Journal of Mathematical Research with Applications
Mar., 2012, Vol. 32, No. 2, pp. 235–240
DOI:10.3770/j.issn:2095-2651.2012.02.011
Http://jmre.dlut.edu.cn

Linear Maps Preserving Projections of Jordan Products on the Space of Self-Adjoint Operators

Meifeng WANG, Guoxing JI*
College of Mathematics and Information Science, Shaanxi Normal University, Shaanxi 710062*, China*

Abstract Let $\mathcal{B}_s(\mathcal{H})$ be the real linear space of all self-adjoint operators on a complex Hilbert space $\mathcal{H}$ with $\dim\mathcal{H} \geq 2$. It is proved that a linear surjective map φ on $\mathcal{B}_s(\mathcal{H})$ preserves the nonzero projections of Jordan products of two operators if and only if there is a unitary or an anti-unitary operator U on $\mathcal{H}$ such that $\varphi(X) = \lambda U^*XU, \forall X \in \mathcal{B}_s(\mathcal{H})$ for some constant λ with $\lambda \in \{1, -1\}$.

Keywords self-adjoint operator; Jordan product; projection; linear map.

MR(2010) Subject Classification 47B49

1. Introduction

Preserver problems are the most extensively studied both on operator algebras and on operator spaces in the last few decades. In recent years, many authors have considered preserver problems concerning certain properties of products or triple Jordan products of operators [1, 2]. For example, maps preserving projections or idempotency of products or triple Jordan products of two operators on operator algebras are characterized. Moreover, those preserver problems on the space of self-adjoint operators attract much attention of many authors [3–7]. In particular, maps preserving zero Jordan products of two operators and linear maps preserving projections of products or triple Jordan products of two operators on the space of Hermitian operators were discussed in [6] and [7], respectively. Motivated by comparing [6] with [7], we study in this paper those linear maps preserving nonzero projections of Jordan products of two operators on the space of self-adjoint operators.

Before starting our main results, we first introduce some notations. Let $\mathcal{H}$ be a Hilbert space over the complex field $\mathbb{C}$. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{B}_s(\mathcal{H})$ the real linear space of all self-adjoint operators in $\mathcal{B}(\mathcal{H})$. $\dim\mathcal{H}$ denotes the dimension of $\mathcal{H}$. For every pair of vectors $x, y \in \mathcal{H}$, $\langle x, y\rangle$ denotes the inner product of x and y. For any vector $x \in \mathcal{H}, \|x\| = \sqrt{\langle x, x\rangle}$ denotes the norm of x. The rank-1 operator $x \otimes x$ is a projection for any unit vector x. For any $A \in \mathcal{B}(\mathcal{H})$, we denote by $rankA$ the rank of A and by $\sigma(A)$ the spectrum of A, respectively. We denote by $\mathscr{P}$ the set of all nonzero projections in $\mathcal{B}(\mathcal{H})$. Given

Received June 30, 2010; Accepted October 31, 2011
Supported by the National Natural Science Foundation of China (Grant No. 10971123) and the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20090202110001).
* Corresponding author
E-mail address: gxji@snnu.edu.cn (Guoxing JI)

two projections $P, Q \in \mathcal{B}(\mathcal{H})$, we say $P \leq Q$ if $PQ = QP = P$ and we say $P < Q$ if $P \leq Q$ and $P \neq Q$. P and Q are orthogonal if $PQ = QP = 0$. We recall that a conjugate linear bijective map U on $\mathcal{H}$ is said to be anti-unitary if $\langle Ux, Uy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. For any $A, B \in \mathcal{B}(\mathcal{H})$, we denote by $A \circ B$ the Jordan product $\frac{1}{2}(AB+BA)$ of A and B. Note that $A \circ B \in \mathcal{B}_s(\mathcal{H})$ whenever $A, B \in \mathcal{B}_s(\mathcal{H})$. A map φ on $\mathcal{B}_s(\mathcal{H})$ is said to preserve nonzero projections of Jordan products of two operators if $\varphi(A) \circ \varphi(B) \in \mathscr{P}$ whenever $A \circ B \in \mathscr{P}$ for all $A, B \in \mathcal{B}_s(\mathcal{H})$. Throughout this paper, we will denote by I the identity operator on a Hilbert space without confusion.

