Characterization of $L_2(16)$ by $\tau_e(L_2(16))$

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Abstract Let G be a group and $\pi_e(G)$ the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Let $\tau_e(G) = \{m_k | k \in \pi_e(G)\}$. In this paper, we prove that $L_2(16)$ is recognizable by $\tau_e(L_2(16))$. In other words, we prove that if G is a group such that $\tau_e(G) = \tau_e(L_2(16)) = \{1, 255, 272, 544, 1088, 1920\}$, then G is isomorphic to $L_2(16)$.

Keywords element orders; recognizable; number of elements; same order; Thompson problem.

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1. Introduction

Let n be an integer. We denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. Denote by $\pi_e(G)$ the set of element orders of G. And we use P_r and n_r to denote a Sylow r-subgroup and the number of Sylow r-subgroups of G, respectively. Let $k \in \pi_e(G)$. Then we denote by m_k the number of elements of order k in G. Let $\tau_e(G) = \{m_k | k \in \pi_e(G)\}$. In 1987, Thompson posed a very interesting problem related to algebraic number fields as follows (see [9] and Problem 12.37 of [6]).

Thompson Problem Let $\Gamma(G) = \{(n, S_n) | n \in \pi_e(G), S_n \in \tau_e(G)\}$, where S_n is the number of elements with order n. Suppose that $\Gamma(G) = \Gamma(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

So far, no one can solve this problem completely, even give a counterexample. We know that $\Gamma(G)$ consists of two sets, that is, $\pi_e(G)$ and $\tau_e(G)$. In 1986, the second author of this note studied the case of the simple group A_5, and he proved an interesting result using only $\pi_e(G)$, that is, a finite group G is isomorphic to A_5 if and only if $\pi_e(G) = \{1, 2, 3, 5\}$ (see [8]). Afterward, many simple groups are characterized using only the set of element orders and there are many relative papers. Of course, the following question is valuable. Consider the sizes of elements of same order but disregard the actual orders of elements in $\Gamma(G)$ of Thompson Problem. In other words, with only $\tau_e(G)$, whether can one characterize finite simple groups? Namely, suppose G is a finite simple group, whether can it be characterized using only the set $\tau_e(G)$?

We denote by $k(\tau_e(G))$ the number of isomorphism classes of finite groups H satisfying $\tau_e(G) = \tau_e(H)$. By using this function we pose the following definition:

Definition 1.1 *Given a natural number n, a finite group G is called n-recognizable by $\tau_e(G)$ if $k(\tau_e(G)) = n$. Usually a 1-recognizable group is called a recognizable group. If there exist infinitely many non-isomorphic finite groups H such that $\tau_e(G) = \tau_e(H)$, then we call G a non-recognizable group by $\tau_e(G)$.*

In [7], it was proved that A_5 is determined by $\tau_e(A_5)$. In [5], it was shown that if G is a group and $\tau_e(G) = \tau_e(PSL(2,q))$, where $q \in \{7,8,11,13\}$, then $G \cong PSL(2,q)$. In fact the authors of [7] and [5] proved that some simple groups can be determined by $\tau_e(G)$ when $|\tau_e(G)|$ is smaller than 6. Is it true that G can be characterized by $\tau_e(G)$ if G is a finite simple group and $|\tau_e(G)| \geq 6$? In this paper we continue this work and we show that $L_2(16)$ is recognizable by $\tau_e(L_2(16))$. And the main result is as follows:

Theorem *$L_2(16)$ is recognizable by $\tau_e(L_2(16))$. In other words, if G is a group such that $\tau_e(G) = \tau_e(L_2(16)) = \{1, 255, 272, 544, 1088, 1920\}$, then G is isomorphic to $L_2(16)$.*

Note that $|\tau_e(L_2(16))| = 6$. We find that this problem is more complicated when $|\tau_e(G)|$ is larger.

2. Preliminaries

Before starting the proof of theorem, we will mention a well-known result of Frobenius [3], which is quoted frequently in the sequel.

