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On Products of Property b_1

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Abstract In this note, we present that: (1) Let $X=\sigma\{X_\alpha : \alpha \in A\}$ be $|A|$-paracompact (resp., hereditarily $|A|$-paracompact). If every finite subproduct of $\{X\alpha : \alpha \in A\}$ has property b_1 (resp., hereditarily property b_1), then so is X. (2) Let X be a P-space and Y a metric space. Then, $X \times Y$ has property b_1 iff X has property b_1. (3) Let X be a strongly zero-dimensional and compact space. Then, $X \times Y$ has property b_1 iff Y has property b_1.

Keywords σ-product; Tychonoff products; property b_1; hereditarily property b_1.

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1. Introduction

It is well known that θ-refinableness [2] $\rightleftarrows\!\!\!\!\!/$ property b_1 (see [1]) $\rightleftarrows\!\!\!\!\!/$ weak θ-refinableness [2]. And in recent years, a great advance has been achieved with the results of σ-products and Tychonoff products of topological spaces characterized by coverings. Particularly, the papers [3, 4, 11–13] were published such that the properties about both θ-refinable and weak θ-refinable spaces had been acquired. But no results on both σ-product and Tychonoff products of property b_1 has ever been seen since the structure of topological spaces with property b_1 is more complex than any one of θ-refinable spaces and weak θ-refinable spaces.

In this paper, the σ-product of property b_1 and hereditarily property b_1 are firstly investigated. Next, some characterizations on products of two spaces with property b_1 are proved.

Throughout this paper, X and X_α ($\alpha \in A$) denote topological spaces (referred to as a space or spaces); ω and κ denote respectively the first infinite ordinal and an arbitrary infinite cardinal. For a set Y, we denote by $\mathcal{P}(Y)$ the collection of all subsets of Y. For $A \in \mathcal{P}(Y)$, $\mathcal{N}(A)$ (resp., $\mathrm{cl}\,A, \mathrm{int}\,A, |A|$) denotes the open neighborhood system (resp., the closure, the interior, the cardinality) of A. For every $\mathcal{U} \in \mathcal{P}(Y)$ and $A \subset Y$, define $(\mathcal{U})_A = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$, $\mathcal{U}|_A = \{U \cap A : U \in \mathcal{U}\}$, $\mathrm{Cl}\mathcal{U} = \{\mathrm{cl}\,U : U \in \mathcal{U}\}$. Let Ω be a set, and Ω^n $(resp., \Omega^\omega)$ denote the Cartesian products of n orders (resp., the family of all infinite sequences of elements) of Ω. Let $\Omega^{<\omega} = \cup_{<\omega}\Omega^n$, $[\Omega]^{<\omega} = \{s \in \mathcal{P}(\Omega) : |s| < \omega\}$, $s|_n = (s_0, s_1, \ldots, s_{n-1})$ if $s \in \Omega^\omega$ and $s \oplus a = (s_0, \ldots, s_{n-1}, a)$ if $s \in \Omega^n$. All spaces are Hausdorff spaces which contain at least two

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points in this paper.

Definition 1.1 ([5]) *Let $s = (s_\alpha)_{\alpha\in A}$ be a fixed point in Tychonoff product $\prod_{\alpha\in A}\{X_\alpha : \alpha \in A\}$. For each $x = (x_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A}\{X_\alpha : \alpha \in A\}$, put $Q(x) = \{\alpha \in A : x_\alpha \neq s_\alpha\}$ and define $\sigma\{X_\alpha : \alpha \in A\} = \{x \in (x_\alpha)_{\alpha\in A} : |Q(x)| < \omega\}$. We call $\sigma\{X_\alpha : \alpha \in A\}$ the σ-product of $\{X_\alpha : \alpha \in A\}$ and s the base point of it. And for every $a \in [A]^{<\omega}$, $\prod_{\alpha\in a} X_\alpha \times \{\{s_\alpha\} : \alpha \in A\backslash a\}$ is called a finite subproduct of $\sigma\{X_\alpha : \alpha \in A\}$.*

Definition 1.2 ([1]) *A space X is said to have property b_1 if each cover $\mathcal{U}$ of X can be refined by a cover $\cup_{n\in\omega}\mathcal{H}_n$ such that, $\mathcal{H}_n$ is a locally finite collection of closed sets in $X\backslash \cup_{i<n} (\cup\mathcal{H}_n)$.*