In this paper, we consider a linear surjective map φ on $\mathcal{B}_s(\mathcal{H})$ which preserves nonzero projections of Jordan products of two operators.

2. Linear maps preserving projections of Jordan products on $\mathcal{B}_s(\mathcal{H})$

Let φ be a linear map on $\mathcal{B}_s(\mathcal{H})$ preserving nonzero projections of Jordan products of two operators, that is, $\varphi(A) \circ \varphi(B)$ is a nonzero projection whenever $A \circ B$ is for all $A, B \in \mathcal{B}_s(\mathcal{H})$ in this section. Our main result is the following theorem.

Theorem 1 *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 2$ and let φ be a linear surjective map on $\mathcal{B}_s(\mathcal{H})$. Then φ preserves nonzero projections of Jordan products of two operators if and only if there exist a unitary or anti-unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a constant λ with $\lambda \in \{1, -1\}$ such that $\varphi(X) = \lambda U^* X U$ for all $X \in \mathcal{B}_s(\mathcal{H})$.*

Proof It is obvious that the "if" part of the theorem is true. We need only to check the "only if" part.

Assume that φ is a linear surjective map and preserves nonzero projections of Jordan products of two operators. In the following, we proceed in steps.

Step 1. φ is injective.

If $\varphi(A) = 0$ for some non-zero operator $A \in \mathcal{B}_s(\mathcal{H})$, then there exists at least a nonzero $\lambda \in \sigma(A)$. Choose a closed subset $\triangle \subseteq \sigma(A)$ such that $\lambda \in \triangle$ and $0 \notin \triangle$. Under the direct sum decomposition $\mathcal{H} = E(\triangle)\mathcal{H} \oplus E(\triangle)^{\perp}\mathcal{H}$, we have $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 is invertible. Put $B = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. Note that $A \circ B = \frac{1}{2}(AB + BA) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{P}$, while $\varphi(A) \circ \varphi(B) = 0$. This is a contradiction. Thus $A = 0$.

Step 2. $\varphi(I) = I$ or $\varphi(I) = -I$.

We first note an elementary fact.

$$AP = PA \in \mathcal{P} \text{ for all } P \in \mathcal{P} \text{ and } A \in \mathcal{B}_s(\mathcal{H}) \text{ with } A \circ P \in \mathcal{P}. \tag{1}$$

In fact, under the direct sum decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we set

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix}.$$

Then

$$A\circ P=\frac{1}{2}(AP+PA)=\begin{pmatrix} A_1 & \frac{1}{2}A_2 \\ \frac{1}{2}A_2^* & 0 \end{pmatrix}\in\mathscr{P}.$$

So we have $A_1\in\mathscr{P}$ and $A_2=0$. Hence

$$PA=AP=\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\in\mathscr{P}.$$

Take any $P,Q\in\mathscr{P}$ with $P\perp Q$. Then for all $\lambda\in\mathbb{R}$ we have that $P\circ(P+\lambda Q)=P(P+\lambda Q)=(P+\lambda Q)P=P\in\mathscr{P}$. It follows that $\varphi(P)\circ\varphi(P+\lambda Q)\in\mathscr{P}$, that is, $\varphi(P)^2+\lambda(\varphi(P)\circ\varphi(Q))\in\mathscr{P}$. In particular, $\varphi(P)^2\in\mathscr{P}$. Thus

