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Multiple Positive Solutions of Nonlocal Boundary Value Problems for p-Laplacian Equations with Fractional Derivative

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Abstract In this paper, we study the multiple positive solutions of integral boundary value problems for a class of p-Laplacian differential equations involving the Caputo fractional derivative. Using a fixed point theorem due to Avery and Peterson, we obtain the existence of at least three positive decreasing solutions of the nonlocal boundary value problems. We give an example to illustrate our results.

Keywords p-Laplacian differential equations; Caputo fractional derivative; integral boundary value problems; positive decreasing solutions; Avery-Peterson fixed point theorem.

MR(2010) Subject Classification 34B15; 26A33

1. Introduction

In this paper, we are concerned with the multiple positive decreasing solutions of integral boundary value problems for a class of p-Laplacian differential equations involving the Caputo fractional derivative

$$\begin{cases} (\varphi_p({}^C D^\alpha x(t)))' + \varphi_p(\lambda) f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(1) = \int_0^1 g(t)x(t)\mathrm{d}t, \\ x^{(k)}(0) = 0, \;\; k = 1, 2, 3, \ldots, [\alpha] \end{cases} \tag{1.1}$$

where φ_p is the p-Laplacian operator, i.e., $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and $\varphi_q = \varphi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$; ${}^C D^\alpha$ is the standard Caputo derivative; $2 \le n-1 = [\alpha] < \alpha < n \in \mathbb{N}$, $0 < \lambda \in \mathbb{R}$ are constants; f and g are given functions. By means of a fixed point theorem due to Avery and Peterson, we obtain some new results on the existence of multiple positive decreasing solutions.

The equation with the p-Laplacian operator arises in the models of different physical, natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, and so on. During the past decades, wide attention has been paid to the study of the boundary value problems with the p-Laplacian operator, and there exist many papers devoted to the existence of positive solutions on this topic (see [1]–[5] and the references therein). In [1], Jin and Lu studied the

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existence for the third-order multi-point boundary value problem with the p-Laplacian operator

$$\begin{cases} (\varphi_p(x''(t)))' = f(t, x(t), x'(t), x''(t)), \quad t \in (0,1), \\ x(0) = 0, \ \ x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i), \ \ x''(0) = 0. \end{cases}$$

In [2], authors considered the existence of multiple positive solutions for second-order impulsive differential equations with the p-Laplacian and integral boundary conditions

$$\begin{cases} -(\varphi_p(u'))' = f(t, u(t)), \quad t \neq t_k, \ \ t \in (0,1), \\ -\Delta u \mid_{t=t_k} = I_k(u(t_k)), \ \ k = 1, 2, \ldots, n, \\ u'(0) = 0, \ \ u(1) = \int_0^1 g(t)u(t)\mathrm{d}t. \end{cases}$$

Recently, many authors are interested in studying the existence of solutions of boundary value problems for fractional differential equation (see [6]–[12] and the references therein). In [12], Jiang and Yuan considered the existence and multiplicity of solutions of boundary value problem for nonlinear fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$ is a real number and D_{0+}^{α} is the standard Riemann-Liouville derivative.

In [7], we studied the existence of multiple solutions for the boundary value problems

$$\begin{cases} {}^{C}D^{q} x(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \\ g_0(x(0), x'(0)) = 0, \\ g_1(x(1), x'(1)) = 0, \\ x^{(k)}(0) = 0, \ \ k = 2, 3, \ldots, [q]. \end{cases}$$

However, there are few results on the existence of multiple positive solutions of the integral boundary value problems for the p-Laplacian operator with fractional derivatives. In this paper, we focus on the multiple positive solutions of nonlocal boundary value problems for the p-Laplacian equation involving fractional differential equations with integral boundary conditions. By applying a theorem due to Avery and Peterson, we obtain a new result on the existence of at least three distinct positive decreasing solutions under the certain conditions.

2. Preliminaries

For the sake of clarity, we list the necessary definitions from fractional calculus theory here. These definitions can be found in the recent literature.