Definition 1.3 ([7]) *A space is called a P-space if for every set Ω and open family $\{G(s) : s \in \cup_{n<\omega}\Omega^n\}$ such that $G(s) \subset G(s\oplus\alpha)$ for $s \in \cup_{n<\omega}\Omega^n$ and $\alpha \in \Omega$, there is a closed set $K(s) \subset G(s)$ for $s \in \cup_{n<\omega}\Omega^n$ such that $\cup_{n<\omega}K(s\,|n) = X$ whenever $\cup_{n<\omega}G(s\,|n) = X$ for $s \in \Omega^\omega$.*

Lemma 1.4 ([6]) *Let X be a space and κ an infinite cardinal. Then the following are equivalent:*

(a) X is κ-paracompact.

(b) X is countably paracompact and every open cover of X with cardinality $\leq \kappa$ has a σ-locally finite open refinement.

Lemma 1.5 ([10]) *Let Y be a metric space. Then there exists a base $\mathcal{L}=\cup_{n<\omega}\mathcal{L}_n$ of Y satisfying the conditions:*

(a) $\mathcal{L}_n = \{L(s)\,|s \in \Omega^n\}$ is a locally finite open cover of Y.

(b) $L(s) = \cup\{L(s \oplus \alpha) : \alpha \in \Omega\}$ for each $s \in \Omega^n$ and $\alpha \in \Omega$.

(c) For each $y \in Y$, there is an $s \in \Omega^n$ such that $\{L(s\,|n) : n < \omega\}$ is a local base of y in Y.

Lemma 1.6 ([9]) *A non-empty normal space X is strongly zero-dimensional iff every open cover $\{U_i\}_i^k$ of the space X has finite open refinement $\{V_i\}_i^m$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$.*

2. σ-product

Now, we shall show our main theorems. The proof of the following lemma is routine and hence, we omit it.

Lemma 2.1 *Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of space X. Then the following are equivalent.*

(1) There is a refinement $\cup_{n\in\omega}\mathcal{H}_n$ of $\mathcal{U}$ such that each $\mathcal{H}_n$ is a locally finite collection of closed sets in $X\backslash \cup_{i<n} (\cup\mathcal{H}_i)$.

(2) There is a refinement $\cup_{n\in\omega}\mathcal{V}_n$ of $\mathcal{U}$, where $\mathcal{V}_n=\{V_{n\lambda} : \lambda \in \Lambda\}$ for each $n \in \omega$, such that $\mathcal{V}_n$ is a locally finite collection of closed sets in $X\backslash \cup_{i<n} (\cup\mathcal{V}_n)$, and $V_{n\lambda} \subset U_\lambda$ for each $\lambda \in \Lambda$.

(3) There is a refinement $\cup_{n\in\omega}\mathcal{V}_n$ of $\mathcal{U}$ such that each $\mathcal{V}_n|_{X\backslash\cup_{i<n}(\cup\mathcal{V}_n)}$ is a locally finite collection of closed sets in $X\backslash \cup_{i<n} (\cup\mathcal{V}_n)$.

Jiang [8] proved that every F_σ-subspace of space Y has property b_1 if Y has property b_1. So every closed subspace of space Y has property b_1. Now we have the following theorem.

Theorem 2.2 *Let X=$\sigma\{X\alpha : \alpha \in A\}$. Suppose X is $|A|$-paracompact and every finite subproduct of $\{X\alpha : \alpha \in A\}$ has property b_1. Then X has property b_1.*

Proof When $|A| < \omega$, clearly X has property b_1 since $X = \prod_{s\in A} X_s$. Without loss of generality, let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of X and $A^* = [A]^{<\omega}$. For each $a \in A^*$ and each $\xi \in \Xi$, let us put $R_{a\xi}$=$\cup\{R : R$ is open in $Y_a, p_a^{-1}(R) \subset G_\xi\}$ and $R_a = \cup\{R_{a\xi} : \xi \in \Xi\}$, where $p_a : X \to Y_a$ is defined as follows: for $x = (x)_{\alpha\in A}$, $(P_a(x))_\alpha = \begin{cases} x_\alpha, & \alpha \in a \\ s_\alpha, & \alpha \in A - \{a\} \end{cases}$. Then it is easily seen that

(i) $\{p_a^{-1}(R_a) : a \in A^*\}$ is an open cover of X and $p_a^{-1}(R_a) \subset p_b^{-1}(R_b)$ if $a \subset b$.