Lemma 2.1 *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m||L_m(G)|$.*

Lemma 2.2 ([7]) *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

From [1] we get the following Lemma.

Lemma 2.3 *Let G be a finite 2-group and $2^n \mid |G|$. Then the number of elements of order 2^n is divisible by 2^n unless G is a cyclic, an elementary abelian or a 2-group of maximal class.*

Lemma 2.4 ([4]) *Let G be a finite 2-group of maximal class. Then G is isomorphic to one of the following groups:*

   (I) A dihedral group: $\langle a,b|a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, $n \geq 2$;

   (II) A generalized quaternion group: $\langle a,b|a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle$, $n \geq 3$;

   (III) A semi-dihedral group: $\langle a,b|a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle$, $n \geq 4$.

Lemma 2.5 ([2]) *Let G be a finite group and let $q \geq 5$ be a prime power, $q \neq 9$. If $\pi_e(G) = \pi_e(L_2(q))$, then $G \cong L_2(q)$.*

3. Proof of the main result

Theorem 3.1 *Let G be a group such that $\tau_e(G) = \tau_e(L_2(16)) = \{1, 255, 272, 544, 1088, 1920\}$. Then G is isomorphic to $L_2(16)$.*

Proof   Let S_m be the number of elements of order m. By Lemma 2.2 we can assume that G is finite. Note that $S_m = k\varphi(m)$, where k is the number of cyclic subgroups of order m and $\varphi(m)$ is Euler totient function. Moreover, if $m > 2$, then $\varphi(m)$ is even.

First we claim that $\pi(G) \subseteq \{2, 3, 5, 17\}$. Since $255 \in \tau_e(G)$, it follows that $2 \in \pi(G)$ and $S_2 = 255$. Suppose that there exists a prime $p > 5$ and $p \in \pi(G)$. Then by Lemma 2.1, we have $p \mid 1 + S_p$ for some $S_p \in \{272, 544, 1088, 1920\}$. Note that $\varphi(p) \mid S_p$. Hence $p = 17$ and $S_{17} = 1920$. Then $\pi(G) \subseteq \{2, 3, 5, 17\}$. In addition, if 3 and $5 \in \pi(G)$, similarly we can get that $S_3 = 272$ or 1088 and $S_5 = 544$.

By a discussion similar to the above we will consider the possibilities of $\pi_e(G)$. By Lemma 2.1 and $\varphi(m) \mid S_m$, it is not hard to get that if $2^i \in \pi_e(G)$, then $i \le 8$; if $3^s \in \pi_e(G)$, then $s \le 1$; if $5^t \in \pi_e(G)$, then $t \le 1$; if $17^k \in \pi_e(G)$, then $k \le 1$; if $2^i \cdot 3 \in \pi_e(G)$, then $i \le 7$; if $2^i \cdot 5 \in \pi_e(G)$, then $i \le 6$; if $2^i \cdot 17 \in \pi_e(G)$, then $i \le 4$. Finally we claim that $85\overline{\in}\pi_e(G)$. If not, then $85 \mid 1 + S_5 + S_{17} + S_{85}$ by Lemma 2.1 for $S_{85} \in \{272, 544, 1088, 1920\}$ and it is easy to see that this is impossible. Thus $85\overline{\in}\pi_e(G)$. Therefore, $\pi_e(G) \subseteq \{1, 2, 2^2, \ldots, 2^8\} \bigcup \{3, 2, \cdot3, \ldots, 2^7 \cdot 3\} \bigcup \{5, 2\cdot5, \ldots, 2^6\cdot5\} \bigcup \{17, 2\cdot17, \ldots, 2^4\cdot17\} \bigcup \{3\cdot5, 3\cdot17\} \bigcup \{2\cdot3\cdot5, \ldots, 2^5\cdot3\cdot5, 2\cdot3\cdot17, \ldots, 2^3\cdot3\cdot17\}$.

