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Weighted Estimates for Marcinkiewicz Integrals with Non-Doubling Measures

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Abstract Let μ be a nonnegative Radon measure on R^d which satisfies the growth condition $\mu(B(x,r)) \le C_0 r^n$ for all $x \in R^d$ and $r > 0$, where C_0 is a fixed constant and $0 < n \le d$. The purpose of this paper is to establish the boundedness of the Marcinkiewicz integrals from $L^p(u)$ to $L^{p,\infty}(u)$, where u is a weight function of Muckenhoupt type associated with μ.

Keywords maximal singular integrals; Muckenhoupt type weights; sharp maximal operators; non-doubling measures.

MR(2010) Subject Classification 42B20; 42B25

1. Introduction

Recent years, more and more people pay considerable attention to the study of function spaces with non-doubling measures [2, 4–6, 8–10, 15–17]. Assuming that the faces (or edges) of the cubes have the measure zero, Orobitg and Pérez in [11] introduced the weights on non-homogeneous spaces and obtained the weighted norm inequalities for the Calderón-Zygmund operators and the corresponding maximal singular operators. Without the above assumption, Hu and Yang in [8] established the weighted boundedness for maximal singular integrals with non-doubling measures from $L^p(u)$ to $L^{p,\infty}(u)$, for $p \in (1,\infty)$ and $u \in A_p^\rho(\mu)$ with $\rho \ge 1$, where $A_p^\rho(\mu)$ consists of the weight functions of Muckenhoupt type associated with μ, see Definition 1 below.

Hu, Lin and Yang in [7] introduced the Marcinkiewicz integrals with non-doubling measures and obtained some boundedness results. The main purpose of this paper is to establish weighted norm inequalities with weights of Munkenhoupt type on non-homogeneous spaces.

Let μ be a nonnegative Radon measure on R^d, which only satisfies the growth condition that there exist positive constants C_0 and $n \in (0, d]$ such that for all $x \in R^d$ and $r > 0$,

$$\mu(B(x,r)) \le C_0 r^n, \tag{1}$$

where $B(x,r)$ is the open ball centered at some point $x \in R^d$ and having radius r. The measure μ in (1) is not assumed to satisfy the doubling condition which is a key assumption in the analysis on spaces of homogeneous type. Some important non-doubling measures satisfying (1)

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and the motivation for developing the analysis related to such measures can be found in [17]. We only point out that analysis with non-doubling measures plays an essential role in solving the long-standing Painlevé open problem by Tolsa in [15].

At first, we recall some notations and definitions and employ the statement used in [8]. By a cube $Q \subset R^d$ we mean a closed cube whose sides are parallel to the axes and which is centered at some point of $\mathrm{supp}(\mu)$, and we denote its side length by $l(Q)$. A μ-measurable function u is said to be a weight if it is nonnegative and μ-locally integrable. The $A_p^\rho(\mu)$ weights of Muckenhoupt type in the setting of non-doubling measures were first introduced by Orobitg and Pérez [11] for $\rho = 1$ and by Komori [2] for $\rho \in [1, \infty)$.

Definition 1 *Let $\rho \in [1,\infty)$, $p \in [1,\infty)$ and $p' = p/(p-1)$. A weight u is said to be an $A_p^\rho(\mu)$ weight if there exists a positive constant C such that for any cube Q,*

$$\Big(\frac{1}{\mu(\rho Q)}\int_Q u(x)\mathrm{d}\mu(x)\Big)\Big(\frac{1}{\mu(\rho Q)}\int_Q u(x)^{1-p'}\mathrm{d}\mu(x)\Big)^{p-1} \le C.$$

Also, a weight u is said to be an $A_1^\rho(\mu)$ weight if there exists a positive constant C such that for any cube Q,

$$\frac{1}{\mu(\rho Q)}\int_Q u(x)\mathrm{d}\mu(x) \le C \inf_{x\in Q} u(x).$$

As in the classical setting, we set $A_\infty^\rho(\mu) = \bigcup_{p=1}^\infty A_p^\rho(\mu)$. For $\rho = 1$, we denote $A_p^\rho(\mu)$, $A_1^\rho(\mu)$ and $A_\infty^\rho(\mu)$ simply by $A_p(\mu)$, $A_1(\mu)$ and $A_\infty(\mu)$, respectively.

As pointed out by Orobitg and Pérez [11], without the assumption that for any cube Q, $\mu(\partial Q) = 0$, where ∂Q is the faces (or edges) of the cube Q, the reverse Hölder inequality, the fact that $u \in A_1(\mu)$ implies $u \in L^{1+\sigma}_{\mathrm{loc}}(\mu)$ with some $\sigma \in (0,\infty)$, and some other important properties enjoyed by the A_p weights in the setting of Euclidean spaces, may not be true.