Since X is $|A|$-paracompact, there exists a locally finite family $\{E_a : a \in A^*\}$ of open sets in X such that $E_a \subset p_a^{-1}(R_a)$ for each $a \in A^*$. Put $\Lambda = [A^*]^{<\omega}$.

Put D_λ=$X - \cup\{\mathrm{cl}\, E_a : a \in A^* - \lambda\}$ for each $\lambda \in \Lambda$. Then

(ii) $\{D_\lambda : \lambda \in \Lambda\}$ is an open cover of X, and $\mathrm{cl}\, D_\lambda \subset p_{a_\lambda}^{-1}(R_{a_\lambda})$ with $a_\lambda = \cup\{a : a \in \lambda\}$.

Let $Z_{a_\lambda} = Y_{a_\lambda} - p_{a_\lambda}(X - \mathrm{cl}\, D_\lambda)$ for each $\lambda \in \Lambda$. Here notice that each Z_{a_λ} is closed in Y_{a_λ} and hence has property b_1. Again let $T_\lambda = \mathrm{int}\, p_{a_\lambda}^{-1}(Z_{a_\lambda})$ for $\lambda \in \Lambda$. Then

(iii) $\cup\{T_\lambda : \lambda \in \Lambda\} = X$.

In fact, let $x \in X$. Then $x \in D_\lambda$ for some $\lambda \in \Lambda$ and there exists an $a \in A^*$ and an open set R in Y_a such that $x \in p_a^{-1}(R) \subset D_\lambda$. Moreover, $p_a^{-1}(R) \subset p_{a_\mu}^{-1}(Z_{a_\mu})$ where $\mu = \lambda \cup \{a\}$. To show this, assume that $y \in P_a^{-1}(R) - p_{a_\mu}^{-1}(Z_{a_\mu})$. Since $y_{a_\mu} \notin Z_{a_\mu}$, $y_{a_\mu} \in p_{a_\mu}(X - \mathrm{cl}\, D_\mu)$ and so $y_{a_\mu} = p_{a_\mu}(z)$ for some $z \in X - \mathrm{cl}\, D_\mu$. Then $p_a(z)$=$y_a \in R$. Furthermore, $z \in p_a^{-1}(R) \subset \mathrm{cl}\, D_\mu$. Thus $z \in \mathrm{cl}\, D_\mu$ which is a contradiction. The proof of (iii) is completed.

By (iii), it follows from Lemma 1.4 that there is a refinement $\cup_{m<\omega}\mathcal{K}_m$, where we may express $\mathcal{K}_m = \{K_{m,\lambda} : \lambda \in \Lambda\}$, such that $\mathrm{CL}\, \mathcal{K}_m$ is locally finite in X and $\mathrm{cl}\, K_{m,\lambda} \subset p_{a_\lambda}^{-1}(Z_{a_\lambda})$ for each $\lambda \in \Lambda$. Since $Z_{a_\lambda} \subset \cup\{R_{a_\lambda\xi} : \xi \in \Xi\}$ for $\lambda \in \Lambda$, there is a refinement $\cup_{n<\omega}\mathcal{W}_{n,a_\lambda}$ of $\{R_{a_\lambda\xi} : \xi \in \Xi\}$ such that

(iv) Each $\mathcal{W}_{n,a_\lambda}$=$\{W_{n,a_\lambda,\xi} : \xi \in \Xi\}$ is a locally finite collection of closed sets in $Z_{a_\lambda} \setminus \cup_{i<n} (\cup\mathcal{W}_{i,a_\lambda})$.

For each $m, n \in \omega$, put $\mathcal{H}_{mn} = \{p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi}) \cap \mathrm{cl}\, K_{m,\lambda} : \xi \in \Xi, \lambda \in \Lambda\}$. Then it is easy to check that $\mathcal{H} = \cup_{m,n\in\omega}\mathcal{H}_{mn}$ refines $\mathcal{G}$. Define a map $f : \omega \times \omega \to \omega$, where $f(m,n) = \frac{1}{2}[(m+n)^2 + n + 3m]$. Then f is an injection and onto map. Let $\mathcal{H}_k = \mathcal{H}_{mn}$ with $k = f(m,n)$. And put $H_k = X \setminus \cup_{i<k} (\cup\mathcal{H}_i), B_{na_\lambda} = Z_{a_\lambda} \setminus \cup_{j<n} (\cup\mathcal{W}_{j,a_\lambda})$.

