Journal of Mathematical Research with Applications
May, 2012, Vol. 32, No. 3, pp. 297–312
DOI:10.3770/j.issn:2095-2651.2012.03.005
Http://jmre.dlut.edu.cn

A Mehrotra-Type Predictor-Corrector Algorithm for $P_*(\kappa)$ Linear Complementarity Problems

Weihua LI, Mingwang ZHANG*, Yiyuan ZHOU
College of Science, China Three Gorges University, Hubei 443002*, P. R. China*

Abstract Mehrotra-type predictor-corrector algorithm, as one of most efficient interior point methods, has become the backbones of most optimization packages. Salahi et al. proposed a cut strategy based algorithm for linear optimization that enjoyed polynomial complexity and maintained its efficiency in practice. We extend their algorithm to $P_*(\kappa)$ linear complementarity problems. The way of choosing corrector direction for our algorithm is different from theirs. The new algorithm has been proved to have an $\mathcal{O}((1+4\kappa)(17+19\kappa)\sqrt{1+2\kappa}n^{\frac{3}{2}}\log\frac{(x^0)^{\rm T}s^0}{\varepsilon})$ worst case iteration complexity bound. An numerical experiment verifies the feasibility of the new algorithm.

Keywords $P_*(\kappa)$ linear complementarity problems; Mehrotra-type predictor-corrector algorithm; polynomial iteration complexity; interior point method.

MR(2010) Subject Classification 90C30; 90C33; 90C51

1. Introduction

Variants of Mehrotra's predictor-corrector algorithm [1, 2] are among the most practical interior-point methods (IPMs) for linear optimization (LO), quadratic optimization(QO) and linear complementarity problems(LCPs) and have become the backbones of most optimization softwares. However, not much about their complexity was known until [3] was presented by Salahi et al. In [3], a numerical example showed that a feasible version of Mehrotra-type predictor-corrector algorithm may be forced to make very small steps to keep the iterates in a certain neighborhood of the central path, which motivated them to combine this algorithm with a simple large-update safeguard that guaranteed polynomial iteration complexity. The authors of [4] analyzed the same algorithm from a different perspective and proposed a cut strategy based algorithm. Their algorithm cuts the maximum step size in the predictor step if it is above a certain threshold, if this cut does not give a desirable step size, then cuts it for the second time which gives a lower bound for the step size in the corrector step. This algorithm enjoys polynomial iteration while its practical efficiency is preserved. The algorithms of [4], having a

Received September 18, 2010; Accepted August 31, 2011
Supported by the Natural Science Foundation of Hubei Province (Grant No. 2008CDZ047).
* Corresponding author
E-mail address: liwh158@126.com (Weihua LI); zmwang@ctgu.edu.cn (Mingwang ZHANG); zyy323@126.com (Yiyuan ZHOU)

stop criteria in the predictor step, are more efficient in solving large scale programs than the safeguard algorithms in [3].

In this paper, the cut strategy based algorithm of [4] is extended to solve the $P_*(\kappa)$ linear complementarity programs. The way of choosing corrector direction is different from the corresponding algorithm for linear optimization. The new algorithm is proved to have an $\mathcal{O}((1+4\kappa)(17+19\kappa)\sqrt{1+2\kappa}n^{\frac{3}{2}}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$ worst case iteration complexity bound. A Matlab numerical experiment indicates that the algorithm is efficient.

Throughout the paper, $\|\cdot\|$ denotes the 2-norm of vectors and e is the all one vector; For $x, s \in R^n$, xs denotes the componentwise product (Hadamard product) of vectors x and s, and so is true for other operations. For simplicity we also use the following notations:

$$\begin{aligned}
&x(\alpha) = x + \alpha\Delta x,\ s(\alpha) = s + \alpha\Delta s,\ \mu_g = \frac{x^{\mathrm{T}}s}{n},\\
&\mathcal{I} = \{1, ..., n\},\ \mathcal{I}_+ = \{i \in \mathcal{I} \mid \Delta x_i^a \Delta s_i^a \ge 0\},\ \mathcal{I}_- = \{i \in \mathcal{I} \mid \Delta x_i^a \Delta s_i^a < 0\},\\
&\mathcal{F} = \{(x, s) \in R^n \times R^n \mid s = Mx + q, (x, s) \ge 0\},\\
&\mathcal{F}^0 = \{(x, s) \in \mathcal{F} \mid (x, s) > 0\},\\
&X = \mathrm{diag}(x),\ S = \mathrm{diag}(s).
\end{aligned}$$

2. Preliminaries

In this paper we consider the following $P_*(\kappa)$ linear complementarity problem (LCP):

$$\begin{cases} s = Mx + q,\\ x^{\mathrm{T}}s = 0,\\ x \ge 0, \quad s \ge 0, \end{cases} \tag{1}$$

where $M \in R^{n\times n}$ is a $P_*(\kappa)$ matrix and $q \in R^n$.

$P_*(\kappa)$ matrix was introduced in [5] and we give the definition as follows.

Definition 2.1. *Let $\kappa \ge 0$ be a nonnegative number. A matrix $M \in R^{n\times n}$ is called a $P_*(\kappa)$ matrix if*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+(x)} x_i(Mx)_i + \sum_{i\in\mathcal{I}_-(x)} x_i(Mx)_i \ge 0,$$

or

$$x^{\mathrm{T}}Mx \ge -4\kappa \sum_{i\in\mathcal{I}_+(x)} x_i(Mx)_i,$$

for all $x \in R^n$, where

$$\mathcal{I}_+(x) = \{i \in \mathcal{I} : x_i(Mx)_i \ge 0\},\ \mathcal{I}_-(x) = \{i \in \mathcal{I} : x_i(Mx)_i < 0\}.$$

Note that for $\kappa = 0$, $P_*(0)$ is the class of positive semidefinite matrices. This implies that the class of $P_*(\kappa)$-matrices includes both the class PSD of positive matrices and the class of P-matrices with all the principal minors positive. Indeed, it is known that by exploiting the first order optimality condition of the optimization problem, any differentiable convex quadratic program can be formulated into a monotone linear complementarity program (MLCP), i.e., $P_*(0)$ LCP, and vice versa [6].

Without loss of generality [7] we may assume that (1) satisfies the interior point condition (IPC), i.e., there exists an (x^0, s^0) such that

$$s^0 = Mx^0 + q, \quad x^0 > 0, \ s^0 > 0.$$

The basic idea of primal-dual IPMs is to replace the second equation of (1) by the parameterized equation $xs = \mu e$. This leads to the following system:

$$\begin{cases} s = Mx + q, \\ xs = \mu e, \\ x \geq 0, \quad s \geq 0. \end{cases} \tag{2}$$

If the IPC holds, the system (2) has a unique solution for each $\mu > 0$. This solution, denoted by $(x(\mu), s(\mu))$, is called the μ-center of (1). The set of μ-centers gives the central path of (1). As $\mu \to 0$, the central path forms a path to the optimal solution of (1) (see [7]).