Now we assume that

$$|G| = 4080 + 272k_1 + 544k_2 + 1088k_3 + 1920k_4 = 2^m \cdot 3^n \cdot 5^i \cdot 17^j, \tag{1}$$

where m, n, i and j are non-negative integers. And we consider the following cases.

Case 1   Let $\pi(G) = \{2\}$. Then $\pi_e(G) \subseteq \{1, 2, \ldots, 2^8\}$ and so $|\pi_e(G)| \le 9$. From the equation (1) it follows that $255 + 17k_1 + 34k_2 + 68k_3 + 120k_4 = 2^{m-4}$. Note that $0 \le k_1 + k_2 + k_3 + k_4 \le 3$ and $2 \mid 255 + 17k_1$, thus k_1 is odd and so $k_1 = 1$ or 3. If $k_1 = 3$, then $k_2 = k_3 = k_4 = 0$. Consequently we have $306 = 2^{m-4}$, which is impossible. Hence $k_1 = 1$ and $136 + 17k_2 + 34k_3 + 60k_4 = 2^{m-5}$, $0 \le k_2 + k_3 + k_4 \le 2$. And so $k_2 = 0$ or 2 since $2 \mid 17k_2$. If $k_2 = 2$, then $k_3 = k_4 = 0$ and it follows that $170 = 2^{m-5}$, which is a contradiction. Hence $k_2 = 0$ and $68 + 17k_3 + 30k_4 = 2^{m-6}$. Since $2 \mid 17k_3$ it follows that $k_3 = 0$ or 2. If $k_3 = 2$, then $k_4 = 0$ and so $102 = 2^{m-6}$, which is impossible. Therefore, $k_3 = 0$ and $68 + 30k_4 = 2^{m-6}$, $0 \le k_4 \le 2$. Similarly, we have $k_4 = 2$ and so $128 = 2^{m-6}$. Therefore, $m = 13$ and $|G| = 2^{13}$. In fact such 2-group does not exist. By Lemma 2.3 we know that G is cyclic, elementary Abelian or a 2-group of maximal class. We have shown that $\exp(G) = 2^8$, thus G is 2-group of maximal class. And it is easy to see that this is impossible by Lemma 2.4.

Case 2   Let $\pi(G) = \{2, 17\}$. If P_{17} is a Sylow 17-subgroup of G, then it follows that $|P_{17}| \mid 1 + S_{17}$ by Lemma 2.1. Namely, $|P_{17}| \mid 1921$, thus $|P_{17}| = 17$ and so $n_{17} = S_{17}/\varphi(17) = 120$, which implies that 3 and $5 \in \pi(G)$. This is a contradiction.

Similarly, we can prove that $\pi(G) \ne \{2, 5\}$, $\{2, 5, 17\}$, $\{2, 3, 17\}$ and $\{2, 3, 5\}$.

Case 3 Let $\pi(G) = \{2, 3\}$.

(3.1) If $S_3 = 272$, then $|P_3| \mid 1 + S_3$ by Lemma 2.1. Namely, $|P_3| \mid 273$. Then $|P_3| = 3$ and it follows that $n_3 = 272/\varphi(3) = 136$. Thus $17 \in \pi(G)$, which is a contradiction.

(3.2) If $S_3 = 1088$, then $|P_3| \mid 1 + S_3$ by Lemma 2.1. Thus $|P_3| \le 9$. If $|P_3| = 3$, then similarly to (3.1) we can get a contradiction. So $|P_3| = 9$ and it follows that $255 + 17k_1 + 34k_2 + 68k_3 + 120k_4 = 2^{m-4} \cdot 3^2$. It is evident that $0 \le k_1 + k_2 + k_3 + k_4 \le 11$ and $m > 8$. Hence $24 \mid 255 + 17k_1 + 34k_2 + 68k_3$, namely, $24 \mid k_1 + 2k_2 + 4k_3 - 9$. We know that $k_1 + 2k_2 + 4k_3 - 9 \le 35$. Therefore, $k_1 + 2k_2 + 4k_3 - 9 = 0$ or 24. If $k_1 + 2k_2 + 4k_3 - 9 = 0$, then $17 + 5k_4 = 2^{m-7} \cdot 3$, which is impossible. If $k_1 + 2k_2 + 4k_3 - 9 = 24$, then $34 + 5k_4 = 2^{m-7} \cdot 3$. Also we can see that it is impossible.