Definition 2.1 ([6]) *Let $\alpha > 0$, for a function $y : (0, +\infty) \to \mathbb{R}$. The fractional integral of order α of y is defined by*

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s,$$

provided the integral exists. The Caputo derivative of a function $y : (0, +\infty) \to R$ is given by

$$^{C}D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}} \mathrm{d}s,$$

provided the right side is pointwise defined on $(0, +\infty)$, *where* $n = [\alpha] + 1$, *and* $[\alpha]$ *denotes the integer part of the real number* α. Γ *denotes the Gamma function:*

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} \mathrm{d}t.$$

Throughout the paper, we suppose that the following condition holds.

(H) $g \in L[0,1]$ is a nonnegative function and $0 < \int_0^1 g(t)\mathrm{d}t < 1$.

Lemma 2.1 *Suppose that (H) holds and* $h \in C[0,1]$. *Then the boundary value problems*

$$\begin{cases} (\varphi_p({}^C D^\alpha x(t)))' + \varphi_p(\lambda) h(t) = 0, \ t \in (0,1), \\ x(1) = \int_0^1 g(t)x(t)\mathrm{d}t, \\ x^{(k)}(0) = 0, \ \ k = 1, 2, 3, \dots, [\alpha] \end{cases} \tag{2.1}$$

is equivalent to integral equation

$$\begin{aligned} x(t) =& \frac{\lambda}{(1 - \int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)} \Big(\int_0^1 g(s) \Big(\int_0^1 (1-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(r)\mathrm{d}r)\mathrm{d}\tau - \\ & \int_0^s (s-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(r)\mathrm{d}r)\mathrm{d}\tau \Big) \mathrm{d}s \Big) + \\ & \frac{\lambda}{\Gamma(\alpha)} \Big(\int_0^1 (1-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau - \int_0^t (t-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau \Big). \end{aligned} \tag{2.2}$$

That is, each solution of (2.1) is also a solution of (2.2) and vice versa.

Proof It is easy to see that the definition of the Caputo derivative implies that ${}^C D^\alpha x(0) = 0$.

By (2.1), we have

$$\varphi_p \left({}^C D^\alpha x\,(t) \right) = -\varphi_p(\lambda) \int_0^t h(s)\mathrm{d}s,$$

and

$${}^C D^\alpha x(t) = -\lambda \varphi_q(\int_0^t h(s)\mathrm{d}s).$$

Then

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \frac{x''(0)}{2!} t^2 + \dots + \frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1} - I^\alpha \lambda \varphi_q(\int_0^t h(s)\mathrm{d}s) \\ &= x(0) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau. \end{aligned} \tag{2.3}$$

From the boundary condition $x(1) = \int_0^1 g(t)x(t)\mathrm{d}t$ of (2.1), we can obtain that

$$x(0) = \int_0^1 g(t)x(t)\mathrm{d}t + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau.$$

Hence,

$$x(t) = \int_0^1 g(s)x(s)\mathrm{d}s + \frac{\lambda}{\Gamma(\alpha)} \Big(\int_0^1 (1-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau - \int_0^t (t-\tau)^{\alpha-1} \varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau \Big). \tag{2.4}$$

By (2.4), it is easy to see that

$$\int_0^1 g(t)x(t)\mathrm{d}t = \frac{\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\Big(\int_0^1 g(s)\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau h(r)\mathrm{d}r)\mathrm{d}\tau - \int_0^s (s-\tau)^{\alpha-1}\varphi_q(\int_0^\tau h(r)\mathrm{d}r)\mathrm{d}\tau\Big)\mathrm{d}s\Big). \tag{2.5}$$

Substituting (2.5) into (2.4), we can get (2.2).

On the other hand, it is easy to verify that each solution of (2.2) is also a solution of (2.1). □

Denote

$$M = \frac{1-\int_0^1 tg(t)\mathrm{d}t}{1-\int_0^1 g(t)\mathrm{d}t} \quad \text{and} \quad N = \frac{\int_0^1 (1-t)g(t)\mathrm{d}t}{1-\int_0^1 tg(t)\mathrm{d}t}.$$

Obviously, $M > 1$, $0 < N < 1$ if (H) holds.