Before proceeding, let us briefly recall a feasible version of Mehrotra's original algorithm for LCPs. In the predictor step one solves the so-called affine scaling system:

$$\begin{aligned} M\Delta x^a &= \Delta s^a, \\ s\Delta x^a + x\Delta s^a &= -xs. \end{aligned} \tag{3}$$

Then the maximum feasible step size in this direction is computed, i.e., the largest $\alpha_a \leq 1$ satisfies

$$(x + \alpha_a \Delta x^a, s + \alpha_a \Delta s^a) \geq 0.$$

However, the algorithm does not make this step right away, it uses the information from the predictor step to compute the corrector direction by solving the following system:

$$\begin{aligned} M\Delta x &= \Delta s, \\ s\Delta x + x\Delta s &= \mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a, \end{aligned} \tag{4}$$

where μ is defined adaptively as

$$\mu = (\frac{g_a}{g})^2 \frac{g_a}{n}, \tag{5}$$

where $g_a = (x + \alpha_a \Delta x^a)^{\mathrm{T}}(s + \alpha_a \Delta s^a)$ and $g = x^{\mathrm{T}} s$.

Remark 2.1 An important ingredient of this paper is that the second equation of (4) is different from the corresponding equation in [4], where it is $s\Delta x + x\Delta s = \mu e - xs - \Delta x^a \Delta s^a$, thus the new corrector direction is also different, which is the key to proving the polynomial complexity of the new algorithm.

Finally, the maximum step size α_c is computed so that the next iterate given by

$$(x + \alpha_c \Delta x, s + \alpha_c \Delta s)$$

belongs to a certain neighborhood of the central path.

3. Algorithm and complexity analysis

In this paper, we consider the negative infinity norm neighborhood defined by

$$\mathcal{N}_\infty^-(\gamma) := \{(x,s) \in \mathcal{F}^0 : \quad x_i s_i \geq \gamma \mu_g, \ \forall i \in \mathcal{I}\},$$

where $\gamma \in (0, \frac{1}{4\kappa+3})$ is a constant independent of n.

We can outline our algorithm as follows:

Algorithm 1

Input:

A proximity $\gamma \in (0, \frac{1}{4\kappa+3})$; a safeguard parameter $\beta \in [\gamma, \frac{1}{4\kappa+3})$;
an accuracy parameter $\varepsilon > 0$; a starting point $(x^0, s^0) \in \mathcal{N}_\infty^-(\gamma)$.

begin
while $x^{\mathrm{T}} s \geq \varepsilon$ do
begin
(Predictor Step)
Solve (3) and compute the maximum step size α_a such that
$(x(\alpha_a), s(\alpha_a)) \in \mathcal{F}$;
If $x(\alpha_a)^{\mathrm{T}} s(\alpha_a) \leq \varepsilon$, then
let $x = x(\alpha_a), s = s(\alpha_a)$ and stop.
end
end
begin
(Corrector Step)
If $\alpha_a > \alpha_1$, where α_1 is given by (13), then let $\alpha_a = \alpha_1$.
end
Solve (4) with μ defined by (5) and compute the
maximum step size α_c such that $(x(\alpha_c), s(\alpha_c)) \in \mathcal{N}_\infty^-(\gamma)$;
If $\alpha_c < \frac{\gamma}{4p_1 n}$, where $p_1 = \frac{17+19\kappa}{32\gamma}\sqrt{(1+4\kappa)(2+4\kappa)}$, then
solve (4) with $\mu = \frac{\beta}{1-\beta}\mu_g$ and compute the
maximum step size α_c such that $(x(\alpha_c), s(\alpha_c)) \in \mathcal{N}_\infty^-(\gamma)$;
end
Set $(x, s) = (x(\alpha_c), s(\alpha_c))$.
end
end

The following technical lemma is used frequently during the analysis.

Lemma 3.1 *Suppose that* $(\Delta x^a, \Delta s^a)$ *is the solution of (3). Then*

1) $\Delta x_i^a \Delta s_i^a \leq \frac{x_i s_i}{4}$, $i \in \mathcal{I}_+$; $-\Delta x_i^a \Delta s_i^a \leq \frac{1}{\alpha_a}(\frac{1}{\alpha_a} - 1) x_i s_i$, $i \in \mathcal{I}_-$;

2) $\sum_{i \in \mathcal{I}_+} \Delta x_i^a \Delta s_i^a \leq \frac{x^{\mathrm{T}} s}{4}$, $\sum_{i \in \mathcal{I}_-} |\Delta x_i^a \Delta s_i^a| \leq \frac{4\kappa+1}{4} x^{\mathrm{T}} s$;

3) $-\kappa x^{\mathrm{T}}s \le (\Delta x^a)^{\mathrm{T}}\Delta s^a \le \frac{x^{\mathrm{T}}s}{4}$.

Proof 1) The proof is analogous to those in Lemma A.1 and Lemma 4.1 in [3].

2) The first conclusion follows from 1). In the following we prove the second one. By (3) and using the fact that M is a $P_*(\kappa)$ matrix, we have

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i^a(M\Delta x^a)_i+\sum_{i\in\mathcal{I}_-}\Delta x_i^a(M\Delta x^a)_i\ge 0,$$

or

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a+\sum_{i\in\mathcal{I}_-}\Delta x_i^a\Delta s_i^a\ge 0. \tag{6}$$

Using the first conclusion in 2) completes the proof.

3) Following from 2), one has

$$(\Delta x^a)^{\mathrm{T}}\Delta s^a\le\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a\le\frac{x^{\mathrm{T}}s}{4}.$$

Moreover, by (6) we have

$$(\Delta x^a)^{\mathrm{T}}\Delta s^a\ge -4\kappa\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a\ge -\kappa x^{\mathrm{T}}s.$$

This completes the proof. □

Following from 3) of Lemma 3.1, and using the definition of μ given by (5), we have

$$\begin{aligned}\mu&=\frac{((1-\alpha_a)x^{\mathrm{T}}s+\alpha_a^2(\Delta x^a)^{\mathrm{T}}\Delta s^a)^3}{n(x^{\mathrm{T}}s)^2}=(1-\alpha_a+\frac{\alpha_a^2(\Delta x^a)^{\mathrm{T}}\Delta s^a}{x^{\mathrm{T}}s})^3\mu_g\\&\le(1-\alpha_a+\frac{\alpha_a^2\cdot\frac{1}{4}x^{\mathrm{T}}s}{x^{\mathrm{T}}s})^3\mu_g=(1-\alpha_a+\frac{1}{4}\alpha_a^2)^3\mu_g\le(1-\frac{3}{4}\alpha_a)^3\mu_g.\end{aligned}$$

Besides, by $(\Delta x^a)^{\mathrm{T}}\Delta s^a\ge-\kappa x^{\mathrm{T}}s$, there holds

$$\mu\ge(1-\alpha_a+\frac{\alpha_a^2(-\kappa x^{\mathrm{T}}s)}{x^{\mathrm{T}}s})^3\mu_g=(1-\alpha_a-\kappa\alpha_a^2)^3\mu_g\ge(1-(1+\kappa)\alpha_a)^3\mu_g.$$

Therefore, we get the bound of μ:

$$(1-(1+\kappa)\alpha_a)^3\mu_g\le\mu\le(1-\frac{3}{4}\alpha_a)^3\mu_g. \tag{7}$$

The following theorem shows that there exists always a guaranteed positive step size in the predictor step of the algorithm.