$$\varphi(P)\circ\varphi(Q)=0. \tag{2}$$

Taking $Q=I-P$, we obtain from (2) another equation

$$\varphi(P)\circ\varphi(I)=\varphi(P)^2. \tag{3}$$

It easily follows that

$$\varphi(P)^2\varphi(I)=\varphi(I)\varphi(P)^2,\ \varphi(P)\varphi(I)^2=\varphi(I)^2\varphi(P) \tag{4}$$

by multiplying (3) from the left and the right by $\varphi(P)$ (resp., $\varphi(I)$), respectively. Since φ is surjective, there exists a non-zero operator $A\in\mathcal{B}_s(\mathcal{H})$ such that $\varphi(A)=B$ for every $B\in\mathcal{B}_s(\mathcal{H})$. It is known that every bounded self-adjoint linear operator A is a real linear combination of eight projections from Theorem 3 in [9]. That is, $A=\sum_{k=1}^{8}\alpha_kP_k$ for some projections $P_1,P_2,\dots,P_8$. Then $B=\varphi(A)=\sum_{k=1}^{8}\alpha_k\varphi(P_k)$. Thus $B\varphi(I)^2=\varphi(I)^2B$ since $\varphi(P_k)\varphi(I)^2=\varphi(I)^2\varphi(P_k)$ for all $k=1,2,\dots,8$ from (4). This implies that $\varphi(I)^2=I$. It follows that $\sigma(\varphi(I))\subseteq\{-1,1\}$. If $\sigma(\varphi(I))=\{-1\}$ or $\{1\}$, then $\varphi(I)=-I$ or $\varphi(I)=I$. If $\sigma(\varphi(I))=\{-1,1\}$, then

$$\varphi(I)=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

by the spectral decomposition. Let $E=e\otimes e$ for any unit vector $e\in\mathcal{H}$. Then there exists a nonzero operator $A\in\mathcal{B}_s(\mathcal{H})$ such that $\varphi(A)=E$. We claim that $\sigma(A)\backslash\{0\}$ is a singleton. Otherwise, if there are λ_1 and λ_2 in $\sigma(A)$ such that $\lambda_1\lambda_2\neq0$ and $\lambda_1\neq\lambda_2$, then there exist closed subsets $\triangle_i\subseteq\sigma(A)$ $(i=1,2)$ such that $0\notin\triangle_1\bigcup\triangle_2$, $\lambda_i\in\triangle_i$ $(i=1,2)$ and $\triangle_1\cap\triangle_2=\emptyset$ $(i=1,2)$. Under the direct sum decomposition $\mathcal{H}=E(\triangle_1)\mathcal{H}\oplus E(\triangle_2)\mathcal{H}\oplus(I-E(\triangle_1)-E(\triangle_2))\mathcal{H}$, we have that

$$A=\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix},$$

where A_1 and A_2 are invertible. Let

$$B_1=\begin{pmatrix} A_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $A\circ B_1$, $A\circ B_2$ and $A\circ(B_1+B_2)$ all are in $\mathscr{P}$. It easily follows that $\varphi(B_1)e=\varphi(B_2)e=(\varphi(B_1)+\varphi(B_2))e=e$. This is a contradiction. Since $\varphi(A)=E$, $\sigma(A)=\{0,\lambda\}$ for some $\lambda\in\mathbb{R}\backslash\{0\}$, we have

$$A=\begin{pmatrix}\lambda I & 0\\ 0 & 0\end{pmatrix}.$$

Since

$$(\frac{1}{\lambda}I)A=A(\frac{1}{\lambda}I)=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}\in\mathscr{P},$$

we have

$$\varphi(\frac{1}{\lambda}I)\circ\varphi(A)=\frac{1}{2}(\frac{1}{\lambda}\varphi(I)\varphi(A)+\frac{1}{\lambda}\varphi(A)\varphi(I))=\frac{1}{\lambda}\varphi(I)e\otimes e\in\mathscr{P}.$$

We thus have $\varphi(I)e=\lambda e$ for every unit vector $e\in\mathcal{H}$. So $\varphi(I)=\lambda I$ for some constant $\lambda\in\mathbb{R}\backslash\{0\}$ and then $\lambda=1$ or $\lambda=-1$ by $\varphi(I)^2=I$.