Lemma 2.2 *Suppose that (H) holds and* $h \in C[0,1]$, $h(t) \geq 0$, $t \in [0,1]$. *Then*

(1) The solutions of the boundary value problems (2.1) are nonnegative, decreasing and concave on $[0,1]$*;*

(2) The solution x *of the boundary value problems (2.1) satisfies*

$$\max_{t\in[0,1]} |x(t)| \leq M \max_{t\in[0,1]} |x'(t)|,$$

and

$$\min_{t\in[0,1]} x(t) = x(1) \geq N \max_{t\in[0,1]} x(t) = Nx(0).$$

Proof (1) Let x be a solution of boundary value problems (2.1). It follows from (2.3) that

$$x'(t) = -\frac{\lambda}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}\varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau \leq 0,$$

and

$$x''(t) = -\frac{\lambda}{\Gamma(\alpha-2)}\int_0^t (t-\tau)^{\alpha-3}\varphi_q(\int_0^\tau h(s)\mathrm{d}s)\mathrm{d}\tau \leq 0.$$

This implies that $x(t)$ is decreasing and concave on [0,1], and

$$\min_{t\in[0,1]} x(t) = x(1) = \frac{\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\int_0^1 g(s)\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau h(r)\mathrm{d}r)\mathrm{d}\tau - \int_0^s (s-\tau)^{\alpha-1}\varphi_q(\int_0^\tau h(r)\mathrm{d}r)\mathrm{d}\tau\Big)\mathrm{d}s \geq 0.$$

Therefore, $x(t) \geq 0$ for $t \in [0,1]$.

(2) It follows from (1)

$$\max_{t\in[0,1]} |x(t)| = x(0), \quad \max_{t\in[0,1]} |x'(t)| = -x'(1),$$

and

$$x(t) \geq tx(1) + (1-t)x(0).$$

Since $x(1) = \int_0^1 g(t)x(t)\mathrm{d}t$, $g(t) \geq 0$, $t \in [0,1]$, $0 < \int_0^1 g(t)\mathrm{d}t < 1$, we have

$$x(0) \leq \frac{1-\int_0^1 tg(t)\mathrm{d}t}{\int_0^1 (1-t)g(t)\mathrm{d}t}x(1). \tag{2.6}$$

On the other hand, $x(1) - x(t) \geq (1-t)x'(1)$, it is easy to get

$$x(1) \leq \frac{\int_0^1 (1-t)g(t)\mathrm{d}t}{1 - \int_0^1 g(t)\mathrm{d}t}(-x'(1)). \tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$x(0) \leq \frac{1 - \int_0^1 tg(t)\mathrm{d}t}{1 - \int_0^1 g(t)\mathrm{d}t}(-x'(1)) = M(-x'(1)).$$

That is $\max_{t\in[0,1]} |x(t)| \leq M \max_{t\in[0,1]} |x'(t)|$, and $x(1) \geq Nx(0)$ from (2.6). $\square$

Definition 2.2 *Let E be a Banach space and $P \subset E$ be a cone. A continuous map γ is called a concave (resp., convex) functional on P if and only if*

$$\gamma(tx + (1-t)y) \geq t\gamma(x) + (1-t)\gamma(y)$$

$$(\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)),$$

for all $x, y \in P$ and $0 < t < 1$.

Let β and ρ be nonnegative continuous convex functionals on a cone P, ω be a nonnegative continuous concave functional on a cone P, and ψ be a nonnegative continuous functional on a cone P. Then for positive real numbers a, b, c and d, we define the following convex sets

$$P(\beta; d) = \{x \in P \mid \beta(x) < d\},\ \overline{P}(\beta; d) = \{x \in P \mid \beta(x) \leq d\};$$

$$P(\beta, \omega; b, d) = \{x \in P \mid \beta(x) \leq d, \omega(x) \geq b\};$$

$$P(\beta, \rho, \omega; b, c, d) = \{x \in P \mid \beta(x) \leq d, \rho(x) \leq c, \omega(x) \geq b\};$$

$$R(\beta, \psi; a, d) = \{x \in P \mid \beta(x) \leq d, \psi(x) \geq a\}.$$

Lemma 2.3 (Avery-Peterson fixed point theorem [13]) *Let P be a cone in a real Banach space E. Let β and ρ be nonnegative continuous convex functionals on P, ω be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(kx) \leq k\psi(x)$ for $0 < k < 1$, such that for some positive numbers M and d, $\omega(x) \leq \psi(x)$ and $||x|| \leq M\beta(x)$, for all $x \in \overline{P}(\beta, d)$.*