Theorem 3.2 *Suppose that the current iterate $(x,s)\in\mathcal{N}_\infty^-(\gamma)$, and $(\Delta x^a,\Delta s^a)$ is the solution of (3). Then the maximum feasible step size, $\alpha_a\in(0,1]$, so that $(x(\alpha_a),s(\alpha_a))\ge 0$, satisfies*

$$\alpha_a\ge\sqrt{\frac{\gamma}{(4\kappa+1)n}}.$$

Proof Since $(x,s)\in\mathcal{N}_\infty^-(\gamma)$, by 2) of Lemma 3.1, we have

$$x_i(\alpha)s_i(\alpha)=(1-\alpha)x_is_i+\alpha^2\Delta x_i^a\Delta s_i^a\ge\gamma(1-\alpha)\frac{x^{\mathrm{T}}s}{n}-\frac{4\kappa+1}{4}\alpha^2x^{\mathrm{T}}s.$$

Our aim is to ensure that $x_i(\alpha)s_i(\alpha) \geq 0$. For this it suffices to require that

$$\gamma(1-\alpha)\frac{x^{\mathrm{T}}s}{n} - \frac{4\kappa+1}{4}\alpha^2 x^{\mathrm{T}}s = ((1-\alpha)\gamma - \frac{4\kappa+1}{4}n\alpha^2)\frac{x^{\mathrm{T}}s}{n} \geq 0,$$

that is equivalent to

$$(4\kappa+1)n\alpha^2 + 4\gamma\alpha - 4\gamma \leq 0.$$

This inequality holds when $\alpha \in [\frac{-2\gamma-2\sqrt{\gamma^2+(4\kappa+1)n\gamma}}{(4\kappa+1)n}, \frac{-2\gamma+2\sqrt{\gamma^2+(4\kappa+1)n\gamma}}{(4\kappa+1)n}]$. So the feasible predictor step satisfies

$$\alpha_a \geq \frac{-2\gamma + 2\sqrt{\gamma^2 + (4\kappa+1)n\gamma}}{(4\kappa+1)n}.$$

Since $\frac{\gamma}{(4\kappa+1)n} < \frac{1}{2}$, we have

$$\alpha_a \geq \frac{-2\gamma + 2\sqrt{\gamma^2 + (4\kappa+1)n\gamma}}{(4\kappa+1)n} = \frac{2}{\sqrt{1 + (4\kappa+1)\frac{n}{\gamma}} + 1} \geq \sqrt{\frac{\gamma}{(4\kappa+1)n}}.$$

This completes the proof. □

Lemma 3.3 *Let $(\Delta x, \Delta s)$ be the solution of (4) with $\mu > 0$. Then*

$$\|\Delta x \Delta s\| \leq \sqrt{(\frac{1}{4}+\kappa)(\frac{1}{2}+\kappa)}\|r\|^2, \quad \sum_{i\in\mathcal{I}_+} \Delta x_i \Delta s_i \leq \frac{1}{4}\|r\|^2,$$

where $\|r\|^2 = \|\mu(xs)^{-\frac{1}{2}} - (xs)^{\frac{1}{2}} - \alpha_a^2 (xs)^{-\frac{1}{2}} \Delta x^a \Delta s^a\|^2$.

Proof The proof is similar to that of Lemma 8 in [8].

The following technical lemma and its corollary will be used in the step size estimation for the corrector step of the new algorithm.

Lemma 3.4 *Suppose that the current iterate $(x,s) \in \mathcal{N}_\infty^-(\gamma)$, and let $(\Delta x, \Delta s)$ be the solution of (4) with $\mu \geq 0$. Then we have*

$$\begin{aligned}\|\Delta x \Delta s\| \leq & \sqrt{(\frac{1}{4}+\kappa)(\frac{1}{2}+\kappa)(\frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{\alpha_a^2 n\mu(4\kappa+1)}{2\gamma} +}\\ & \overline{\frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(4\kappa+1)(1-\alpha_a) + 16}{16}n\mu_g)},\\ \Delta x^{\mathrm{T}}\Delta s \leq & \frac{1}{4}(\frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{\alpha_a^2 n\mu(4\kappa+1)}{2\gamma} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16}n\mu_g).\end{aligned}$$

Proof Expanding $\|r\|^2$ denoted in Lemma 3.3, we have

$$\|r\|^2 = \mu^2 \sum_{i\in\mathcal{I}} \frac{1}{x_i s_i} + \sum_{i\in\mathcal{I}} x_i s_i - 2n\mu + \alpha_a^4 \sum_{i\in\mathcal{I}} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} - 2\mu\alpha_a^2 \sum_{i\in\mathcal{I}} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} + 2\alpha_a^2 \sum_{i\in\mathcal{I}} \Delta x_i^a \Delta s_i^a.$$

Since $(x,s) \in \mathcal{N}_\infty^-(\gamma)$, there holds

$$\mu^2 \sum_{i\in\mathcal{I}} \frac{1}{x_i s_i} \leq \frac{n\mu^2}{\gamma\mu_g}.$$

Furthermore, by 1) and 2) of Lemma 3.1, we have

$$\begin{aligned}\sum_{i\in\mathcal{I}}\frac{(\Delta x_i^a\Delta s_i^a)^2}{x_is_i} &\le \sum_{i\in\mathcal{I}_+}\frac{(\frac{1}{4}x_is_i)^2}{x_is_i}+\sum_{i\in\mathcal{I}_-}\frac{\frac{1}{\alpha_a}(\frac{1}{\alpha_a}-1)x_is_i}{x_is_i}(-\Delta x_i^a\Delta s_i^a)\\ &= \frac{1}{16}x^{\mathrm{T}}s+\frac{1-\alpha_a}{\alpha_a^2}\sum_{i\in\mathcal{I}_-}|\Delta x_i^a\Delta s_i^a|\\ &\le (\frac{1}{16}+\frac{(4\kappa+1)(1-\alpha_a)}{4\alpha_a^2})n\mu_g.\end{aligned}$$

Besides, there hold

$$-2\mu\sum_{i\in\mathcal{I}}\frac{\Delta x_i^a\Delta s_i^a}{x_is_i}\le 2\mu\sum_{i\in\mathcal{I}_-}\frac{|\Delta x_i^a\Delta s_i^a|}{x_is_i}\le\frac{2\mu}{\gamma\mu_g}\sum_{i\in\mathcal{I}_-}|\Delta x_i^a\Delta s_i^a|\le\frac{(4\kappa+1)n\mu}{2\gamma}$$

and

$$2\sum_{i\in\mathcal{I}}\Delta x_i^a\Delta s_i^a\le 2\sum_{i\in\mathcal{I}_+}\Delta x_i^a\Delta s_i^a\le\frac{n\mu_g}{2}.$$

Therefore, we have

$$\|r\|^2\le\frac{n\mu^2}{\gamma\mu_g}-2n\mu+\frac{\alpha_a^2n\mu(4\kappa+1)}{2\gamma}+\frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16}n\mu_g.$$

We get the conclusion following from Lemma 3.3.

The following corollary gives an explicit upper bound for $\|\Delta x\Delta s\|$ and $\Delta x^{\mathrm{T}}\Delta s$ when μ is chosen adaptively as given by (5).