Claim 1 $\mathrm{cl}\, K_{m\lambda} \cap H_k \subset p_{a_\lambda}^{-1}(B_{na_\lambda})$ for each $\lambda \in \Lambda$ with k=$f(m,n)$.

Assume that there is an $x \in \mathrm{cl}\, K_{m\lambda} \cap H_k - p_{a_\lambda}^{-1}(B_{na_\lambda})$. Then $x_{a_\lambda} \in Z_{a_\lambda}$ and $x \in p_{a_\lambda}^{-1}(W_{j_o,a_\lambda,\xi_o})$ for some $j_o < n$ and some $\xi_o \in \Xi$. Put $t = f(m, j_o) < k$. Then $x \in \cup\mathcal{H}_t$. This contradicts $x \in H_k$ and completes the proof of Claim 1.

Claim 2 Each $\mathcal{H}_k$ is locally finite in H_k with $k = f(m,n)$.

Let $x \in H_k$. Since $\{\mathrm{cl}\, K_{m,\lambda} : \lambda \in \Lambda\}$ is locally finite in X, take a $U \in \mathcal{N}(x)$ such that

$(\mathrm{CL}\,\mathcal{K}_{m,\lambda})_{U\cap H_k}$ is finite, say $\{\lambda\in\Lambda:\mathrm{cl}\,K_{m,\lambda}\cap U\cap H_k\neq\emptyset\}=\Delta$ for some $\Delta\subset\Lambda$. Let $(\lambda,\xi)\in\Lambda\times\Xi$. We can consider the following two cases.

Case 1 If $x\notin\cup_{\mu\in\Delta}\mathrm{cl}\,K_{m,\mu}$, it follows that $V=U\cap\cap_{\mu\in\Delta}(X\backslash\mathrm{cl}\,K_{m,\mu})\cap H_k$ is an open neighborhood of x in H_k. Then it is easy to check $V\cap p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi})\cap\mathrm{cl}\,K_{m,\lambda}=\emptyset$ for each $\lambda\in\Lambda$.

Case 2 If $x\in\cup_{\mu\in\Delta}\mathrm{cl}\,K_{m,\mu}$, there exists a finite set $\Delta_0\subset\Delta$ such that $x\in\mathrm{cl}\,K_{m,\mu}$ for each $\mu\in\Delta_0$. Since $x_{a_\mu}\in B_{na_\mu}$ by Claim 1, Ξ_μ=$\{\xi\in\Xi:O(\mu)\cap B_{na_\mu}\cap W_{n,a_\mu,\xi}\neq\emptyset\}$ is finite for some $O(\mu)\in\mathcal{N}(x_{a_\mu})$ by (v). Put

$$V=U\cap\cap_{\mu\in\Delta\backslash\Delta_0}(X\backslash\mathrm{cl}\,K_{m\mu})\cap\cap_{\mu\in\Delta_0}p_{a_\mu}^{-1}(O(\mu))\cap H_k.$$

Then, $V\cap p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi})\cap\mathrm{cl}\,K_{m,\lambda}\subset\cap_{\mu\in\Delta_0}p_{a_\mu}^{-1}(O(\mu))\cap\mathrm{cl}\,K_{m\lambda}\cap H_k\cap p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi})\subset p_{a_\mu}^{-1}(O(\mu)\cap W_{n,a_\mu,\xi}\cap B_{na_\mu})=\emptyset$ for all $(\lambda,\xi)\in\Delta_0\times(\Xi\backslash\Xi_\mu)$; if $\lambda\in\Lambda\backslash\Delta_0$, then $V\cap p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi})\cap\mathrm{cl}\,K_{m,\lambda}=\emptyset$ for each $\xi\in\Xi$. Hence the proof of Claim 2 is completed.

Finally, put $\mathcal{F}_k=\mathcal{H}_k\,|_{H_k}$ for each $k\in\omega$ with $k=f(m,n)$.