Let K be a locally integral function on $R^d \times R^d \backslash \{x = y\}$ such that for any $x \ne y$,

$$|K(x,y)| \le C|x-y|^{-(n-1)} \tag{2}$$

and for any x, y and $y' \in R^d$ with $|x-y| \ge 2|y-y'|$,

$$\int_{|x-y|\ge 2|y-y'|} [|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)|]\frac{1}{|x-y|}\,\mathrm{d}\mu(x) \le C, \tag{3}$$

for any $y, y' \in R^d$. The Marcinkiewicz integral $\mathcal{M}$ associated to the above kernel K and the measure μ as in (1) is defined by

$$\mathcal{M}(f)(x) = \Big(\int_0^\infty \Big|\int_{|x-y|\le t} K(x,y)f(y)\,\mathrm{d}\mu(y)\Big|^2 \frac{\mathrm{d}t}{t^3}\Big)^{1/2}, \quad x \in R^d. \tag{4}$$

In their remarkable work [7], Hu, Lin and Yang successfully established the boundedness of $\mathcal{M}$ with kernel K satisfying (2) and (3), respectively, from the Lebesgue space $L^1(\mu)$ to the weak Lebesgue space $L^{1,\infty}(\mu)$, from the Hardy space $H^1(\mu)$ to $L^1(\mu)$ and from the Lebesgue space $L^\infty(\mu)$ to the space $\mathrm{RBLO}(\mu)$ (see [12]). In this note, we make some modification for the kernel K. Besides K satisfies the size condition (2), it also satisfies that for any x, y and $y' \in R^d$ with

$|y-y'| \le |x-y|/2$, there exists $0<\varepsilon\le 1$ such that

$$|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)| \le \frac{|y-y'|^{\varepsilon}}{|x-y|^{n-1+\varepsilon}}. \tag{5}$$

Obviously, when the kernel satisfies condition (5), it also satisfies (3). Under above conditions, we shall prove that $\mathcal{M}$ is bounded from $L^p(u)$ to $L^p(u)$, $u\in A_p^{\rho}(\mu)$.

The following theorem is our main result.

Theorem 1 *Let $\rho\in[1,\infty)$. Let K satisfy (2) and (5), and $\mathcal{M}$ be the Marcinkiewicz integral defined as in (4). If $\mathcal{M}$ is bounded on $L^2(\mu)$, for any $p\in[1,\infty)$ and $u\in A_p^{\rho}(\mu)$, then $\mathcal{M}$ is also bounded from $L^p(u)$ to $L^{p,\infty}(u)$, that is, there exists a positive constant C such that for any $\lambda>0$ and all bounded functions f with compact support and $x\in R^d$,*

$$u(\{x\in R^d : \mathcal{M}f(x)>\lambda\}) \le C\lambda^{-p}\int_{R^d}|f(x)|^p u(x)\mathrm{d}\mu(x),$$

where, for a weight u and a μ-measurable set E, $u(E)=\int_E u(x)d\mu(x)$ and C only depends on d, ρ and p.

Throughout the paper, C denotes a positive constant that is independent of the main parameters involved but whose value may vary from line to line. Constants with subscript such as C_1 do not change in different occurrences. Let a cube α and β be positive constants such that $\beta>\alpha^n$. For a cube Q, we say that Q is (α,β)-doubling if $\mu(\alpha Q)\le\beta\mu(Q)$, where αQ denotes the cube concentric with Q which has side length $\alpha l(Q)$. It was pointed out by Tolsa [14] that there exists a large constant $\beta=\beta_{\alpha,d}>0$ such that for any $x\in\operatorname{supp}(\mu)$ and $H>0$, there exists some $(\alpha,\beta_{\alpha,d})$-doubling cube centered at x with $l(Q)>H$ and for μ-almost all $x\in R^d$, and there exists a sequence $\{Q_k\}_{k\in N}$ of (α,β)-doubling cubes centered at x with $l(Q_k)\to 0$ as $k\to\infty$. In what follows, by a doubling cube Q we mean that Q is a $(2\rho,\beta_{2\rho,d})$ doubling cube, where $\rho\ge 1$. Moreover, for a cube Q, $\widetilde{Q}$ denotes the smallest doubling cube of the form $(2\rho)^kQ$ with $k\in\mathbb{N}\cup\{0\}$. For any two cubes $Q\subset R$, set

$$\delta_{Q,R}^{\rho}=1+\sum_{k=1}^{N_{Q,R}^{\rho}}\frac{\mu((2\rho)^kQ)}{[l((2\rho)^kQ)]^n},$$

where $N_{Q,R}^{\rho}$ is the least positive integer k such that $l((2\rho)^kQ)\ge l(R)$.

2. Some lemmas

At first, we recall the John-Strömberg maximal operator and the John-Strömberg sharp maximal operator related to the measure in (1), and the weighted norm inequalities with $A_\infty^{\rho}(\mu)$-weights related to these two operators, where $\rho\in[1,\infty)$.

For a cube with $\mu(Q)\ne 0$ and a real valued μ-measurable function f, we define the median value of f on the cube Q, denoted by $m_f(Q)$, to be one of the numbers such that

$$\mu(\{y\in Q : f(y)>m_f(Q)\}) \le \frac{1}{2}\mu(Q),$$

and

$$\mu(\{y \in Q : f(y) < m_f(Q)\}) \le \frac{1}{2}\mu(Q).$$

For the case $\mu(Q) = 0$, we define $m_f(Q) = 0$. If f is complex-valued, the median value $m_f(Q)$ of f is defined by

$$m_f(Q) = m_{\mathrm{Re} f}(Q) + i m_{\mathrm{Im} f}(Q),$$

where $i^2 = -1$.