Corollary 3.5 *Let μ be defined by (5), where $\gamma\in(0,\frac{1}{4\kappa+3})$. Then*

$$\|\Delta x\Delta s\|\le p_1n\mu_g,\ \ \Delta x^{\mathrm{T}}\Delta s\le p_2n\mu_g,$$

where $p_1=\frac{17+19\kappa}{32\gamma}\sqrt{(1+4\kappa)(2+4\kappa)}$, $p_2=\frac{17+19\kappa}{32\gamma}$.

Proof By (7), and using the fact that $\gamma<\frac{1}{4\kappa+3}\le\frac{1}{3}$ and $0<\alpha_a\le 1$, then

$$\begin{aligned}&\frac{n\mu^2}{\gamma\mu_g}-2n\mu+\frac{\alpha_a^2n\mu(4\kappa+1)}{2\gamma}+\frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16}n\mu_g\\ &\le(\frac{1}{\gamma}(1-\frac{3}{4}\alpha_a)^6+\frac{(4\kappa+1)\alpha_a^2}{2\gamma}(1-\frac{3}{4}\alpha_a)^3+\frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16})n\mu_g\\ &\le(\frac{1}{\gamma}+\frac{4\kappa+1}{2\gamma}+\frac{1+8+4(4\kappa+1)+16}{16})n\mu_g\\ &=\frac{16+32\kappa+8+(29+16\kappa)\gamma}{16\gamma}n\mu_g\le\frac{17+19\kappa}{8\gamma}n\mu_g.\end{aligned}$$

By Lemma 3.4 we complete the proof of the corollary.

For simplicity the following notation is used in the rest of our development:

$$t=\max_{i\in\mathcal{I}_+}\{\frac{\Delta x_i^a\Delta s_i^a}{x_is_i}\}. \tag{8}$$

Remark 3.1 Since M is a $P_*(\kappa)$ matrix, there is $\mathcal{I}_+\neq\emptyset$. Besides, by 1) of Lemma 3.1, we have $t\in[0,\frac{1}{4}]$.

The next theorem provides an upper bound for α_a that ensures a positive step size in the corrector step, which also indicates that a larger step size in the predictor step might result in a very small or zero step size in the corrector step.

Theorem 3.6 *Suppose that the current iterate $(x,s)\in\mathcal{N}_\infty^-(\gamma)$, and let $(\Delta x,\Delta s)$ be the solution of (4) with μ as defined by (5). Then for $\alpha_a\in(0,1]$ satisfying*

$$\alpha_a<\frac{1}{1+\kappa}(1-(\frac{\frac{1}{1+\kappa}\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}}) \tag{9}$$

the maximum step size in the corrector step is strictly positive.

Proof Our goal is to find a lower bound for the maximal $\alpha\in(0,1]$ such that

$$x_i(\alpha)s_i(\alpha)\geq\gamma\mu_g(\alpha),\quad \forall i\in\mathcal{I}, \tag{10}$$

where

$$\mu_g(\alpha)=\frac{x(\alpha)^{\mathrm{T}}s(\alpha)}{n}=(1-\alpha)\mu_g+\alpha\mu-\frac{\alpha\alpha_a^2(\Delta x^a)^{\mathrm{T}}\Delta s^a}{n}+\frac{\alpha^2\Delta x^{\mathrm{T}}\Delta s}{n}. \tag{11}$$

By (4), we conclude that (10) is equivalent to

$$(1-\alpha)x_is_i+\alpha\mu-\alpha\alpha_a^2\Delta x_i^a\Delta s_i^a+\alpha^2\Delta x_i\Delta s_i\geq\gamma(1-\alpha)\mu_g+\alpha\gamma\mu-\frac{\alpha\alpha_a^2\gamma(\Delta x^a)^{\mathrm{T}}\Delta s^a}{n}+\frac{\alpha^2\gamma\Delta x^{\mathrm{T}}\Delta s}{n}$$

or

$$(1-\alpha)x_is_i+(1-\gamma)\alpha\mu-\alpha\alpha_a^2\Delta x_i^a\Delta s_i^a+\alpha^2\Delta x_i\Delta s_i+\frac{\alpha\alpha_a^2\gamma(\Delta x^a)^{\mathrm{T}}\Delta s^a}{n}-\frac{\alpha^2\gamma\Delta x^{\mathrm{T}}\Delta s}{n}\geq\gamma(1-\alpha)\mu_g.$$

Note that $(x,s)\in\mathcal{N}_\infty^-(\gamma)$. It follows from 3) of Lemma 3.1 that the above inequality holds if

$$(1-\gamma)\frac{\mu}{\mu_g}-\frac{\alpha_a^2\gamma\Delta x_i^a\Delta s_i^a}{\mu_g}-\alpha_a^2\gamma\kappa+\frac{\alpha\Delta x_i\Delta s_i}{\mu_g}-\frac{\alpha\gamma\Delta x^{\mathrm{T}}\Delta s}{n\mu_g}\geq 0. \tag{12}$$

In the following, we consider (12) for two cases.

i) For $i\in\mathcal{I}_+$, by using (8), (7) and $(x,s)\in\mathcal{N}_\infty^-(\gamma)$, it follows from Corollary 3.5 that (12) holds when

$$-\alpha_a^2\gamma(t+\kappa)+(1-\gamma)(1-\alpha_a(1+\kappa))^3-\alpha(p_1n+p_2\gamma)\geq 0.$$

Obviously, for α_a satisfying (9), the above inequality holds if α satisfies

$$\begin{aligned}&-\frac{\gamma(t+\kappa)}{1+\kappa}(1-(\frac{\frac{1}{1+\kappa}\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}})+\frac{\gamma(t+\kappa)}{1+\kappa}-\alpha(p_1n+p_2\gamma)\\&\quad=(1-\gamma)^{-\frac{1}{3}}(\frac{\gamma(t+\kappa)}{1+\kappa})^{\frac{4}{3}}-\alpha(p_1n+p_2\gamma)\geq 0\end{aligned}$$

or

$$\alpha<\frac{1}{(p_1n+p_2\gamma)(1-\gamma)^{\frac{1}{3}}}(\frac{\gamma(t+\kappa)}{1+\kappa})^{\frac{4}{3}},$$

that is

$$\alpha_c\geq\frac{1}{(p_1n+p_2\gamma)(1-\gamma)^{\frac{1}{3}}}(\frac{\gamma(t+\kappa)}{1+\kappa})^{\frac{4}{3}}>0.$$

ii) For $i \in \mathcal{I}_-$ and α_a satisfying (9), by Corollary 3.5, (12) holds for α satisfying

$$\begin{aligned} &-\alpha_a^2\gamma\kappa + (1-\gamma)(1-\alpha_a(1+\kappa))^3 - \alpha(p_1 n + p_2\gamma) \\ &> -\frac{\gamma\kappa}{1+\kappa}(1-(\frac{\frac{1}{1+\kappa}\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}}) + \frac{\gamma(t+\kappa)}{1+\kappa} - \alpha(p_1 n + p_2\gamma) \\ &= \frac{\gamma t}{1+\kappa} + \frac{\gamma\kappa}{1+\kappa}(\frac{\frac{1}{1+\kappa}\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}} - \alpha(p_1 n + p_2\gamma) \geq 0 \end{aligned}$$

or

$$\alpha < \frac{1}{p_1 n + p_2\gamma}(\frac{\gamma t}{1+\kappa} + \frac{\gamma\kappa}{1+\kappa}(\frac{\frac{1}{1+\kappa}\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}})$$

i.e.,

$$\alpha_c \geq \frac{1}{p_1 n + p_2\gamma}(\frac{\gamma t}{1+\kappa} + \frac{\gamma\kappa}{1+\kappa}(\frac{\frac{1}{1+\kappa}\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}}) > 0.$$