Case 4 Let $\pi(G) = \{2, 3, 5, 17\}$.

(4.1) If $S_3 = 272$, then $|P_3| \mid 1 + S_3$ by Lemma 2.1. Namely, $|P_3| \mid 273$. Then $|P_3| = 3$. Similarly, we can get that $|P_5| = 5$ and $|P_{17}| = 17$. Then from the equation (1) we have $|G| = 4080 + 272k_1 + 544k_2 + 1088k_3 + 1920k_4 = 2^m \cdot 3 \cdot 5 \cdot 17$, $0 \le k_1 + k_2 + k_3 + k_4 \le 33$. Therefore, $17 \mid k_4$ and it follows that $k_4 = 0$ or 17. If $k_4 = 17$, then $135 + k_1 + 2k_2 + 4k_3 = 2^{m-4} \cdot 3 \cdot 5$. Thus $15 \mid k_1 + 2k_2 + 4k_3$. Note that $k_1 + 2k_2 + 4k_3 \le 64$ since $0 \le k_1 + k_2 + k_3 \le 16$. Consequently, $k_1 + 2k_2 + 4k_3 = 0$, 15, 30 or 45. If $k_1 + 2k_2 + 4k_3 = 0$, then $135 = 2^{m-4} \cdot 3 \cdot 5$, which is impossible. Similarly, $k_1 + 2k_2 + 4k_3 \ne 15$, 30 and 45. So $k_4 = 0$ and we have $15 + k_1 + 2k_2 + 4k_3 = 2^{m-4} \cdot 3 \cdot 5$.

If $6 \in \pi_e(G)$, then $6 \mid 1 + S_2 + S_3 + S_6$ by Lemma 2.1 for $S_6 \in \{272, 544, 1088, 1920\}$. Hence $S_6 = S_{17} = 1920$, which is a contradiction since $k_4 = 0$. And so $6 \overline{\in} \pi_e(G)$. By the same reason 2^8 and $10 \overline{\in} \pi_e(G)$. Thus $|\pi_e(G)| \le 17$. Therefore, $0 \le k_1 + k_2 + k_3 \le 11$, which implies that $0 \le k_1 + 2k_2 + 4k_3 \le 44$. Then $k_1 + 2k_2 + 4k_3 = 0$, 15 or 30 since $15 \mid k_1 + 2k_2 + 4k_3$.

If $k_1 + 2k_2 + 4k_3 = 0$, then $k_1 = k_2 = k_3 = 0$. Thus $|\pi_e(G)| = 6$. If $15 \overline{\in} \pi_e(G)$, then we consider P_5 acts point freely on the set of elements of order 3. Therefore, $|P_5| \mid S_3$. Namely, $5 \mid 272$, which is a contradiction. Thus $15 \in \pi_e(G)$ and so $\pi_e(G) = \{1, 2, 3, 5, 15, 17\}$. And it follows that $G \cong L_2(16)$ by Lemma 2.5. If $k_1 + 2k_2 + 4k_3 = 15$, then $m = 5$. We consider P_2 acts point freely on the set of elements of order 3. Thus $|P_2| \mid S_3$, namely $2^5 \mid 272$, which is a contradiction. If $k_1 + 2k_2 + 4k_3 = 30$, then $45 = 2^{m-4} \cdot 3 \cdot 5$, which is also a contradiction.

(4.2) If $S_3 = 1088$, then $|P_3| \le 9$ since $|P_3| \mid 1 + S_3$. We claim that $|P_3| = 9$.