Claim 3 $\mathcal{F}_k$ is a collection of closed sets in H_k.

Pick a $W\in\mathcal{F}_k$. By Claim 1, $W=p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi})\cap\mathrm{cl}\,K_{m,\lambda}\cap H_k=p_{a_\lambda}^{-1}(F)\cap p_{a_\lambda}^{-1}(B_{na_\lambda})\cap\mathrm{cl}\,K_{m,\lambda}=p_{a_\lambda}^{-1}(F)\cap\mathrm{cl}\,K_{m,\lambda}\cap H_k$ for some $\lambda\in\Lambda$ and some $\xi\in\Xi$, where F is a closed subset in $Y_{a\lambda}$. Then W is closed in H_k.

The family $\cup_{k\in\omega}\mathcal{H}_k$ satisfies the conditions in Lemma 2.1 by Claims 1 and 3. Hence, X has property b_1. □

Recall that a space X is called hereditarily property b_1 if every subspace of X has property b_1. And we can easily check the following by Lemma 2.1.

Lemma 2.3 *A space X is said to have hereditarily property b_1 iff every open subspace of X has property b_1.*

Similarly, we have σ-products for hereditarily property b_1.

Theorem 2.4 *Let X=$\sigma\{X\alpha:\alpha\in A\}$. Suppose X is hereditarily $|A|$-paracompact and every finite subproduct of $\{X\alpha:\alpha\in A\}$ has hereditarily property b_1. Then X has hereditarily property b_1.*

Proof Let G be an arbitrary open subspace of X. Assume that $\mathcal{G}=\{G_\xi:\xi\in\Xi\}$ is an open cover of G and $A^*=[A]^{<\omega}$. For each $a\in A^*$ and $\xi\in\Xi$, put $R_{a\xi}=\cup\{R:R$ is open in $Y_a,p_a^{-1}(R)\cap G\subset G_\xi\}$ and $R_a=\cup\{R_{a\xi}:\xi\in\Xi\}$. Then

(1) $\{p_a^{-1}(R_a):a\in A^*\}$ is an open cover of G and $p_a^{-1}(R_a)\subset p_b^{-1}(R_a)$ if $a\subset b$.

Since X is hereditarily $|A|$-paracompact, there is a locally finite family $\{E_a:a\in A^*\}$ of open sets in G such that $\mathrm{cl}_G E_a\subset p_a^{-1}(R_a)$ for each $a\in A^*$. Put $\Lambda=[A^*]^{<\omega}$. For each $\lambda\in\Lambda$, let us put $D_\lambda=G-\cup\{\mathrm{cl}_G E_a:a\in A^*-\lambda\}$ and put $a_\lambda=\cup\{a:a\in\lambda\}$. Then it is easily seen that

(2) $\{D_\lambda:\lambda\in\Lambda\}$ is an open cover of G and $\mathrm{cl}_G D_\lambda\subset p_{a_\lambda}^{-1}(R_{a_\lambda})$ for each $\lambda\in\Lambda$.

For each $\lambda\in\Lambda$, let us put $O_\lambda=\cup\{O:O$ is open in G_{a_λ} and $p_{a_\lambda}^{-1}(O)\subset D_\lambda\}$. Then

(3) $p_{a_\lambda}^{-1}(O_\lambda)\subset D_\lambda\subset p_{a_\lambda}^{-1}(R_{a_\lambda})$ for each $\lambda\in\Lambda$ and $p_{a_\lambda}^{-1}(O_\lambda)\subset p_{a_\mu}^{-1}(O_\mu)$ if $\lambda\subset\mu$.

It can be easily checked that $\cup\{p_{a_\lambda}^{-1}(O_\lambda) : \lambda \in \Lambda\} = G$. Put $F_\lambda = \mathrm{cl}\, O_\lambda \cap (\mathrm{cl}\, R_{a_\lambda} - R_{a_\lambda})$. Then, we have the following.

(5) $p_{a_\lambda}^{-1}(F_\lambda) \cap G = \emptyset$.