This completes the proof. □

Following from the proof of Theorem 3.6, to have an explicit strictly positive lower bound for the maximum step size α_c in the corrector step, instead of (9), we use the following inequality:

$$\alpha_a \leq \frac{1}{1+\kappa}(1-(\frac{\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}}) := \alpha_1. \tag{13}$$

Lemma 3.7 *For sufficiently small μ_g there holds $t \leq \mathcal{O}(\mu_g)$.*

Proof Analogously to the proof of Theorem 3.6 of [9], we have $\|\Delta x^a\| = \mathcal{O}(\mu_g)$ and $\|\Delta s^a\| = \mathcal{O}(\mu_g)$ (see Appendix for the proof). Therefore, there is

$$|\Delta x^a \Delta s^a| \leq \mathcal{O}(\mu_g^2).$$

This implies the statement of the lemma by the definition of t.

Corollary 3.8 *For sufficiently small μ_g one has $\alpha_a \geq 1 - \mathcal{O}(\mu_g)$.*

Proof The proof is analogous to the interpretation of the section 5 in [3].

Remark 3.2 By Theorem 3.6 we see that for sufficiently small μ_g, we can guarantee a positive step size in the corrector step for $\alpha_a \leq \frac{1}{1+\kappa}(1-\mathcal{O}(\mu_g^{\frac{1}{3}}))$. However, following the Corollary 3.8 $\alpha_a \geq 1 - \mathcal{O}(\mu_g)$, which is greater than or equal to $\frac{1}{1+\kappa}(1-\mathcal{O}(\mu_g^{\frac{1}{3}}))$ for sufficiently small μ_g. In other words, in asymptotic case we might need to cut α_a, but still have a reasonably big α_a.

Remark 3.3 From (13) it is obvious that when $\kappa = 0$ and t approaches to zero, α_1 approaches to one. In other words, our cut does not block the convergence of the affine scaling step size to one, it just reduces the speed of convergence in order to guarantee a positive step size for the corrector step.

In the following corollary, we discuss the specific case that $t = 0$.

Corollary 3.9 *If $t = 0$ at a certain iteration, then the algorithm can make a full Newton step in the predictor step and stop with an optimal solution.*

Proof From (8) it is obvious that when $t=0$, there is $\Delta x_i^a \Delta s_i^a = 0$, $\forall i \in \mathcal{I}_+$. Thus, by the first equation of (3) and using the fact that M is a $P_*(\kappa)$ matrix, we have

$$\begin{aligned}0 &\le (1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i^a (M\Delta x^a)_i + \sum_{i\in\mathcal{I}_-}\Delta x_i^a (M\Delta x^a)_i \\ &= (1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i^a \Delta s_i^a + \sum_{i\in\mathcal{I}_-}\Delta x_i^a \Delta s_i^a = \sum_{i\in\mathcal{I}_-}\Delta x_i^a \Delta s_i^a.\end{aligned}$$

Subsequently $\Delta x_i^a \Delta s_i^a = 0$, $\forall i \in \mathcal{I}$, i.e., for all $i \in \mathcal{I}$, there is $\Delta x_i^a = 0$ or $\Delta s_i^a = 0$. In the following, the proof of $(x+\Delta x^a, s+\Delta s^a) \in \mathcal{F}$ is given.

a) It is obvious that $(x+\Delta x^a, s+\Delta s^a)$ satisfies $M(x+\Delta x^a)+q = s+\Delta s^a$.

b) There are both $x+\Delta x^a \ge 0$ and $s+\Delta s^a \ge 0$. By contradiction, we assume that $x_i+\Delta x_i^a < 0$ or $s_i+\Delta s_i^a < 0$ for some $i \in \mathcal{I}$. Let us suppose $x_i+\Delta x_i^a < 0$ here. Then one has $\Delta s_i^a = 0$ since $\Delta x_i^a < 0$. Moreover, using the second equation of (3), we have $(x_i+\Delta x_i^a)(s_i+\Delta s_i^a) = 0$, that is, $s_i+\Delta s_i^a = s_i = 0$, which would contradict $(x,s)\in\mathcal{F}^0$.

Therefore, a full Newton step in the predictor step leads to an optimal solution since $(x+\Delta x^a)^{\mathrm{T}}(s+\Delta s^a) = 0$ by (3). This completes the proof. □

Therefore, if α_a violates (13), we let $\alpha_a = \alpha_1$ and proceed with the corrector step. If the maximum step size in the corrector step is still below a certain threshold depending only on the dimension, then we let $\alpha_a = \frac{1}{1+\kappa}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})$, where $\gamma \le \beta < \frac{1}{4\kappa+3}$. By (7) one can see that this choice implies $\mu \ge \frac{\beta}{1-\beta}\mu_g$. By $\mu = \frac{\beta}{1-\beta}\mu_g$ with $\alpha_a = \frac{1}{1+\kappa}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})$ one further can guarantee a lower bound for the maximum step size in the corrector step which is independent of t. Subsequently the polynomial iteration complexity of the algorithm can be proved. In the next corollary and the subsequent theorem, we discuss this particular case.

Corollary 3.10 *Let $\mu = \frac{\beta}{1-\beta}\mu_g$, where $\gamma \le \beta < \frac{1}{4\kappa+3}$ and $\gamma \in (0, \frac{1}{4\kappa+3})$. Then*

$$\|\Delta x \Delta s\| \le p_1 n\mu_g,\ \Delta x^{\mathrm{T}}\Delta s \le p_2 n \mu_g.$$

Proof Using $0<\gamma\le\beta<\frac{1}{4\kappa+3}\le\frac{1}{3}$, one has $\frac{\beta}{1-\beta}\in(0,\frac{1}{2})$. Moreover, by $\alpha_a\in(0,1]$, there is

$$\begin{aligned}&\frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{\alpha_a^2 n\mu(4\kappa+1)}{2\gamma} + \frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16}n\mu_g \\ &\le (\frac{1}{\gamma}(\frac{\beta}{1-\beta})^2 + \frac{(4\kappa+1)\alpha_a^2}{2\gamma}\cdot\frac{\beta}{1-\beta} + \frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16})n\mu_g \\ &\le (\frac{1}{4\gamma} + \frac{4\kappa+1}{4\gamma} + \frac{1+8+4(4\kappa+1)+16}{16})n\mu_g = \frac{4+4(4\kappa+1)+\gamma(29+16\kappa)}{16\gamma}n\mu_g \\ &\le \frac{11\kappa+9}{8\gamma}n\mu_g \le \frac{19\kappa+17}{8\gamma}n\mu_g.\end{aligned}$$