Suppose $T : \overline{P}(\beta; d) \to \overline{P}(\beta; d)$ is completely continuous and there exist positive numbers a, b and c with $a < b$, such that

(A1) $\{x \in P(\beta, \rho, \omega; b, c, d) | \omega(x) > b\} \neq \phi$, and $\omega(Tx) > b$ for all $x \in P(\beta, \rho, \omega; b, c, d)$;

(A2) $\omega(Tx) > b$, for all $x \in P(\beta, \omega; b, d)$ with $\rho(Tx) > c$;

(A3) $\theta \notin R(\beta, \psi; a, d)$ and $\psi(Tx) < a$ for $x \in R(\beta, \psi; a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P}(\beta; d)$, such that $\beta(x_i) < d$, $i = 1, 2, 3$; $\omega(x_1) > b$; $\psi(x_2) > a$, with $\omega(x_2) < b$; and $\psi(x_3) < a$.

3. Main results and their proof

Let $E = C^1[0,1]$, $\| x \| = \max\left\{\max_{t\in[0,1]} | x(t) |,\ \max_{t\in[0,1]} | x'(t) |\right\}$, and

$$P = \Big\{x \in E\ \mid x(t) \geq 0,\ \ t \in [0,1],\ \ \min_{t\in[0,1]} x(t) \geq N \max_{t\in[0,1]} x(t),\ \ \max_{t\in[0,1]} x(t) \leq M \max_{t\in[0,1]} |x'(t)|,$$

$$x \text{ is decreasing and concave on } [0,1]\Big\}.$$

Then $(E, \|\cdot\|)$ is a Banach space and P is a cone in E.

Denote

$$r_1 = \frac{\Gamma(\alpha+q-1)}{\lambda\Gamma(q)},\ r_2 = \frac{\Gamma(\alpha+q)}{\lambda N\Gamma(q)} \text{ and } r_3 = \frac{\Gamma(\alpha+q)(1-\int_0^1 g(t)\mathrm{d}t)}{\lambda\Gamma(q)}.$$

Then r_1, r_2, $r_3 > 0$, and it is easy to see that $r_1 d > r_2 b$ for $\frac{N}{\alpha+q-1}d > b$.

Theorem 3.1 *Suppose (H) holds and $f \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, there exist constants a, b, c and d with $0 < a < b < Nc \le \frac{N}{\alpha+q-1}d < d$, such that*

(C1) $0 \le f(t,u,v) \le \varphi_p(r_1 d)$, *for any* $(t,u,v) \in [0,1]\times[0,Md]\times[-d,0]$;

(C2) $f(t,u,v) > \varphi_p(r_2 b)$, *for any* $(t,u,v) \in [0,1]\times[b,c]\times[-d,0]$;

(C3) $0 \le f(t,u,v) < \varphi_p(r_3 a)$, *for any* $(t,u,v) \in [0,1]\times[Na,a]\times[-d,0]$.

Then the boundary value problem (1.1) has at least three positive decreasing solutions x_1, x_2, $x_3 \in \overline{P}(\beta;d)$ such that

$$\max_{t\in[0,1]} |x_i'(t)| < d,\ i=1,2,3;\ \min_{t\in[0,1]} |x_1(t)| > b;$$

$$\max_{t\in[0,1]} x_2(t) > a \ \text{ with } \ \min_{t\in[0,1]} x_2(t) < b; \text{ and } \max_{t\in[0,1]} x_3(t) < a.$$

Proof Define nonnegative continuous convex functionals β, ρ, nonnegative continuous functional ψ, and nonnegative continuous concave functional ω on the cone P by

$$\beta(x) = \max_{t\in[0,1]} |x'(t)|,\ \ \rho(x) = \psi(x) = \max_{t\in[0,1]} |x(t)|, \text{ and } \omega(x) = \min_{t\in[0,1]} |x(t)|,$$

respectively. Obviously, $\psi(kx) = k\psi(x)$ for any $k \in (0,1)$ and $x \in P$, and

$$\|x\| = \max\{\max_{t\in[0,1]} |x(t)|,\ \max_{t\in[0,1]} |x'(t)|\} = \max\{\rho(x),\ \beta(x)\} \le M\beta(x), \text{ for } x \in P.$$