We may replace φ by $-\varphi$ if $\varphi(I)=-I$. Without loss of generality, we next assume that $\varphi(I)=I$. Then φ preserves nonzero projections and thus preserves the order as well as the orthogonality of projections.

Step 3. φ preserves rank-1 projections as well as the orthogonality of rank-1 projections in both directions.

As in Step 2, set $\varphi(A)=e\otimes e$ for any unit vector $e\in\mathcal{H}$. Then we know that A is a nonzero projection. If $\operatorname{rank}A\geq 2$, then there exists two unit vectors $f_1,f_2\in\mathcal{H}$ with $f_1\perp f_2$ such that $A\geq f_1\otimes f_1$ and $A\geq f_2\otimes f_2$ and then

$$e\otimes e\geq\varphi(f_1\otimes f_1),e\otimes e\geq\varphi(f_2\otimes f_2).$$

On the other hand, $\varphi(f_1\otimes f_1)\perp\varphi(f_2\otimes f_2)$. This is a contradiction. Thus $\operatorname{rank}A=1$.

On the other hand, let $P=\varphi(f\otimes f)$ for any unit vector $f\in\mathcal{H}$. It is clear that $P\in\mathscr{P}$. We prove that P is also of rank-1 . In fact, if $\operatorname{rank}P\geq 2$, put $P=P_1+P_2$, where P_1 is a rank-1 projection and $P_1\perp P_2$, then $P_2\in\mathscr{P}$ and there exists a unit vector $f_1\in\mathcal{H}$ such that $\varphi(f_1\otimes f_1)=P_1$. We know that $\varphi^{-1}(P)=\varphi^{-1}(P_1)+\varphi^{-1}(P_2)$. So $\varphi^{-1}(P_2)=f\otimes f-f_1\otimes f_1$. If $\sigma(f\otimes f-f_1\otimes f_1)$ contains two different real scalars, say, λ_1 and λ_2, then $f\otimes f-f_1\otimes f_1=\lambda_1e_1\otimes e_1+\lambda_2e_2\otimes e_2$, where e_1,e_2 are two orthogonal unit vectors. Since $(\frac{1}{\lambda_i}e_i\otimes e_i)\circ(f\otimes f-f_1\otimes f_1)=e_i\otimes e_i\in\mathscr{P}$, we get $\varphi(\frac{1}{\lambda_i}e_i\otimes e_i)\circ\varphi(f\otimes f-f_1\otimes f_1)\in\mathscr{P}$ for $i=1,2$. It follows that $\frac{1}{\lambda_i}\varphi(e_i\otimes e_i)P_2\in\mathscr{P}$ from (1). We thus have that $\lambda_i=1$ for $i=1,2$. This is a contradiction. Hence $\operatorname{rank}P=1$. This shows that φ preserves rank-1 projections as well as orthogonality.

We next show that φ^{-1} preserves the orthogonality of rank-1 projections as well. Take any two orthogonal unit vectors $e_1,e_2\in\mathcal{H}$. Then there exists two unit vectors $f_1,f_2\in\mathcal{H}$ such that $\varphi(f_1\otimes f_1)=e_1\otimes e_1$ and $\varphi(f_2\otimes f_2)=e_2\otimes e_2$. It is clear that f_1 and f_2 are linearly independent. Note that $f_1\otimes f_1+f_2\otimes f_2=\alpha_1x_1\otimes x_1+\alpha_2x_2\otimes x_2$ for two orthogonal unit vectors $x_i\in\mathcal{H}$ and some nonzero constants $\alpha_i\in\mathbb{R}\setminus\{0\}$ for $i=1,2$. We know that there exist two unit vectors $y_i\in\mathcal{H}$ such that $\varphi(x_i\otimes x_i)=y_i\otimes y_i$ $(i=1,2)$ from the above proof. Note that $(\frac{1}{\alpha_i}x_i\otimes x_i)(f_1\otimes f_1+f_2\otimes f_2)=(f_1\otimes f_1+f_2\otimes f_2)(\frac{1}{\alpha_i}x_i\otimes x_i)=x_i\otimes x_i\in\mathscr{P}$ for $i=1,2$. We have $\frac{1}{\alpha_i}y_i \otimes y_i(e_1 \otimes e_1 + e_2 \otimes e_2) \in \mathscr{P}$ by (1) for $i = 1, 2$. We must have $\alpha_1 = \alpha_2 = 1$. It follows that