Let $\rho \in [1, \infty)$ and $s \in (0, \beta_{2\rho,d}^{-1}/4)$. For any fixed cube Q and μ-measurable function f, we define the quantity $m_{0,s;Q}^{\rho}(f)$ by

$$m_{0,s;Q}^{\rho}(f) = \inf\{t > 0 : \mu(\{y \in Q : |f(y)| > t\}) < s\mu(\frac{3}{2}\rho Q)\},$$

if $\mu(\frac{3}{2}\rho Q) \neq 0$, and $m_{0,s;Q}^{\rho}(f) = 0$, if $\mu(\frac{3}{2}\rho Q) = 0$. The John-Strömberg maximal operator $M_{0,s}^{\rho,d}$ is defined by setting, for all $x \in R^d$,

$$M_{0,s}^{\rho,d} f(x) = \sup_{\substack{Q \ni x, \\ Q \text{ doubling}}} m_{0,s;Q}^{\rho}(f).$$

And the John-Strömberg sharp maximal function $M_{0,s}^{\rho,\sharp} f$ for any μ-measurable function f is defined by

$$M_{0,s}^{\rho,\sharp} f(x) = \sup_{Q \ni x} m_{0,s;Q}^{\rho}(f - m_f(\widetilde{Q})) + \sup_{\substack{R \supset Q \ni x \\ Q, R \text{ doubling}}} \frac{|m_f(Q) - m_f(R)|}{\delta_{Q,R}^{\rho}}.$$

For the case that μ is the d-dimensional Lebesgue measure, this sharp maximal operator was introduced by John [3] and then rediscovered by Strömberg [14] and Lerner [4, 5]. It is easy to check that for any cube $Q \ni x$ and $\varepsilon > 0$,

$$\mu(\{y \in Q : |f(y) - m_f(\widetilde{Q})| > M_{0,s}^{\rho,\sharp} f(x) + \varepsilon\}) < s\mu(\frac{3}{2}\rho Q).$$

Let $\rho \in [1, \infty)$ be fixed. For $\eta \in (1, \infty)$, we define the maximal by setting, for all $x \in R^d$,

$$M_\eta f(x) = \sup_{Q \ni x} \frac{1}{(\eta Q)} \int_Q |f(y)| \mathrm{d}\mu(y). \tag{6}$$

A result of Komori [2] states that for any $\eta > \rho$, $p \in [1, \infty)$ and $u \in A_p^{\rho}(\mu)$, M_η is bounded from $L^p(u)$ to $L^{p,\infty}(u)$. Let $M^{\rho,d}$ be the doubling maximal operator defined by setting, for all $x \in R^d$,

$$M^{\rho,d} f(x) = \sup_{\substack{Q \ni x \\ Q \text{ doubling}}} \frac{1}{\mu(\rho Q)} \int_Q |f(y)| \mathrm{d}\mu(y). \tag{7}$$

Notice that for any doubling cube Q,

$$\frac{1}{\mu(Q)} \int_Q |f(y)| \mathrm{d}\mu(y) \le \beta_{2\rho,d} \frac{1}{(2\rho Q)} \int_Q |f(y)| \mathrm{d}\mu(y) \le C \inf_{x \in Q} M_{2\rho} f(x).$$

Hu and Yang [8] proved the following result:

Theorem HY *Let $\rho \in [1, \infty)$, $s_1 \in (0, \beta_{2\rho,d}^{-1}/4)$, $p \in (0, \infty)$ and $u \in A_\infty^{\rho}(\mu)$. Then there exists a constant $C_1 \in (0, 1)$, depending on s_1 and u, and a positive constant C such that for any $s_2 \in (0, C_1 s_1)$,* *(i) If* $\mu(R^d) = \infty$, $f \in L^{p_0,\infty}(\mu)$ *with* $p_0 \in [1,\infty)$, *and for any* $R > 0$,

$$\sup_{0<\lambda<R} \lambda^p u(\{x \in R^d : |f(x)| > \lambda\}) < \infty,$$

then

$$\sup_{\lambda>0} \lambda^p u(\{x \in R^d : M^{\rho,d}_{0,s_1} f(x) > \lambda\}) \leq \sup_{\lambda>0} \lambda^p u(\{x \in R^d : M^{\rho,\sharp}_{0,s_2} f(x) > \lambda\});$$

(ii) If $\mu(R^d) < \infty$, *and* $f \in L^{p_0,\infty}(\mu)$ *with* $p_0 \in [1,\infty)$, *then*

$$\begin{aligned} &\sup_{\lambda>0} \lambda^p u(\{x \in R^d : M^{\rho,d}_{0,s_1} f(x) > \lambda\}) \\ &\leq \sup_{\lambda>0} \lambda^p u(\{x \in R^d : M^{\rho,\sharp}_{0,s_2} f(x) > \lambda\}) + Cu(R^d)(s_1\mu(R^d))^{-p/p_0} \|f\|^p_{L^{p_0,\infty}(\mu)}. \end{aligned}$$

Hu and Yang [8] still introduced the sharp maximal operator $M^{\rho,\sharp}_r$. Let $r \in (0,\infty)$. The sharp maximal operator $M^{\rho,\sharp}_r$ was defined by setting, for all $x \in R^d$,

$$M^{\rho,\sharp}_r f(x) = \sup_{Q \ni x} \Big(\frac{1}{\mu(\frac{3}{2}\rho Q)} \int_Q |f(y) - m_f(\widetilde{Q})|^r \mathrm{d}\mu(y) \Big)^{\frac{1}{r}} + \sup_{\substack{x \in Q \subset R \\ Q\ R\ \text{doubling}}} \frac{|m_f(Q) - m_f(Q)|}{\delta^\rho_{Q,R}}.$$

For $r = 1$ and $\rho = 1$, this operator is the sharp maximal operator introduced by Tolsa [15]. It is easy to check that for any cube Q and $r \in (0,\infty)$,

$$m^\rho_{0,s;Q}(f - m_f(\widetilde{Q})) \leq s^{-1/r} \Big(\frac{1}{\mu(\frac{3}{2}\rho Q)} \int_Q |f(y) - m_f(\widetilde{Q})|^r \mathrm{d}\mu(y) \Big)^{\frac{1}{r}}.$$

Therefore, for all $x \in R^d$,

$$M^{\rho,\sharp}_{0,s} f(x) \leq s^{-1/r} M^{\rho,\sharp}_r f(x). \tag{8}$$

To prove our theorem, we need some Lemmas.