To show this, assume the contrary. Take an $x \in p_{a_\lambda}^{-1}(F_\lambda) \cap G$. Then $x_{a_\lambda} \in F_\lambda \subset \mathrm{cl}\, R_{a_\lambda} - R_{a_\lambda}$, $x \notin \mathrm{cl}_G p_{a_\lambda}^{-1}(O_\lambda)$ since $\mathrm{cl}_G p_{a_\lambda}^{-1}(O_\lambda) \subset p_{a_\lambda}^{-1}(R_{a_\lambda})$. Next, we have $x \in \mathrm{cl}_G p_{a_\lambda}^{-1}(O_\lambda)$. To observe this, take an $H \in \mathcal{N}_G(x)$. Then there are some $a \in A^*$ and some open set W_a in Y_a such that $x \in p_a^{-1}(W_a) \subset H$. Let $\mu = \lambda \cup \{a\}$. Put $K = (p_a^{a_\mu})^{-1}(W_a)$. Then $x \in p_{a_\mu}^{-1}(K) = (p_a^{a_\mu} p_{a_\mu})^{-1}(W_a) \subset H$. Since $x_{a_\lambda} \in F_\lambda \subset \mathrm{cl}\, O_\lambda$, $O_\lambda \cap p_a^{a_\mu}(K) \neq \emptyset$. Pick a $z \in O_\lambda \cap p_a^{a_\mu}(K)$. Then $z = p_a^{a_\mu}(b)$ for some $b \in K$. There is a $y \in X$ such that $p_{a_\mu}(y) = b$. i.e., $y_{a_\lambda} = p_a^{a_\mu} p_{a_\mu}(y) = b \in O_\lambda$. Moreover, $y \in p_{a_\mu}^{-1}(K) \cap p_{a_\lambda}^{-1}(O_\lambda) \subset H \cap p_{a_\lambda}^{-1}(O_\lambda) \neq \emptyset$. Thus, $x \in \mathrm{cl}_G p_{a_\lambda}^{-1}(O_\lambda)$. This is a contradiction. Then (5) is true.

We can easily see that $R_{a_\lambda} \cap O_\lambda = \mathrm{cl}\, O_\lambda - F_\lambda \subset \cup_{\xi\in\Xi} R_{a_\lambda \xi}$ for each $\lambda \in \Lambda$. By property b_1 space of $R_{a_\lambda} \cap O_\lambda$, there is a cover $\cup_{n<\omega} \mathcal{W}_{n,a_\lambda}$ of $\{R_{a_\lambda \xi} : \xi \in \Xi\}$ such that each $\mathcal{W}_{n,a_\lambda} = \{W_{n,a_\lambda,\xi} : \xi \in \Xi\}$ is a locally finite collection of closed sets in $R_{a_\lambda} \cap O_\lambda \setminus \cup_{i<n} (\cup \mathcal{W}_{i,a_\lambda})$.

For each $m, n \in \omega$, put $\mathcal{H}_{mn} = \{p_{a_\lambda}^{-1}(W_{n,a_\lambda,\xi}) \cap \mathrm{cl}\, K_{m,\lambda} : \xi \in \Xi, \lambda \in \Lambda\}$. Then it is easy to check that $\mathcal{H} = \cup_{m,n\in\omega} \mathcal{H}_{mn}$ refines $\mathcal{G}$. Define a map $f : \omega \times \omega \to \omega$, where $f(m,n) = \frac{1}{2}[(m+n)^2 + n + 3m]$. Then f is an injection and onto map. Let $\mathcal{H}_k = \mathcal{H}_{mn}$ with $k = f(m,n)$. And put $H_k = G \setminus \cup_{i<k} (\cup \mathcal{H}_i)$, $B_{na_\lambda} = R_{a_\lambda} \cap O_\lambda \setminus \cup_{j<n} (\cup \mathcal{W}_{j,a_\lambda})$.

Then, we assert that

(6) Each $\mathcal{H}_k$ is locally finite in H_k;

(7) $\mathcal{H}_k |_{H_k}$ is a collection of closed sets in H_k.

The proofs of (6), (7) are quite similar to Claims 1–3 of Theorem 2.2. So, the details are omitted.

Hence, G has property b_1. Furthermore, X has hereditarily property b_1. $\square$

3. Tychonoff products

It is well known that most covering properties poorly maintain its products. Now we prove the following theorems.

