta)\cap|\eta-\zeta|<\frac{\delta}{2}}\frac{1}{|\eta-\zeta|^{n-1}}|\mathrm{d}\sigma_\eta|\frac{1}{|\eta-\xi|^{n-1}}|\mathrm{d}\sigma_\xi||\eta-\zeta|^\beta\\
\le& \frac{2^{n-1}M_1M_2}{\omega_n^2}\int_{\partial\Omega_\xi}[\int_{(\partial\Omega_\eta\backslash|\eta-\xi|<2\delta)\cap|\eta-\zeta|<\frac{\delta}{2}}\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta|\frac{1}{|\xi-\zeta|^{n-1}}|\mathrm{d}\sigma_\xi|\\
\le& \frac{2^{n-1}M_0^2M_1M_2}{\omega_n^2}\int_0^{\frac{\delta}{2}}\rho_1^{\beta-1}\mathrm{d}\rho_1\int_\delta^L\frac{1}{\rho_2}\mathrm{d}\rho_2\le\frac{2^{n-1}M_0^2M_1M_2}{\omega_n^2\beta}(\frac{\delta}{2})^\beta(|\ln L|+|\delta^{-\varepsilon}|)=M_{15}\delta^{\beta-\varepsilon}.
\end{aligned}
$$

Hence $\lim_{\delta\to 0}|I_4'|=0$. Then $|I_1-I_1'|=\lim_{\delta\to 0}|I_1-I_1'|=\lim_{\delta\to 0}|I_4-I_4'|\le\lim_{\delta\to 0}|I_4|+\lim_{\delta\to 0}|I_4'|=0$. Therefore, $I_1=I_1'$.

By Lemma 4 we know $\lim_{\delta\to 0}(\int_{\partial\Omega_\xi\cap|\eta-\xi|<2\delta}E(\xi,\eta)\mathrm{d}\sigma_\xi)=0$. Then for any $\varepsilon>0$, we can find a number $\delta'>0$, such that $|\int_{\partial\Omega_\xi\cap|\eta-\xi|<2\delta}E(\xi,\eta)\mathrm{d}\sigma_\xi|<\varepsilon$ when $0<\delta<\delta'$. Then when $0<\delta<\delta'$, we obtain

$$\begin{aligned}|I_2|&=|\int_{\partial\Omega_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi\cap|\eta-\xi|<2\delta}E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|\\&=|\int_{\partial\Omega_\eta}E(\eta,\zeta)\mathrm{d}\sigma_\eta(\int_{\partial\Omega_\xi\cap|\eta-\xi|<2\delta}E(\xi,\eta)\mathrm{d}\sigma_\xi)[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]|\\&\le M_2\varepsilon\int_{\partial\Omega_\eta}|E(\eta,\zeta)||\mathrm{d}\sigma_\eta||\varphi(\eta,\eta)-\varphi(\zeta,\zeta)|\le\frac{M_1M_2\varepsilon}{\omega_n}\int_0^L\frac{1}{|\eta-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\eta|\\&=\frac{M_0M_1M_2\varepsilon}{\omega_n}\int_0^L\rho_1^{\beta-1}\mathrm{d}\rho_1=\frac{M_0M_1M_2L^\beta}{\beta\omega_n}\varepsilon=M_{16}\varepsilon.\end{aligned}$$

Hence $\lim_{\delta\to 0}|I_2|=0$.

$$\begin{aligned}I_2'=&\int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]]\\=&\int_{\partial\Omega_\xi\cap|\xi-\zeta|<3\delta}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]]+\\&\int_{\partial\Omega_\xi\backslash|\xi-\zeta|<3\delta}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]]=N_1+N_2.\\N_2=&\int_{\partial\Omega_\xi\backslash|\xi-\zeta|<3\delta}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\xi,\xi)]]+\\&\int_{\partial\Omega_\xi\backslash|\xi-\zeta|<3\delta}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\zeta,\zeta)]]=\tau_1'+\tau_2'.\end{aligned}$$

When $|\xi-\zeta|>3\delta,|\eta-\xi|<2\delta$, we have

$$|\eta-\zeta|\ge|\xi-\zeta|-|\eta-\xi|>|\xi-\zeta|-2\delta>|\xi-\zeta|-\frac{2}{3}|\xi-\zeta|=\frac{1}{3}|\xi-\zeta|.$$