Lemma 1 ([8]) *Let* $\rho \in [1,\infty)$, M_η *and* $M^{\rho,d}$ *be the maximal operators defined (6) and (7), respectively. For any* $p \in [1,\infty)$ *and* $u \in A^\rho_p(\mu)$, *both* M_η *with* $\eta \in (\rho,\infty)$ *and* $M^{\rho,d}$ *are bounded from* $L^p(u)$ *to* $L^{p,\infty}(u)$.

Lemma 2 ([8]) *Let* $\rho, p \in [1,\infty), u \in A^\rho_p(\mu)$, *and* $\eta \in (\rho,\infty)$. *Then there exist constants* $C_1, C_2 \geq 1$ *such that*

(i) For any cube Q *and* μ*-measurable set* $E \subset Q$, $\frac{u(E)}{u(Q)} \geq C_1^{-1}(\frac{\mu(E)}{\mu(\eta Q)})^p$;

(ii) For any doubling cube Q *and* μ*-measurable set* $E \subset Q$, $\frac{u(E)}{u(Q)} \geq C_2^{-1}(\frac{\mu(E)}{\mu(Q)})^p$;

(iii) For any doubling cube Q *and* μ*-measurable set* $E \subset Q$, $\frac{u(E)}{u(Q)} \leq 1 - C_2^{-1}(\frac{1-\mu(E)}{\mu(Q)})^p$.

Lemma 3 ([8]) *Let* $\rho, p \in [1,\infty), s \in (0, \beta^{-1}_{2\rho,d}/4)$. *Then for all* μ*-measurable functions* f *and* $\lambda > 0$,

(i) $\{x \in R^d : |f(x)| > \lambda\} \subset \{x \in R^d : M^{\rho,d}_{0,s} f(x) > \lambda\} \cup \Theta$ *with* $\mu(\Theta) = 0$;

(ii) For $u \in A^\rho_p(\mu)$,

$$u(\{x \in R^d : M^{\rho,d}_{0,s} f(x) > \lambda\}) \leq Cs^{-p} u(\{x \in R^d : |f(x)| > \lambda\}),$$

where C *is a positive constant depending on* d *and* ρ, *but not on* s *and the weight* u.

Lemma 4 ([8]) *Let $\rho \in [1,\infty)$, $s \in (0, \beta_{2\rho,d}^{-1}/4)$ and Q be a doubling cube with $\mu(Q) \neq 0$. For any constant $c \in \mathbb{C}$ and μ-locally integrable function f,*

$$|m_{0,s;Q}^{\rho}(f) - |c|| \leq m_{0,s;Q}(f-c).$$

Lemma 5 ([8]) *Let $\rho \in [1,\infty)$, $s \in (0, \beta_{2\rho,d}^{-1}/4)$ and $r \in (0,\infty)$. For any cube Q and μ-locally integrable function f,*

$$m_{0,s;Q}^{\rho}(f - m_f(Q)) \leq 3s^{-1/r} \inf_{c\in\mathbb{C}} \Big(\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |f(y)-c|^r \mathrm{d}\mu(y)\Big)^{\frac{1}{r}}.$$

Lemma 6 *Let $\rho \in [1,\infty)$, and $r \in (0,1)$. Let K satisfy (2) and (5), and $\mathcal{M}$ be the Marcinkiewicz integral defined as in (4). If $\mathcal{M}$ is bounded on $L^2(\mu)$, there exists a positive constant C such that for all bounded functions f with compact support and $x \in R^d$,*

$$M_r^{\rho;\sharp}(\mathcal{M}f)(x) \leq CM_{\frac{9}{8}\rho}f(x). \tag{9}$$

Proof For each cube Q and each bounded function f with compact support, set

$$h_Q = m_Q(\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})).$$

Here and in what follows, for any μ-locally integrable function h,

$$m_Q(f) = \frac{1}{\mu(Q)}\int_Q h(z)\mathrm{d}\mu(z).$$

It follows from Lemmas 4 and 5 that for any cube Q, $s \in (0, \beta_{2\rho,d}^{-1}/4)$, and an elementary inequality,