By Lemma 3.4, we complete the proof. □

Theorem 3.11 *Suppose that the current iterate $(x,s)\in\mathcal{N}_\infty^-(\gamma)$ and $(\Delta x, \Delta s)$ is the solution of (4) with $\mu = \frac{\beta}{1-\beta}\mu_g$ and $\alpha_a = \frac{1}{1+\kappa}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})$. Then*

$$\alpha_c \ge \frac{\gamma}{4p_1 n}.$$

Proof Following the proof of Theorem 3.6, in the worst case, as given by (12), it suffices to have

$$\alpha\frac{\Delta x_i\Delta s_i}{\mu_g}-\alpha\frac{\gamma\Delta x^{\mathrm{T}}\Delta s}{n\mu_g}-\alpha_a^2\gamma t+(1-\gamma)\frac{\beta}{1-\beta}+\frac{\alpha_a^2\gamma(\Delta x^a)^{\mathrm{T}}\Delta s^a}{n\mu_g}\geq 0.$$

By Lemma 3.1 and Corollary 3.10, and using the fact that $\frac{\beta}{1-\beta}\geq\frac{\gamma}{1-\gamma}$, the above inequality holds as long as

$$-(np_1+\gamma p_2)\alpha-\alpha_a^2\gamma t+\gamma-\alpha_a^2\gamma\kappa=-(np_1+\gamma p_2)\alpha+\gamma(1-\alpha_a^2(t+\kappa))\geq 0.$$

Note that $\alpha_a=\frac{1}{1+\kappa}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})\leq\frac{1}{1+\kappa}$, then the previous inequality holds for α satisfying

$$-(np_1+\gamma p_2)\alpha+\gamma(1-\frac{t+\kappa}{(1+\kappa)^2})\geq 0$$

or

$$\alpha\leq\frac{\gamma(1-\frac{t+\kappa}{(1+\kappa)^2})}{np_1+\gamma p_2}.$$

Since $0\leq t\leq\frac{1}{4}$ and $\gamma p_2\leq np_1$, there is

$$\alpha_c\geq\frac{\gamma(1-\frac{t+\kappa}{(1+\kappa)^2})}{np_1+\gamma p_2}\geq\frac{\frac{1}{2}\gamma}{2p_1n}=\frac{\gamma}{4p_1n},$$

which completes the proof. □

For the worst case, i.e., when $i\in\mathcal{I}_+,\ (\Delta x^a)^{\mathrm{T}}\Delta s^a=-\kappa x^{\mathrm{T}}s,\ \Delta x^{\mathrm{T}}\Delta s>0$, by Lemma 3.1 and Corollary 3.5 or 3.10, there is

$$\mu_g(\alpha)\leq(1-\alpha+\alpha\frac{\mu}{\mu_g}+\alpha\alpha_a^2\kappa+p_2\alpha^2)\mu_g.$$

When $\mu=\frac{\beta}{1-\beta}\mu_g$ and $\alpha_a=\frac{1}{1+\kappa}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})$, if $\alpha\leq\frac{1}{6p_2}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})$, then we can prove that $\mu_g(\alpha)<\mu_g$, that is to say, the dual gap is decreased after the iteration; When $\alpha_a>\alpha_1$ and $\mu=(\frac{g_a}{g})^2\frac{g_a}{n}$, if $\alpha\leq\frac{13\times 0.37}{160p_2(1+\kappa)}$, there is $\mu_g(\alpha)<\mu_g$, too; Similarly, when $\alpha_a\leq\alpha_1$ and $\mu=(\frac{g_a}{g})^2\frac{g_a}{n}$, if $\alpha\leq\frac{13\gamma^{\frac{1}{2}}}{160p_2(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}$, we can keep $\mu_g(\alpha)<\mu_g$. Hence, in order to guarantee the dual gap decreased after each iteration, we assume

$$\alpha\leq\min\{\frac{1}{6p_2}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}}),\ \frac{13\times 0.37}{160p_2(1+\kappa)},\ \frac{13\gamma^{\frac{1}{2}}}{160p_2(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}\}.\tag{14}$$

Obviously, when $n\geq 2$, there is

$$\frac{\gamma}{4p_1n}<\min\{\frac{1}{6p_2}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}}),\ \frac{13\times 0.37}{160p_2(1+\kappa)},\ \frac{13\gamma^{\frac{1}{2}}}{160p_2(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}\},$$

which implies that the conclusion of Theorem 3.1 still holds. On the other hand, there is

$$\max\{\frac{1}{6p_2}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}}),\ \frac{13\times 0.37}{160p_2(1+\kappa)},\ \frac{13\gamma^{\frac{1}{2}}}{160p_2(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}\}<1.$$

Therefore, (14) is well defined.

The following theorem gives the maximum number of iterations for Algorithm 1 to find an ε-approximate solution.

Theorem 3.12 *After at most*

$$O((1+4\kappa)(17+19\kappa)\sqrt{1+2\kappa}n^{\frac{3}{2}}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$$

number of iterations Algorithm 1 stops with a solution for which $x^{\mathrm{T}}s \leq \varepsilon$.

Proof By (11) and Lemma 3.1, and following from Corollary 3.5 or 3.10, we have

$$\mu_g(\alpha) \leq (1-\alpha+\alpha\frac{\mu}{\mu_g}+\alpha\alpha_a^2\kappa+\frac{\alpha^2\Delta x^{\mathrm{T}}\Delta s}{n\mu_g})\mu_g \leq (1-\alpha+\alpha\frac{\mu}{\mu_g}+\alpha\alpha_a^2\kappa+p_2\alpha^2)\mu_g.$$

1) If $\alpha_a > \alpha_1$, and $\alpha_c \geq \frac{\gamma}{4p_1n}$, the algorithm uses the cut strategy, i.e., it cuts α_a to $\alpha_1 = \frac{1}{1+\kappa}(1-(\frac{\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}})$. Using $\gamma < \frac{1}{4\kappa+3}$ and $t \in [0, \frac{1}{4}]$, there is

$$\alpha_1 = \frac{1}{1+\kappa}(1-(\frac{\gamma(t+\kappa)}{1-\gamma})^{\frac{1}{3}}) \geq \frac{0.37}{1+\kappa}.$$

So as to prove the polynomial complexity of the algorithm, we discuss $\mu_g(\alpha)$ for two cases, i.e., $\kappa > \frac{2}{3}$ and $\kappa \leq \frac{2}{3}$. When $\kappa > \frac{2}{3}$, there holds $\frac{0.37}{1+\kappa} \leq \alpha_1 \leq \frac{1}{1+\kappa} \leq \frac{3}{5}$. By (7) and (14), and noting that $\alpha_1\kappa \leq \frac{\kappa}{1+\kappa} \leq 1$, we have

$$\begin{aligned}
\mu_g(\alpha) &\leq (1-\alpha+(1-\frac{3}{4}\alpha_1)^3\alpha+\alpha\alpha_1^2\kappa+p_2\alpha^2)\mu_g \\
&\leq (1-\alpha+(1-\frac{3}{4}\alpha_1)^2\alpha+\alpha_1\alpha+p_2\alpha^2)\mu_g \\
&\leq (1-\frac{1}{2}\alpha_1\alpha+\frac{9}{16}\cdot\frac{3}{5}\alpha_1\alpha+p_2\alpha^2)\mu_g \\
&\leq (1-\frac{13\times 0.37}{80(1+\kappa)}\alpha+p_2\alpha^2)\mu_g \\
&\leq (1-(\frac{13\times 0.37}{80(1+\kappa)}-\frac{13\times 0.37}{160(1+\kappa)})\cdot\frac{\gamma}{4p_1n})\mu_g \\
&\leq (1-\frac{\gamma}{160p_1(1+\kappa)n})\mu_g.
\end{aligned}$$