Then

$$\begin{aligned}\lim_{\delta\to 0}|\tau_1'|&\le\lim_{\delta\to 0}\frac{M_1M_2}{\omega_n^2}\int_{\partial\Omega_\xi\backslash|\xi-\zeta|<3\delta}\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}\frac{1}{|\eta-\zeta|^{n-1}}\frac{1}{|\eta-\xi|^{n-1-\beta}}|\mathrm{d}\sigma_\eta||\mathrm{d}\sigma_\xi|\\&\le M_{17}\lim_{\delta\to 0}\delta^\beta\int_{3\delta}^L\frac{1}{|\xi-\zeta|^{n-1}}|\mathrm{d}\sigma_\xi|\le\lim_{\delta\to 0}M_{18}[\ln L-(3\delta)^{-\varepsilon}]\delta^\beta=0,\end{aligned}$$

where $0<\varepsilon<\beta$.

$$\begin{aligned}\tau_2'&=\int_{\partial\Omega_\xi\backslash|\xi-\zeta|<3\delta}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\zeta,\zeta)]]\\&=\int_{\partial\Omega_\xi\backslash|\xi-\zeta|<3\delta}[\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)]\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\zeta,\zeta)].\end{aligned}$$

For $\lim_{\delta\to 0}\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta}E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)=0$, $\lim_{\delta\to 0}\tau_2'=0$. Hence $\lim_{\delta\to 0}N_2=0$.

Similarly to the proof of Theorem 1, we have

$$|\int_{\partial\Omega_\eta\cap|\eta-\xi|<2\delta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]| \le M_{19}\frac{1}{|\xi-\zeta|^{n-1-\beta}}|\mathrm{d}\sigma_\xi|.$$

Then we have $\lim_{\delta\to 0}|N_1| \le \lim_{\delta\to 0}\int_0^{3\delta} M_{19}M_0M_2\frac{\rho_1^{n-1}}{\rho_1^{n-1-\beta}}\mathrm{d}\rho_1 = \lim_{\delta\to 0}\frac{M_0M_{19}M_2}{\beta}(3\delta)^\beta = 0$. Hence $\lim_{\delta\to 0} I_2^* = 0$. The proof is completed. □

Similarly we can prove the other equations.

**Remark 5** Essentially, Theorem 3 draws the following conclusion: If there is a weak singular integral in the twice integrals (i.e., it is convergent in the sense of generalized integral), the integral order can be commuted although the other integral is convergent in the sense of principal value. The proving method is outlined as follows: at the beginning we try to prove that the integrals are convergent in the sense of principal value and then prove that the equations exist.

**Theorem 4** *Let $\partial\Omega$ be as stated above. Then we have*

$$\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\eta,\xi) = \int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\eta,\xi)+\frac{1}{4}\varphi(\zeta,\zeta).$$

**Proof** By Theorem 3 and Lemma 1, we can obtain

$$\begin{aligned}
&\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\eta,\xi)\\
&=\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\xi)-\varphi(\xi,\xi)]+\\
&\quad\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\eta,\eta)]+\\
&\quad\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]+\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\zeta,\zeta)\\
&=\int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\xi)-\varphi(\xi,\xi)]+\\
&\quad\int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\xi,\xi)-\varphi(\eta,\eta)]+\\
&\quad\int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi[\varphi(\eta,\eta)-\varphi(\zeta,\zeta)]+\frac{1}{2}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\varphi(\zeta,\zeta)\\
&=\int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\eta,\xi)-\int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\zeta,\zeta)+\frac{1}{4}\varphi(\zeta,\zeta).
\end{aligned}$$

From Lemma 3 we have $\int_{\partial\Omega_\xi}[\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)]\mathrm{d}\sigma_\xi\varphi(\zeta,\zeta)=0$. Hence

$$\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta\int_{\partial\Omega_\xi} E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\eta,\xi) = \int_{\partial\Omega_\xi}\int_{\partial\Omega_\eta} E(\eta,\zeta)\mathrm{d}\sigma_\eta E(\xi,\eta)\mathrm{d}\sigma_\xi\varphi(\eta,\xi)+\frac{1}{4}\varphi(\zeta,\zeta).$$

**Remark 6** The last theorem shows that when the twice integrals are convergent in the sense of Cauchy-type principal values, there is an extra item after the integral exchanging order and this agrees with the result in complex analysis.

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