$$\begin{aligned}
&\int_Q |\mathcal{M}(f)(y) - m_{\mathcal{M}(f)}(\widetilde{Q})|^r \mathrm{d}\mu(y)\\
&\leq \int_Q |\mathcal{M}(f)(y) - h_Q|^r \mathrm{d}\mu(y) + |h_Q - h_{\widetilde{Q}}|^r\mu(Q) +\\
&\quad |m_{0,s;\widetilde{Q}}^{\rho}(\mathcal{M}(f)) - m_{\mathcal{M}(f)}(\widetilde{Q})|^r\mu(Q) + |m_{0,s;\widetilde{Q}}^{\rho}(\mathcal{M}(f)) - h_{\widetilde{Q}}|^r\mu(Q)\\
&\leq \int_Q |\mathcal{M}(f)(y) - h_Q|^r \mathrm{d}\mu(y) + |h_Q - h_{\widetilde{Q}}|^r\mu(Q) +\\
&\quad (m_{0,s;\widetilde{Q}}^{\rho}(\mathcal{M}(f) - m_{\mathcal{M}(f)}(\widetilde{Q})))^r\mu(Q) + (m_{0,s;\widetilde{Q}}^{\rho}(\mathcal{M}(f) - h_{\widetilde{Q}}))^r\mu(Q)\\
&\leq \int_Q |\mathcal{M}(f)(y) - h_Q|^r \mathrm{d}\mu(y) + |h_Q - h_{\widetilde{Q}}|^r\mu(Q) +\\
&\quad C(3^r s^{-1} + s^{-1})\frac{\mu(Q)}{\mu(\widetilde{Q})}\int_{\widetilde{Q}} |\mathcal{M}(f)(y) - h_{\widetilde{Q}}|^r \mathrm{d}\mu(y),
\end{aligned}$$

where C is a positive constant, and for any two doubling cubes $Q \subset R$,

$$\begin{aligned}
&|m_{\mathcal{M}(f)}(Q) - m_{\mathcal{M}(f)}(R)|\\
&\quad\leq |m_{0,s;Q}^{\rho}(\mathcal{M}(f)) - h_Q| + |h_Q - h_R| + |m_{0,s;R}^{\rho}(\mathcal{M}(f)) - h_R| +\\
&\qquad |m_{0,s;Q}^{\rho}(\mathcal{M}(f)) - m_{\mathcal{M}(f)}(Q)| + |m_{0,s;R}^{\rho}(\mathcal{M}(f)) - m_{\mathcal{M}(f)}(R)|\\
&\quad\leq m_{0,s;Q}^{\rho}(\mathcal{M}(f) - h_Q) + |h_Q - h_R| + m_{0,s;R}^{\rho}(\mathcal{M}(f) - h_R) +\\
&\qquad m_{0,s;Q}^{\rho}(\mathcal{M}(f) - m_{\mathcal{M}(f)}(Q)) + m_{0,s;R}^{\rho}(\mathcal{M}(f) - m_{\mathcal{M}(f)}(R))
\end{aligned}$$

$$\leq 4s^{-1/r}\Big(\frac{1}{\mu(\frac{3}{2}\rho Q)}\int_Q |\mathcal{M}(f)(y)-h_Q|^r \mathrm{d}\mu(y)\Big)^{1/r}+|h_Q-h_R|+$$
$$4s^{-1/r}\Big(\frac{1}{\mu(\frac{3}{2}\rho R)}\int_R |\mathcal{M}(f)(y)-h_R|^r \mathrm{d}\mu(y)\Big)^{1/r}.$$

Thus, the proof of (9) can be reduced to proving that for any cube Q,

$$\Big(\frac{1}{\mu(\frac{3}{2}\rho Q)}\int_Q |\mathcal{M}(f)(y)-h_Q|^r \mathrm{d}\mu(y)\Big)^{1/r}\leq C\inf_{x\in Q} M_{\frac{9}{8}\rho}f(x), \tag{10}$$

and for any two cubes $Q\subset R$ with R a doubling cube,

$$|h_Q-h_R|\leq C\delta^{\rho}_{Q,R}\inf_{x\in Q} M_{\frac{9}{8}\rho}f(x), \tag{11}$$

where C is a positive constant.

We first consider (10). For any cube Q, write

$$\begin{aligned}
&\int_Q |\mathcal{M}(f)(y)-h_Q|^r \mathrm{d}\mu(y)\\
&\leq \int_Q |\mathcal{M}(f)(y)-\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(y)|^r \mathrm{d}\mu(y)+\int_Q |\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(y)-h_Q|^r \mathrm{d}\mu(y)\\
&\leq \int_Q |\mathcal{M}(f\chi_{\frac{4}{3}Q})(y)|^r \mathrm{d}\mu(y)+\int_Q |\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(y)-h_Q|^r \mathrm{d}\mu(y),
\end{aligned}$$

where we use the fact that $|\mathcal{M}(f)(y)-\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(y)|\leq|\mathcal{M}(f\chi_{\frac{4}{3}Q})(y)|$. Recall that $\mathcal{M}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ (see [7]). It follows from the Kolmogorov inequality that

$$\Big(\frac{1}{\mu(\frac{3}{2}\rho Q)}\int_Q |\mathcal{M}(f\chi_{\frac{4}{3}Q})(y)|^r \mathrm{d}\mu(y)\Big)^{1/r}\leq\frac{C}{\mu(\frac{3}{2}\rho Q)}\|f\chi_{\frac{4}{3}Q}\|_{L^1(\mu)}\leq C\inf_{x\in Q} M_{\frac{9}{8}\rho}f(x).$$

On the other hand, following the method of [7], for any $x,y\in Q$, let $f_*=f\chi_{R^d\setminus\frac{4}{3}Q}$ and set

$$\mathrm{A}_1=\Big(\int_0^\infty\Big[\int_{|y-z|\leq t\leq|x-z|}|K(y,z)||f_*(z)|\,\mathrm{d}\mu(z)\Big]^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2},$$
$$\mathrm{A}_2=\Big(\int_0^\infty\Big[\int_{|x-z|\leq t\leq|y-z|}|K(y,z)||f_*(z)|\,\mathrm{d}\mu(z)\Big]^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2},$$

and

$$\mathrm{A}_3=\Big(\int_0^\infty\Big[\int_{\max(|y-z|,|x-z|)\leq t}|K(y,z)-k(x,z)||f_*(z)|\,\mathrm{d}\mu(z)\Big]^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}.$$