Define the operator $T: P \to E$ by

$$\begin{aligned}
&(Tx)(t)\\
&= \frac{\lambda}{\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(s,x(s),x'(s))\mathrm{d}s)\mathrm{d}\tau -\\
&\quad \int_0^t (t-s)^{\alpha-1}\varphi_q(\int_0^\tau f(s,x(s),x'(s))\mathrm{d}s)\mathrm{d}\tau\Big) +\\
&\quad \frac{\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\int_0^1 g(s)\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau -\\
&\quad \int_0^s (s-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau\Big)\mathrm{d}s. \qquad (3.1)
\end{aligned}$$

It is clear that the fixed points of the operator T are the solutions of the boundary value problems (1.1).

Since $x \in \overline{P}(\beta;\, d)$ implies that $x(t) \ge 0$ and

$$\rho(x) = \max_{t\in[0,1]} |x(t)| \le M \max_{t\in[0,1]} |x'(t)| = \beta(x) \le M\beta(x) \le Md,$$

by condition (C1), we have $0 \le f(t,x(t),x'(t)) \le \varphi_p(r_1 d)$. It follows from Lemma 2.2 that

$$\beta(Tx) = -(Tx)'(1) = \frac{\lambda}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\varphi_q(\int_0^\tau f(s,x(s),x'(s))\mathrm{d}s)\mathrm{d}\tau$$
$$\le \frac{\lambda r_1 d}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\tau^{q-1}\mathrm{d}\tau = \frac{\lambda\Gamma(q)}{\Gamma(\alpha+q-1)} r_1 d = d.$$

Therefore, $T : \overline{P}(\beta; d) \to \overline{P}(\beta; d)$. By Lemma 2.2 and (3.1), it is easy to see that T is a completely continuous operator.

Let $x_0 = \frac{b+c}{2}$. Then $\rho(x_0) = x_0 = \frac{b+c}{2} \le c$, $\beta(x_0) = 0 \le d$, and $\omega(x_0) = x_0 = \frac{b+c}{2} > b$, so

$$x_0 \in \{x \in P(\beta,\rho,\omega;\, b,c,d) | \omega(x) > b\} \ne \phi.$$

For $x \in P(\beta,\rho,\omega; b,c,d)$, it follows from condition (C2) that $f(t,x(t),x'(t)) > \varphi_p(r_2 b)$. By Lemma 2.2 and (3.1), we can get

$$\begin{aligned}
\omega(Tx) = &\min_{t\in[0,1]} |(Tx)(t)| = (Tx)(1) \ge N(Tx)(0)\\
=&\frac{N\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau -\\
&\int_0^1 g(s)\Big(\int_0^s (s-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau\Big)\mathrm{d}s\Big)\\
\ge&\frac{N\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau -\\
&\int_0^1 g(s)\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau\Big)\mathrm{d}s\Big)\\
=&\frac{N\lambda}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau\\
>&\frac{\lambda r_2 N b}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{q-1}\mathrm{d}\tau\\
=&\frac{\lambda r_2 N b}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)\Gamma(q)}{\Gamma(\alpha+q)} = b.
\end{aligned}$$

Thus, condition (A1) in Avery-Peterson theorem is satisfied.

For $x \in P(\beta,\omega; b,d)$ with $\rho(Tx) > c$, we get

$$\rho(Tx) = \max_{t\in[0,1]} |(Tx)(t)| = (Tx)(0) > c.$$

Then

$$\omega(Tx) = \min_{t\in[0,1]} |(Tx)(t)| = (Tx)(1) \ge N(Tx)(0) > Nc > b.$$

Consequently, condition (A2) in Avery- Peterson theorem is satisfied.

It is clear that $\theta = 0 \notin R(\beta,\psi;\, a,d)$.