$$f_1 \otimes f_1 + f_2 \otimes f_2 = x_1 \otimes x_1 + x_2 \otimes x_2$$

is a projection. It is elementary that the sum of two projections is also a projection if and only if they are orthogonal. Thus f_1 and f_2 are orthogonal.

Step 4. There are a unitary or an anti-unitary operator U such that $\varphi(E) = UEU^*$ for any rank-1 projection E.

In fact, we have that φ is a bijection on the set of all rank-1 projections and preserves orthogonality in both directions. If $\dim \mathcal{H} \geq 3$, then it follows from the Uhlhorn's theorem in [8] that there is a unitary or anti-unitary operator U on $\mathcal{H}$ such that $\varphi(E) = UEU^*$ for any rank-1 projection E.

Next we assume that $\dim \mathcal{H} = 2$. Let E_1 and E_2 be two orthogonal rank-1 projections. Then so are $\varphi(E_1)$ and $\varphi(E_2)$. Without loss of generality, we may assume that $\varphi(E_i) = E_i$ for $i = 1, 2$. Let $E \neq E_1$ be a rank-1 projection such that $EE_1 \neq 0$. Then $E = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & z \\ \overline{z} & |z|^2 \end{pmatrix}$ for some nonzero complex constant $z \in \mathbb{C}$ in terms of the decomposition $\mathcal{H} = E_1\mathcal{H} \oplus E_1^{\perp}\mathcal{H}$. Let $A = \begin{pmatrix} 0 & \frac{z}{|z|} \\ \frac{\overline{z}}{|z|} & 0 \end{pmatrix}$. We claim that $\varphi(A) = \begin{pmatrix} 0 & w \\ \overline{w} & 0 \end{pmatrix}$ for some $w \in \mathbb{C}$. Put $\varphi(A) = \begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & a_{22} \end{pmatrix}$. It is easy to see that $\varphi(A)^2 = I$ since $A^2 = I$. Hence we have $a_{11}^2 + |a_{12}|^2 = a_{22}^2 + |a_{12}|^2 = 1$ and $a_{11}a_{12} + a_{12}a_{22} = 0$. On the other hand, $\begin{pmatrix} 1 & \frac{z}{|z|} \\ \frac{\overline{z}}{|z|} & -1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I$. Then

$$\begin{aligned} &\varphi(\begin{pmatrix} 1 & \frac{z}{|z|} \\ \frac{\overline{z}}{|z|} & -1 \end{pmatrix}) \circ \varphi(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \\ &= \frac{1}{2}((E_1 - E_2 + \varphi(A))(E_1 - E_2) + (E_1 - E_2)(E_1 - E_2 + \varphi(A)) \\ &= (E_1 - E_2)^2 + \frac{1}{2}(\varphi(A)(E_1 - E_2) + (E_1 - E_2)\varphi(A)) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \in \mathscr{P}. \end{aligned}$$