By the Minkowski inequality, we have

$$\begin{aligned}
\mathcal{M}(f_*)(x)&\leq\Big(\int_0^\infty\Big|\int_{|x-z|\leq t}K(x,z)f_*(z)\mathrm{d}\mu(z)-\int_{|y-z|\leq t}K(y,z)f_*(z)\,\mathrm{d}\mu(z)\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}+\mathcal{M}(f_*)(y)\\
&\leq \mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3+\mathcal{M}(f_*)(y).
\end{aligned}$$

This together with symmetry gives

$$|\mathcal{M}(f_*)(x)-\mathcal{M}(f_*)(y)|\leq \mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3. \tag{12}$$

Applying the Minkowski inequality and (2), we get for $x,y\in Q$

$$\mathrm{A}_1\leq\int_{|y-z|\leq|x-z|}\frac{|f_*(z)|}{|y-z|^{n-1}}\Big[\int_{|y-z|\leq t\leq|x-z|}\frac{\mathrm{d}t}{t^3}\Big]^{\frac{1}{2}}\mathrm{d}\mu(z)$$

$$\le C\int_{|y-z|\le|x-z|}\frac{|f_*(z)|}{|x_Q-z|^{n+\frac{1}{2}}}|x-y|^{\frac{1}{2}}\mathrm{d}\mu(z)$$
$$\le Cl(Q)^{\frac{1}{2}}\sum_{k=1}^{\infty}\int_{(\frac{4}{3})^{k+1}Q\setminus(\frac{4}{3})^kQ}\frac{|f(z)|}{|x_Q-z|^{n+\frac{1}{2}}}\mathrm{d}\mu(z)$$
$$\le Cl(Q)^{\frac{1}{2}}\sum_{k=1}^{\infty}\frac{\mu(\frac{9}{8}\rho(\frac{4}{3})^{k+1}Q)}{(\frac{4}{3}^k l(Q))^{n+\frac{1}{2}}}\frac{1}{\mu(\frac{9}{8}\rho(\frac{4}{3})^{k+1}Q)}\int_{(\frac{4}{3})^{k+1}Q}|f(z)|\mathrm{d}\mu(z)$$
$$\le C\sum_{k=1}^{\infty}(\frac{4}{3})^{-k}M_{\frac{9}{8}\rho}f(x)\le CM_{\frac{9}{8}\rho}f(x).$$

By symmetry, we have

$$A_2\le CM_{\frac{9}{8}\rho}f(x).$$

And by (5),

$$\mathrm{A}_3\le\int_{R^d}|K(y,z)-K(x,z)||f_*(z)|\Big[\int_{max(|y-z|,|x-z|)\le t}\frac{\mathrm{d}t}{t^3}\Big]^{1/2}\mathrm{d}\mu(z)$$
$$\le C\int_{R^d\setminus(\frac{4}{3})Q}\frac{|x-y|^{\delta}}{|x-z|^{n+\delta-1}}|f(z)|\frac{1}{|x_Q-z|}\mathrm{d}\mu(z)$$
$$\le C\sum_{k=1}^{\infty}\int_{(\frac{4}{3})^{k+1}Q\setminus(\frac{4}{3})^kQ}\frac{l(Q)^{\delta}}{|x_Q-z|^{n+\delta}}|f(z)|\mathrm{d}\mu(z)$$
$$\le C\sum_{k=1}^{\infty}\int_{(\frac{4}{3})^{k+1}Q}\frac{l(Q)^{\delta}\mu(\frac{9}{8}\rho(\frac{4}{3})^{k+1}Q)}{((\frac{4}{3})^k l(Q))^{n+\delta}}\frac{1}{\mu(\frac{9}{8}\rho(\frac{4}{3})^{k+1}Q)}|f(z)|\mathrm{d}\mu(z)$$
$$\le C\sum_{k=1}^{\infty}(\frac{4}{3})^{-k}M_{\frac{9}{8}\rho}f(x)\le CM_{\frac{9}{8}\rho}f(x).$$

Combining these estimates for $\mathrm{A}_1,\mathrm{A}_2$ and A_3, we have

$$\Big(\frac{1}{\mu(\frac{3}{2}\rho Q)}\int_Q|\mathcal{M}(f)(y)-h_Q|^r\mathrm{d}\mu(y)\Big)^{1/r}\le C\inf_{x\in Q}M_{\frac{9}{8}\rho}f(x),$$

and obtain the estimate (10).