When $\kappa \leq \frac{2}{3}$, we have

$$\begin{aligned}
\mu_g(\alpha) &\leq (1-\alpha+(1-\frac{3}{4}\alpha_1)^2\alpha+\frac{2}{3}\alpha_1^2\alpha+p_2\alpha^2)\mu_g \\
&\leq (1-\frac{3}{2}\alpha_1\alpha+(\frac{9}{16}+\frac{2}{3})\alpha_1\alpha+p_2\alpha^2)\mu_g \\
&\leq (1-\frac{13\times 0.37}{48(1+\kappa)}\alpha+p_2\alpha^2)\mu_g \\
&\leq (1-(\frac{13\times 0.37}{48}-\frac{13\times 0.37}{160})\cdot\frac{\gamma}{(1+\kappa)4p_1n})\mu_g \\
&\leq (1-\frac{7\gamma}{480p_1(1+\kappa)n})\mu_g.
\end{aligned}$$

2) If $\alpha_a > \alpha_1$, and $\alpha_c < \frac{\gamma}{4p_1n}$, then our algorithm cuts α_a for the second time, i.e., $\alpha_a = \frac{1}{1+\kappa}(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})$ and $\mu = \frac{\beta}{1-\beta}\mu_g$, which guarantees a lower bound for α_c by Theorem 3.11, that is, $\alpha_c \geq \frac{\gamma}{4p_1n}$. Thus, there is

$$\alpha_a^2\kappa \leq (1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})^2 \leq 1-(\frac{\beta}{1-\beta})^{\frac{1}{3}}.$$

Moreover, by (14) and $\beta < \frac{1}{4\kappa+3} \le \frac{1}{3}$, we have

$$\begin{aligned}
\mu_g(\alpha) &\le (1-\alpha+\frac{\beta}{1-\beta}\alpha+\alpha\alpha_a^2\kappa+p_2\alpha^2)\mu_g \\
&\le (1-\alpha+\frac{\beta}{1-\beta}\alpha+(1-(\frac{\beta}{1-\beta})^{\frac{1}{3}})\alpha+p_2\alpha^2)\mu_g \\
&\le (1-(\frac{\beta}{1-\beta})^{\frac{1}{3}}(1-(\frac{\beta}{1-\beta})^{\frac{2}{3}})\alpha+p_2\alpha^2)\mu_g \\
&\le (1-\frac{1}{3}(\frac{\beta}{1-\beta})^{\frac{1}{3}}\alpha+p_2\alpha^2)\mu_g \\
&\le (1-(\frac{1}{3}-\frac{1}{6})(\frac{\beta}{1-\beta})^{\frac{1}{3}}\cdot\frac{\gamma}{4p_1n})\mu_g \\
&= (1-\frac{\gamma(\frac{\beta}{1-\beta})^{\frac{1}{3}}}{24p_1n})\mu_g.
\end{aligned}$$

3) If $\alpha_a \le \alpha_1$, and $\alpha_c \ge \frac{\gamma}{4p_1n}$, the algorithm uses the Mehrotra's strategy, i.e., $\mu = (\frac{g_a}{g})^2\frac{g_a}{n}$. When $\kappa > \frac{2}{3}$, there is $\alpha_a \le \alpha_1 \le \frac{3}{5}$. By Theorem 3.2, using (7) and (14) and $\alpha_a\kappa \le \alpha_1\kappa \le 1$, we have

$$\begin{aligned}
\mu_g(\alpha) &\le (1-\alpha+(1-\frac{3}{4}\alpha_a)^2\alpha+\alpha_a\alpha+p_2\alpha^2)\mu_g \\
&\le (1-(\frac{1}{2}-\frac{9}{16}\cdot\frac{3}{5})\alpha_a\alpha+p_2\alpha^2)\mu_g \\
&\le (1-\frac{13}{80}\cdot\frac{\gamma^{\frac{1}{2}}}{(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}\alpha+p_2\alpha^2)\mu_g \\
&\le (1-\frac{13\gamma^{\frac{1}{2}}}{160(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}\cdot\frac{\gamma}{4p_1n})\mu_g \\
&\le (1-\frac{13\gamma^{\frac{3}{2}}}{640p_1(4\kappa+1)^{\frac{1}{2}}n^{\frac{3}{2}}})\mu_g.
\end{aligned}$$

When $\kappa \le \frac{2}{3}$, we have

$$\begin{aligned}
\mu_g(\alpha) &\le (1-\alpha+(1-\frac{3}{4}\alpha_a)^2\alpha+\frac{2}{3}\alpha_a\alpha+p_2\alpha^2)\mu_g \\
&\le (1-\frac{13}{48}\alpha_a\alpha+p_2\alpha^2)\mu_g \\
&\le (1-(\frac{13}{48}-\frac{13}{160})\cdot\frac{\gamma^{\frac{1}{2}}\alpha}{(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}})\mu_g \\
&\le (1-\frac{91\gamma^{\frac{1}{2}}}{480(4\kappa+1)^{\frac{1}{2}}n^{\frac{1}{2}}}\cdot\frac{\gamma}{4p_1n})\mu_g \\
&= (1-\frac{91\gamma^{\frac{3}{2}}}{1920p_1(4\kappa+1)^{\frac{1}{2}}1000n^{\frac{3}{2}}})\mu_g.
\end{aligned}$$

We complete the proof by Theorem 5.4 in [4].

4. Numerical result

In this section, we verify our algorithm using Matlab 7.6. For $P_*(\kappa)$ linear complementarity problems, there has not been a polynomial algorithm for which we can calculate the value of the parameter κ for a given $P_*(\kappa)$ matrix. So in this paper we consider the $P_*(\kappa)$ LCPs with $\kappa=0$, where

$$M=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix},\quad q=\left(\frac{1}{11},-4,-\frac{3}{11}\right)^{\mathrm{T}}.$$

Let $\gamma=0.02,\ \beta=0.03$ and $\varepsilon=10^{-6}$. Starting from the feasible point

$$x^0=(2.5,\ 2.5,\ 1)^{\mathrm{T}},\ s^0=(\frac{1}{11},\ 1,\ \frac{19}{11})^{\mathrm{T}},$$

the algorithm stops after 17 iterations with a solution meeting accuracy. The solution is

$$x=(1.909090912570709,\ 1.954545457571089,\ 0.136365040653625)^{\mathrm{T}},$$

$$s=10^{-5}\times(0.000090833143495,\ 0.000514293719393,\ 0.280857997644044)^{\mathrm{T}}.$$

The dual gap is $x^{\mathrm{T}}s=3.947783144852814e-007$, which shows that our algorithm is feasible.