Therefore, $\begin{pmatrix} -a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$ is a projection with rank less than 1. If either $-a_{11}$ or a_{22} is 1, then we have $a_{12} = 0$ and $\varphi(A)$ is of rank-1. This is a contradiction. Thus $a_{11} = a_{22} = 0$. Again we have

$$E = \frac{1}{1+|z|^2}E_1 + \frac{|z|^2}{1+|z|^2}E_2 + \frac{|z|}{1+|z|^2}A$$

and

$$\varphi(E) = \frac{1}{1+|z|^2}E_1 + \frac{|z|^2}{1+|z|^2}E_2 + \frac{|z|}{1+|z|^2}\varphi(A).$$

An elementary calculation shows that $\mathrm{tr}(EE_1) = \frac{1}{1+|z|^2} = \mathrm{tr}(\varphi(E)\varphi(E_1))$, where tr is the trace of matrices. It follows from the Winger's theorem in [10] that there is a unitary or anti-unitary operator U on $\mathcal{H}$ such that

$$\varphi(E) = UEU^*$$

for any rank-1 projection E.

We now complete the proof. Without loss of generality, we may assume that $\varphi(E) = E$ for every rank-1 projection E. Otherwise, we consider a map

$$\psi(A) = U^*\varphi(A)U = A, \quad \forall A \in \mathcal{B}_s(\mathcal{H}).$$

Then ψ preserves nonzero projections of Jordan products such that $\psi(E) = E$ for every rank-1 projection E. In this case, we have $\varphi(E) = E$ for every finite rank projection E. Let P be an infinite rank projection. Then we have

$$P = \sup\{E : E \leq P, E \text{ is a finite rank projection}\}.$$

Note that φ preserves the order of projections. It follows that

$$\varphi(P) \geq \sup\{F : F \leq P, F \text{ is a finite rank projection}\} = P.$$

We similarly have $\varphi(I - P) = I - \varphi(P) \geq I - P$. Hence $\varphi(P) = P$ for every projection P. It now follows that $\varphi(X) = X$ for any $X \in \mathcal{B}_s(\mathcal{H})$ since X is a real linear combination of eight projections from Theorem 3 in [9] and φ is linear. The proof is completed. $\square$

Acknowledgment The authors would like to thank the referees for the valuable comments and suggestions.

References

[1] Guoxing JI, Yaling GAO. *Maps preserving operator pairs whose products are projections*. Linear Algebra Appl., 2010, **433**(7): 1348–1364.

[2] Meili WANG, Li FANG, Guoxing JI. *Linear maps preserving idempotency of products or triple Jordan products of operators*. Linear Algebra Appl., 2008, **429**(1): 181–189.

[3] L. MOLNÁR, P. ŠEMRL. *Nonlinear commutativity preserving maps on self-adjoint operators*. Q. J. Math., 2005, **56**(4): 589–595.

[4] Jinchuan HOU, Liankuo ZHAO. *Zero-product preserving additive maps on symmetric operator spaces and self-adjoint operator spaces*. Linear Algebra Appl., 2005, **399**: 235–244.

[5] Huanhuan SHI, Guoxing JI. *Maps preserving norms of Jordan products on the space of self-adjoint operators*. J. Shandong Univ. Nat. Sci., 2010, **45**(3): 80–84. (in Chinese)

[6] M. A. CHEBOTAR, W. F. KE, P. H. LEE. *Maps preserving zero Jordan products on Hermitian operators*. Illinois J. Math., 2005, **49**(2): 445–452.

[7] Guoxing JI, Fanlian QU. *Linear maps preserving projections of products of operators*. Acta Math. Sinica (Chin. Ser.), 2010, **53**(2): 315–322. (in Chinese)

[8] U. UHLHORN. *Representation of symmetry transformations in quantum mechanics*. Ark Fysik., 1963, **23**: 307–340.

[9] C. PEARCY, D. TOPPING. *Sums of small numbers of idempotents*. Michigan Math. J., 1967, **14**: 453–465.

[10] L. MOLNÁR. *Maps on states preserving the relative entropy*. J. Math. Phys., 2008, **49**(3): 032114, 4 pp.