Next we turn to (11). Let $N=N_{Q,R}^{\rho}+1$ and by Minkowski inequality, we have,

$$\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(x)\le\mathcal{M}(f\chi_{R^d\setminus(2\rho)^NQ})(x)+\mathcal{M}(f\chi_{(2\rho)^NQ\setminus\frac{4}{3}Q})(x)$$

and

$$\mathcal{M}(f\chi_{R^d\setminus(2\rho)^NQ})(x)\le\mathcal{M}(f\chi_{(2\rho)^NQ\setminus\frac{4}{3}Q})(x)+\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(x).$$

By the same estimate as above, we obtain

$$\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}R})(y)\le\mathcal{M}(f\chi_{R^d\setminus(2\rho)^NQ})(y)+\mathcal{M}(f\chi_{(2\rho)^NQ\setminus\frac{4}{3}R})(y)$$

and

$$\mathcal{M}(f\chi_{R^d\setminus(2\rho)^NQ})(y)\le\mathcal{M}(f\chi_{(2\rho)^NQ\setminus\frac{4}{3}R})(y)+\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}R})(y).$$

Then, it is easy to check that, for any $x,y\in\mathbb{R}^d$,

$$\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(x)-\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}R})(y)\le\mathcal{M}(f\chi_{R^d\setminus(2\rho)^NQ})(x)+\mathcal{M}(f\chi_{(2\rho)^NQ\setminus\frac{4}{3}Q})(x)+$$

$$\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}R})(y)-\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(y),$$

and

$$\begin{aligned}\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}R})(y)-\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(x)\le&\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(y)+\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}R})(y)+\\&\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}Q})(x)-\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(x).\end{aligned}$$

It follows that,

$$\begin{aligned}|\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}Q})(x)-\mathcal{M}(f\chi_{R^d\setminus\frac{4}{3}R})(y)|\le&\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}Q})(x)+\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}R})(y)+\\&|\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(x)-\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(y)|.\end{aligned}$$

Then we have

$$\begin{aligned}|h_Q-h_R|\le&m_Q(\mathcal{M}(f\chi_{2\rho Q\setminus\frac{4}{3}Q}))+m_Q(\mathcal{M}(f\chi_{(2\rho)^N Q\setminus 2\rho Q}))+\\&\frac{1}{\mu(Q)\mu(R)}\int_Q\int_R|\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})-\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})|\mathrm{d}\mu(y)\mathrm{d}\mu(x)+\\&m_R(\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}R}))\\=&I_1+I_2+I_3+I_4.\end{aligned}$$

The size condition (2) and the Minkowski inequality, along with the growth condition (1) implies that for any $x\in Q$,

$$\begin{aligned}\mathcal{M}(f\chi_{2\rho Q\setminus\frac{4}{3}Q})(x)&\le\int_{2\rho Q\setminus\frac{4}{3}Q}|f(z)K(x,z)|\Big(\int_{l(Q)/6}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}\mu(z)\\&\le\frac{C}{[l(Q)]^n}\int_{2\rho Q}|f(z)|\mathrm{d}\mu(z)\le CM_{\frac{9}{8}\rho}f(x)\end{aligned}$$

and for any $y\in R$,

$$\begin{aligned}\mathcal{M}(f\chi_{(2\rho)^N Q\setminus\frac{4}{3}R})(y)&\le\int_{4\rho R\setminus\frac{4}{3}R}|f(z)K(y,z)|\Big(\int_{l(R)/6}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}\mu(z)\\&\le\frac{C}{[l(R)]^n}\int_{4\rho R}|f(z)|\mathrm{d}\mu(z)\le CM_{\frac{9}{8}\rho}f(x).\end{aligned}$$

Therefore, there exists a positive constant C such that

$$I_1+I_4\le C\inf_{x\in Q}M_{\frac{9}{8}\rho}f(x).$$

For the term I_2, the Minkowski inequality, the size condition (2) and the growth condition (1) indicate that for any $x\in Q$,

$$\begin{aligned}\mathcal{M}(f\chi_{(2\rho)^N Q\setminus 2\rho Q})(x)&\le\int_{(2\rho)^N Q\setminus 2\rho Q}|f(z)K(y,z)|\Big(\int_{|x-z|\le t}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}\mu(z)\\&\le C\sum_{k=1}^{N-1}\int_{(2\rho)^{k+1}Q\setminus(2\rho)^k Q}\frac{|f(z)|}{|x-z|^n}\mathrm{d}\mu(z)\\&\le C\sum_{k=1}^{N-1}\frac{\mu((2\rho)^{k+2}Q)}{[l((2\rho)^k Q)]^n}\inf_{x\in Q}M_{2\rho}f(x)\\&\le C\delta^{\rho}_{Q,R}\inf_{x\in Q}M_{\frac{9}{8}\rho}f(x).\end{aligned}$$

So, we have

$$I_2 \le C\delta^{\rho}_{Q,R} \inf_{x\in Q} M_{\frac{9}{8}\rho} f(x).$$

Finally, as in the inequality (12), a familiar argument involving the condition (5) gives, for any $x \in Q$ and $y \in R$,

$$|\mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(x) - \mathcal{M}(f\chi_{R^d\setminus(2\rho)^N Q})(y)| \le C \inf_{x\in Q} M_{\frac{9}{8}\rho} f(x)$$

and

$$I_3 \le C \inf_{x\in Q} M_{\frac{9}{8}\rho} f(x).$$

Then, the inequality (11) holds, and the proof of Lemma 6 is completed. □

3. Proof of Theorem 1

Now we turn to prove Theorem 1.