Theorem 3.1 *Let X be a P-space and Y a metric space. Then, $X \times Y$ has property b_1 iff X has property b_1.*

Proof $\Leftarrow$. Let $\mathcal{L} = \cup_{n<\omega} \mathcal{L}_n$ be a base of Y satisfying the conditions (a)–(c) in Lemma 1.5. Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of $X \times Y$ and $\Omega^* = \cup_{n<\omega} [\Omega]^n$. For each $s \in \Omega^*$ and $\xi \in \Xi$, put $L(s,\xi) = \cup\{E : E \text{ is open in } X, E \times L(s) \subset G_\xi\}$ and $E(s) = \cup\{E(s,\xi) : \xi \in \Xi\}$. Then $E(s) \subset E(s \oplus \alpha)$ for $\alpha \in \Omega$. Since X is a P-space, there is a closed set $F(s)$ in X such that

(1) $F(s) \subset E(s)$ for each $s \in \Omega^*$;

(2) $\cup_{n<\omega} F(s|n) = X$ whenever $\cup_{n<\omega} E(s|n) = X$ for $s \in \Omega^\omega$.

Since each $F(s)$ has property b_1, there exists an open cover $\mathcal{B}_s = \cup_{n<\omega} \mathcal{B}_{s,n}$, where $\mathcal{B}_{s,n} = \{B(s,n,\xi) : \xi \in \Xi\}$, such that

(3) Each $\mathcal{B}_{s,n}$ is a locally finite collection of closed sets in $F(s)\backslash\cup_{i<n}(\cup\mathcal{B}_{s,i})$, and $B(s,n,\xi)\subset E(s,n,\xi)$ for each $\xi\in\Xi$.

For each $m,n\in\omega$, put $\mathcal{D}_{m,n}=\{B(s,n,\xi)\times\mathrm{cl}\,L(s):s\in\Omega^m,\xi\in\Xi\}$. Then it is easily seen that

(4) $\mathcal{D}=\cup_{m,n\in\omega}\mathcal{D}_{m,n}$ is a refinement of $\mathcal{G}$.

Define a map $f:\omega\times\omega\to\omega$, where $f(m,n)=\frac{1}{2}[(m+n)^2+n+3m]$. Then, f is a bijection map. Put $\mathcal{H}_k=\mathcal{D}_{mn}$ when $k=f(m,n)$. Then, we prove that

(5) Each $\mathcal{H}_k$ is locally finite in $X\times Y\backslash\cup_{i<k}(\cup\mathcal{H}_i)$ with $k=f(m,n)$.

In fact, let $(x,y)\in X\times Y\backslash\cup_{i<k}(\cup\mathcal{H}_i)$. Since $\{\mathrm{cl}\,L(s):s\in\Omega^m\}$ is locally finite in Y, there exists $W\in N(y)$ such that $\Delta=\{s\in\Omega^m:W\cap\mathrm{cl}\,L(s)\neq\emptyset\}$ is a finite set. Let $(s,\xi)\in\Omega^m\times\Xi$. We can concern about the two cases.

(a) If $y\notin\cup_{t\in\Delta}\mathrm{cl}\,L(t)$, then $V=X\times(W\cap_{t\in\Delta}(Y\backslash\mathrm{cl}\,L(t)))\in N(y)$. It is easy to check $V\cap(B(s,n,\xi)\times\mathrm{cl}\,L(s))=\emptyset$ for each s $\in\Omega^m$.

(b) If $y\in\cup_{t\in\Delta}\mathrm{cl}\,L(t)$, there is a finite set $\Delta_0\subset\Delta$ such that $y\in\mathrm{cl}\,L(t)$ for each $t\in\Delta_0$. By (3), each $\mathcal{B}_{t,n}$ is a locally finite collection of closed sets in $X\backslash\cup_{i<n}(\cup\mathcal{B}_{t,i})$ since $F(t)$ is closed in X. Take some neighborhood U_t of x in $X\backslash\cup_{i<n}(\cup\mathcal{B}_{t,i})$ for $t\in\Delta_0$. Then $\Xi_t=\{\xi\in\Xi:U_t\cap B(t,n,\xi)\neq\emptyset\}$ is a nonempty finite set. Then it follows that $V=\cap_{t\in\Delta_0}U_t\times(W\cap_{s\in\Delta\backslash\Delta_0}(Y\backslash\mathrm{cl}\,L(s)))$ is a neighborhood of (x,y) in $X\times Y\backslash\cup_{i<k}(\cup\mathcal{H}_i)$. Moreover, $V\cap(B(s,n,\xi)\times\mathrm{cl}\,L(s))\subset(U_t\cap B(t,n,\xi))\times(W\cap\mathrm{cl}\,L(t))=\emptyset$ for $(s,\xi)\in\Delta_0\times(\Xi\backslash\Xi_t)$; if $s\in\Omega^m\backslash\Delta_0$, then $V\cap(B(s,n,\xi)\times\mathrm{cl}\,L(s))=\emptyset$ for each $\xi\in\Xi$.