If $|P_3| = 3$, then by a discussion similar to (4.1) we can get that $|G| = 4080 + 272k_1 + 544k_2 + 1088k_3 + 1920k_4 = 2^m \cdot 3 \cdot 5 \cdot 17$. Also we have 6, 10, $51 \overline{\in} \pi_e(G)$, and $k_4 = 0$, which implies that $15 + k_1 + 2k_2 + 4k_3 = 2^{m-4} \cdot 3 \cdot 5$.

By Lemma 2.1 we know that $102 \mid 1 + S_2 + S_3 + S_{17} + S_{34}$ for $S_{34} \in \{272, 544, 1088, 1920\}$. Therefore, $102 \mid S_{34}$, which is impossible. Hence $34 \overline{\in} \pi_e(G)$, and it follows that $|\pi_e(G)| \le 12$, which implies that $k_1 + k_2 + k_3 \le 6$, and so $k_1 + 2k_2 + 4k_3 \le 24$. Then $k_1 + 2k_2 + 4k_3 = 0$ or 15 since $15 \mid k_1 + 2k_2 + 4k_3$.

If $k_1 + 2k_2 + 4k_3 = 0$, then similarly to (4.1) we get that $G \cong L_2(16)$, which is a contradiction since we know that the number of elements of order 3 of $L_2(16)$ is 272. If $k_1 + 2k_2 + 4k_3 = 15$, then $m = 5$. If $2^5 \in \pi_e(G)$, then $\pi(n_2) \subseteq \{2, 17\}$ since $k_4 = 0$ and $S_{17} = 1920$. Thus 3 and $5 \in$

$\pi(N_G(P_2))$. Note that $N_G(P_2)/C_G(P_2) \lesssim \mathrm{Aut}(P_2)$ and $\pi(C_G(P_2)) = \{2\}$, then $15 \mid |\mathrm{Aut}(P_2)|$, which is a contradiction. Consequently, $2^5 \overline{\in} \pi_e(G)$ and so $|\pi_e(G)| \leq 9$. Hence $k_1 + k_2 + k_3 \leq 3$, which implies $k_1 + 2k_2 + 4k_3 \leq 12$. We get a contradiction. Therefore $|P_3| \neq 3$ and so $|P_3| = 9$. Since $51 \overline{\in} \pi_e(G)$ we consider P_3 acts point freely on the set of elements of order 17. Then $|P_3| \mid S_{17}$, namely, $9 \mid 1920$, which is a contradiction.

Now the proof of Theorem 3.1 is completed. □

Remark By [4, Chap. 2, Theorems 8.2–8.5] we can get the following statements:

(i) If $2 \nmid q$, then $\tau_e(L_2(q)) = \{1, \varphi(d) \cdot q \cdot (q+1)/2, 1 < d \mid (q-1)/2, \varphi(s) \cdot q \cdot (q-1)/2, 1 < s \mid (q+1)/2, q^2 - 1\}$.

(ii) If $2 \mid q$, then $\tau_e(L_2(q)) = \{1, \varphi(d) \cdot q \cdot (q+1)/2, 1 < d \mid (q-1), \varphi(s) \cdot q \cdot (q-1)/2, 1 < s \mid (q+1), q^2 - 1\}$, where φ is Euler's totient function.

Problem 1 We try to make a further study to the problem of characterization of finite simple groups by $\tau_e(G)$, thus we give the above remark. Now from [5], [7] and this paper we know that $L_2(2^n)$ can be characterized by $\tau_e(L_2(2^n))$, $n = 2, 3, 4$. Is it true that $L_2(2^m)$ can be characterized by $\tau_e(L_2(2^m))$ for an arbitrary natural number m?

Problem 2 Let G be a finite simple group. Then from Lemma 2.2 we know that G is n-recognizable by $\tau_e(G)$ for some natural number n. Do there exist two finite simple groups G and H not isomorphic to each other such that $\tau_e(G) = \tau_e(H)$?

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