Proof of Theorem 1 By Lemma 1, it suffices to show that

$$u(\{x \in R^d : \mathcal{M}(f)(x) > \lambda\}) \le Cu(\{x \in R^d : M_{\frac{9}{8}\rho} f(x) > \lambda\}). \tag{13}$$

Using Theorem HY, we first prove that for $p \in [1,\infty)$ and any bounded function f with compact support and for any $R > 0$,

$$\mathcal{M}(f) \in L^{p,\infty}(\mu), \tag{14}$$

and for any $\rho \in [1,\infty)$ and $u \in A^{\rho}_p(\mu)$,

$$\sup_{0<\lambda<R} u(\{x \in R^d : \mathcal{M}(f)(x) > \lambda\}) < \infty. \tag{15}$$

The fact (14) was proved in [7]. So, we need only to prove (15). Let $t > 2$ be large enough such that the support of f is contained in the ball $B(0,t)$. It is obvious that

$$\sup_{0<\lambda<R} \lambda^p u(\{x \in B(0,2t) : \mathcal{M}(f)(x) > \lambda\}) \le R^p u(B(0,2t)) \le \infty.$$

On the other hand, it is easy to see that if $x \in R^d\setminus B(0,2t)$ and $y \in B(0,t)$, then we obtain $|x| \sim |x-y|$ and by the Minkowski inequality and the size condition (2),

$$\mathcal{M}(f)(x) \le \int_{R^d} \frac{|f(y)|}{|x-y|^n} \mathrm{d}\mu(y) \le \frac{C_4}{|x|^n} \|f\|_{L^1(\mu)}.$$

Lemma 2 (i) and the growth condition (1) imply that if $\lambda \le C_4\|f\|_{L^1(\mu)}/2$,

$$\begin{aligned}
&u(\{x \in R^d\setminus B(0,2t) : \mathcal{M}(f)(x) > \lambda\}) \le u(\{x \in R^d : |x|^n > \lambda/(C_4\|f\|_{L^1(\mu)})\}) \\
&\quad \le u(B(0, \frac{9}{8}\rho(C_4\|f\|_{L^1(\mu)})^{1/n}\lambda^{-1/n})) \\
&\quad \le Cu(B(0,1))\Big(\frac{\mu(B(0,\frac{9}{8}\rho(C_4\|f\|_{L^1(\mu)})^{1/n}\lambda^{-1/n}))}{\mu(B(0,1))}\Big) \\
&\quad \le C_f \frac{u(B(0,1))}{[\mu(B(0,1))]^p}\lambda^{-p},
\end{aligned}$$

where C_f is a positive constant depending on f.

Notice that for $\lambda > C_4\|f\|_{L^1(\mu)}/2$, there exists no point $x \in R^d \backslash B(0,2t)$ satisfying $\mathcal{M}(f)(x) > \lambda$. Therefore,

$$\begin{aligned}
&\sup_{\lambda>0}\lambda^p u(\{x \in R^d \backslash B(0,2t) : \mathcal{M}(f)(x) > \lambda\}) \\
&\quad = \sup_{C_4\|f\|_{L^1(\mu)}/2 \geq \lambda > 0} \lambda^p u(\{x \in R^d \backslash B(0,2t) : \mathcal{M}(f)(x) > \lambda\}) \\
&\quad \leq C_f \frac{u(B(0,1))}{[\mu(B(0,1))]^p},
\end{aligned}$$

which yields (15).

Now we conclude the proof of (13).

If $\mu(R^d) = \infty$, by Lemma 3 (i), Theorem HY with $s_1 = \beta_{2\rho,d}^{-1}/5$ and $p_0 = 1$, (8) and Lemma 6, we have that

$$\begin{aligned}
u(\{x \in R^d : \mathcal{M}(f)(x) > \lambda\}) &\leq Cu(\{x \in R^d : M_{0,s}^{\rho,d}\mathcal{M}(f)(x) > \lambda\}) \\
&\leq Cu(\{x \in R^d : M_{0,s}^{\rho,\sharp}\mathcal{M}(f)(x) > \lambda\}) \\
&\leq Cu(\{x \in R^d : M_r^{\rho,d}\mathcal{M}(f)(x) > \lambda\}) \\
&\leq Cu(\{x \in R^d : M_{\frac{9}{8}\rho} f(x) > \lambda\}).
\end{aligned}$$

If $\mu(R^d) < \infty$, $p, \rho \in [1,\infty)$ and $u \in A_p^\rho(\mu)$, then for a positive constant C,

$$\begin{aligned}
u(R^d)[\mu(R^d)]^{-p}\|\mathcal{M}(f)\|_{L^{1,\infty}(\mu)}^p &\leq Cu(R^d)[\mu(R^d)]^{-p}\|f\|_{L^1(\mu)}^p \\
&\leq Cu(R^d)(\inf_{x\in R^d} M_{\frac{9}{8}\rho} f(x))^p \\
&\leq C\sup_{\lambda>0}[u(\{x \in R^d : M_{\frac{9}{8}\rho} f(x) > \lambda\})],
\end{aligned}$$

where in the first inequality, we have invoked the estimate

$$\|\mathcal{M}(f)\|_{L^{1,\infty}(\mu)} \leq C\|f\|_{L^1(\mu)},$$

(see [7]), and the second inequality follows from the fact that

$$\frac{1}{\mu(R^d)}\int_{R^d}|f(y)|\mathrm{d}\mu(y) = \lim_{l(Q)\to\infty}\frac{1}{\mu(\frac{9}{8}\rho Q)}\int_Q |f(y)|\mathrm{d}\mu(y) \leq \inf_{x\in R^d} M_{\frac{9}{8}\rho} f(x).$$

The desired result again follows from Lemma 3 (i), Theorem HY with $s_1 = \beta_{2\rho,d}^{-1}/5$ and $p_0 = 1$, (8) and Lemma 6. This completes the proof of Theorem 1. □

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