Since $\cup_{i<n}((\cup_{\xi\in\Xi}B(s,i,\xi))\times\mathrm{cl}\,L(s))\subset\cup_{r<k}(\cup\mathcal{H}_r)$ whenever $i<n$ with $r=f(m,i)$ for each $s\in\Omega^m$, each $\mathcal{H}_k\big|_{X\times Y\backslash\cup_{i<k}(\cup\mathcal{H}_i)}$ is a closed family in $X\times Y\backslash\cup_{i<k}(\cup\mathcal{H}_i)$. Since $\mathcal{D}$ refines $\mathcal{G}$, by Lemma 2.1, X has property b_1.

$\Rightarrow$. The conclusion holds trivially since $X\times\{y\}$ is closed in $X\times Y$ for a fixed point $y\in Y$. $\Box$

Theorem 3.2 *Let X be a strongly zero-dimensional and compact space. Then, $X\times Y$ has property b_1 iff Y has property b_1.*

Proof Let $\mathcal{U}$ be an open cover of $X\times Y$. Take some $U(x,y)\in\mathcal{U}$ for each $(x,y)\in X\times Y$. There exist $S(x,y)\in\mathcal{N}(x)$ and $T(x,y)\in\mathcal{N}(y)$ such that $(x,y)\in S(x,y)\times T(x,y)\subset U(x,y)$. Since $X\times\{y\}$ is compact, there exists an open cover $\{S(x,y):x\in A(y)\}$ of X where $A(y)$ is a finite set. By Lemma 1.6, there is a disjoint collection $\{E(x,y):x\in A(y)\}$ such that

(i) $E(x,y)$ is open-and-closed in X and $E(x,y)\subset S(x,y)$ for each $x\in A(y)$;

Put $T(y)=\cap\{T(x,y):x\in A(y)\}$. Then

(ii) $X\times\{y\}\subset\cup\{E(x,y)\times T(y):x\in A(y)\}$ and $Y=\cup\{T(y):y\in Y\}$;

Since Y has property b_1, there is a cover $\cup_{n<\omega}\mathcal{B}_n$ of Y, where $\mathcal{B}_n=\{B_{n,\alpha}:\alpha\in\Lambda_n\}$, such that

(iii) $\mathcal{B}_n$ is a locally finite collection of closed sets in $Y\backslash\cup_{i<n}(\cup\mathcal{B}_i)$;

(iv) Each $B_{n,\alpha}\subset T(y_{n,\alpha})$ for some $y_{n,\alpha}\in Y$.

Put $\mathcal{H}_n=\{E(x,y_{n,\alpha})\times B_{n,\alpha}:x\in A(y_{n,\alpha}),\alpha\in\Omega_n\}$ for each $n\in\omega$. Then it is easy to check $\mathcal{H}=\cup_{n<\omega}\mathcal{H}_n$ refines $\mathcal{U}$. Put $(X\times Y)_n=X\times Y\backslash\cup_{i<n}(\cup\mathcal{H}_i)$ and $Y_n=Y\backslash\cup_{i<n}\mathcal{H}_i$. Then

$(X \times Y)_n = X \times Y_n$ since $\cup\mathcal{H}_n = X \times (\cup\mathcal{H}_n)$. By the similar way to that of Theorem 3.1 (5), we have each $\mathcal{H}_n$ is locally finite in $(X \times Y)_n$ and $\mathcal{H}_n \big|_{(X\times Y)_n}$ is a collection of closed subsets of $(X \times Y)_n$. Hence, $X \times Y$ has property b_1 since $\cup_{n\in\omega}\mathcal{H}_n$ satisfies the conditions in Lemma 2.1.

$\Rightarrow$. The conclusion holds trivially since $\{x\} \times Y$ is closed in $X \times Y$ for a fixed point $x \in X$. $\square$

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