5. Conclusion

In this paper, we have extended the Mehrotra-type predictor-corrector algorithm for linear optimization to $P_*(\kappa)$ LCPs. Since the search directions Δx and Δs are not orthogonal for $P_*(\kappa)$ LCPs, the new technical lemmas are needed and the analysis is different from the corresponding algorithm for linear optimization in [4]. There is an $O((1+4\kappa)(17+19\kappa)\sqrt{1+2\kappa}n^{\frac{3}{2}}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$ worst case iteration complexity bound for our algorithm. Unfortunately, up to now, the parameter κ of the matrix M is not known appropriately and there is no polynomial algorithm to decide whether a matrix is $P_*(\kappa)$ matrix or not [8], so our algorithm is not suitable to solve practical problems directly.

6. Appendix

Proof of Lemma 3.7 For a $P_*(\kappa)$ LCP possessing a strictly complementary solution, a unique partition B and N, where $B\bigcup N=\{1,2,\ldots,n\}$ and $B\bigcap N=\emptyset$, exists such that $x^*_N=0$ and $s^*_B=0$ in every complementarity solution and at least one complementarity solution has $x^*_B>0$ and $x^*_N>0$. Since the sequence generated by Algorithm 1 is contained in a wide neighborhood, we have

$$\gamma\mu_g\le x_is_i\le n\mu_g. \tag{15}$$

The Lemma 2 of Güler and Ye [10] has shown that for all $(x,s)\in\mathcal{N}^-_\infty(\gamma)$, relation (15) implies that

$$\xi\le x_j\le 1/\xi\ \text{ for } j\in B,\ \ \xi\le s_j\le 1/\xi\ \text{ for } j\in N, \tag{16}$$

where $0<\xi<1$ is a positive constant.

Let $z=xs$ and $Z=\mathrm{diag}(z)$. Note from (15) that we must have

$$\gamma\mu_g\le z_j\le n\mu_g\ \text{ for } j=1,2,\ldots,n.$$

Define $D=(XS^{-1})^{\frac{1}{2}}$. We now introduce several lemmas to estimate $\|\Delta x^a\|$ and $\|\Delta s^a\|$. We start by characterizing the solution to (3).

Lemma 6.1 *If $\|\Delta x^a\|$ and $\|\Delta s^a\|$ are obtained from the system (3) and $\mu_g=\frac{x^{\mathrm{T}}s}{n}$, then*

$$\|D^{-1}\Delta x^a\|\le\sqrt{(1+2\kappa)n\mu_g},\quad \|D\Delta s^a\|\le\sqrt{(1+2\kappa)n\mu_g}.$$

Proof Multiply the second equation by $(XS)^{-\frac{1}{2}}$ and square both sides of it, then by 3) of Lemma 3.1, we have

$$\|D^{-1}\Delta x^a\|^2+\|D\Delta s^a\|^2=\|(XS)^{\frac{1}{2}}e\|^2-2(\Delta x^a)^{\mathrm{T}}\Delta s^a\le x^{\mathrm{T}}s+2\kappa x^{\mathrm{T}}s=(1+2\kappa)n\mu_g.$$

Thus, we have the conclusion $\|D^{-1}\Delta x^a\|\le\sqrt{(1+2\kappa)n\mu_g}$, $\|D\Delta s^a\|\le\sqrt{(1+2\kappa)n\mu_g}$.

Lemma 6.2 *If $\|\Delta x^a\|$ and $\|\Delta s^a\|$ are obtained from (3), and $\mu_g=\frac{x^{\mathrm{T}}s}{n}$, then*

$$\|(\Delta x^a)_N\|=O(\mu_g),\quad \|(\Delta s^a)_B\|=O(\mu_g).$$

Proof From Lemma 6.1 and (16), we have

$$\begin{aligned}\|(\Delta x^a)_N\|&=\|D_ND_N^{-1}(\Delta x^a)_N\|\le\|D_N\|\|D_N^{-1}(\Delta x^a)_N\|\le\|D_N\|O(\sqrt{\mu_g})\\&=\|Z_N^{\frac{1}{2}}S_N^{-1}\|O(\sqrt{\mu_g})\le\|Z_N^{\frac{1}{2}}\|O(1/\xi)O(\sqrt{\mu_g})\\&=O(\sqrt{\mu_g})O(\sqrt{\mu_g})=O(\mu_g).\end{aligned}$$

This proves $\|(\Delta x^a)_N\|=O(\mu_g)$. The proof that $\|(\Delta s^a)_B\|=O(\mu_g)$ is similar.

Following from the second equation of (3), we have

$$\Delta x^a=-S^{-1}(xs)-D^2\Delta s^a,\quad \Delta s^a=-X^{-1}(xs)-D^{-2}\Delta s^a. \tag{17}$$

Theorem 6.3 *If $\|\Delta x^a\|$ and $\|\Delta s^a\|$ are obtained from (3), and $\mu_g=\frac{x^{\mathrm{T}}s}{n}$, then*

$$\|\Delta x^a\|=O(\mu_g),\quad \|\Delta s^a\|=O(\mu_g).$$

Proof Due to Lemma 6.2, we only need to prove

$$\|(\Delta x^a)_B\|=O(\mu_g),\quad \|(\Delta s^a)_N\|=O(\mu_g).$$

Using Lemma 6.1 and (17), we have

$$\begin{aligned}\|(\Delta x^a)_B\|&=\|S_B^{-1}(xs)_B+D_B(D_B\Delta s_B^a)\|\\&\le\|S_B^{-1}\|\|(XS)_B\|+\|D_B\|\|D_B\Delta s_B^a\|\\&=O(1/\xi)O(\mu_g)+O(\sqrt{\mu_g})O(\sqrt{\mu_g})=O(\mu_g).\end{aligned}$$

Similarly, one has $\|(\Delta s^a)_N\|=O(\mu_g)$. The proof is completed. □

References

[1] S. MEHROTRA. *On finding a vertex solution using interior point methods*. Linear Algebra Appl., 1991, **152**: 233–253.

[2] S. MEHROTRA. *On the implementation of a (primal-dual) interior point method*. SIAM Journal on Optimization, 1992, **2**: 576–601.

[3] M. SALAHI, J. PENG, T. TERLAKY. *On Mehrotra-type predictor-corrector algorithms*. SIAM J. Optim., 2007, **18**(4): 1377–1397.

[4] M. SALAHI, T. TERLAKY. *Mehrotra-type predictor-corrector algorithm revisited*. Optim. Methods Softw., 2008, **23**(2): 259–273.

[5] M. KOJIMA, N. MEGIDDO, T. NOMA. *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*. Springer-Verlag, Berlin, 1991.

[6] S. J. WRIGHT. *Primal-Dual Interior-Point Methods*. Philadelphia, PA, 1997.

[7] C. ROOS, T. TERLAKY, J. P. VIAL. *Interior Point Methods for Linear Optimization*. Springer, New York, 2006.

[8] T. ILLÉS, M. NAGY. *A Mizuno-Todd-Ye type predictor-corrector algorithm for sufficient linear complementarity problems*. European J. Oper. Res., 2007, **181**(3): 1097–1111.

[9] Yinyu YE, K. ANSTREICHER. *On quadratic and $O(\sqrt{n}L)$ convergence of a predictor-corrector algorithm for LCP*. Math. Programming, Ser. A, 1993, **62**(3): 537–551.

[10] O. GÜLER, Yinyu YE. *Convergence behavior of interior-point algorithms*. Math. Programming, Ser. A, 1993, **60**(2): 215–228.