For $x \in R(\beta,\psi; a,d)$ with $\psi(x) = a$, we have

$$\beta(x) = \max_{t\in[0,1]} |x'(t)| \le d \quad\text{and}\quad \max_{t\in[0,1]} x(t) = a.$$

It is easy to see

$$\min_{t\in[0,1]} x(t) \geq N \max_{t\in[0,1]} x(t) = Na,$$

for $x \in P$. By condition (C3), we have

$$\begin{aligned}
\psi(Tx) &= \max_{t\in[0,1]} |(Tx)(t)| = (Tx)(0)\\
&= \frac{\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau -\\
&\quad \int_0^1 g(s)\Big(\int_0^s (s-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau\Big)\mathrm{d}s\Big)\\
&\leq \frac{\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-1}\varphi_q(\int_0^\tau f(r,x(r),x'(r))\mathrm{d}r)\mathrm{d}\tau\Big)\\
&\leq \frac{r_3 a\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{q-1}\mathrm{d}\tau\\
&= \frac{r_3 a\lambda}{(1-\int_0^1 g(s)\mathrm{d}s)\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)\Gamma(q)}{\Gamma(\alpha+q)} = a.
\end{aligned}$$

So, condition (A3) in Avery-Peterson theorem holds.

By Avery-Peterson theorem, the operator T has at least three fixed points x_1, x_2, $x_3 \in \overline{P}(\beta, d)$ and

$$\beta(x_i) < d,\ i = 1,2,3;\ \ \omega(x_1) > b;\ \psi(x_2) > a, \text{ with } \omega(x_2) < b; \text{ and } \psi(x_3) < a.$$

Thus, the boundary value problem (1.1) has at least three positive decreasing solutions x_1, x_2, $x_3 \in \overline{P}(\beta; d)$ such that

$$\max_{t\in[0,1]} |x_i'(t)| < d,\ \ i = 1,2,3;\ \min_{t\in[0,1]} |x_1(t)| > b;$$

$$\max_{t\in[0,1]} x_2(t) > a,\ \text{ with } \min_{t\in[0,1]} x_2(t) < b;\ \text{and } \max_{t\in[0,1]} x_3(t) < a.\ \ \Box$$

4. Illustration

In this section, we give an example to demonstrate the application of Theorem 3.1 in Section 3.

Example 4.1 Let $p = \frac{5}{2}$. We consider the following boundary value problems

$$\begin{cases}
(\varphi_p({}^C D^{3.5}x(t)))' + \varphi_p(\frac{2}{3})\Big(\frac{5000\left(\pi+2x^4(\arctan(2.5+x))^3\right)}{100+x^5} + \sin^2(tx'(t))\Big) = 0, \quad t \in (0,1),\\
x(1) = \int_0^1 e^{-t}x(t)\mathrm{d}t,\\
x^{(k)}(0) = 0,\ \ k = 1,2,3.
\end{cases} \tag{4.1}$$

Conclusion The boundary value problem (4.1) has at least three positive decreasing solutions x_1, x_2, x_3, such that

$$\max_{t\in[0,1]} |x_i'(t)| < 30,\ \ i = 1,2,3;\ \min_{t\in[0,1]} |x_1(t)| > 2;$$

$$\max_{t\in[0,1]} x_2(t) > 1, \text{ with } \min_{t\in[0,1]} x_2(t) < 2; \text{ and } \max_{t\in[0,1]} x_3(t) < 1.$$

Proof Let $f(t,u,v) = \frac{5000\left(\pi+2u^4(\arctan(2.5+u)^3\right)}{100+u^5} + \sin^2(tv)$, $g(t) = e^{-t}$, $\lambda = \frac{2}{3}$, $\alpha = \frac{7}{2}$.

Obviously, $\alpha > 2$, $g \in L[0,1]$ is a nonnegative function, and $0 < \int_0^1 g(t)\mathrm{d}t = \int_0^1 e^{-t}\mathrm{d}t = 1 - \frac{1}{e} < 1$. Then (H) holds,

$$M = \frac{1 - \int_0^1 tg(t)\mathrm{d}t}{1 - \int_0^1 g(t)\mathrm{d}t} = 2 \text{ and } N = \frac{\int_0^1 (1-t)g(t)\mathrm{d}t}{1 - \int_0^1 tg(t)\mathrm{d}t} = \frac{1}{2},$$

$$r_1 = \frac{\Gamma(\alpha+q-1)}{\lambda\Gamma(q)} = \frac{3\Gamma(\frac{25}{6})}{2\Gamma(\frac{5}{3})} = 12.3391, \ r_2 = \frac{\Gamma(\alpha+q)}{\lambda N\Gamma(q)} = \frac{3\Gamma(\frac{31}{6})}{\Gamma(\frac{5}{3})} = 102.826, \text{ and}$$

$$r_3 = \frac{\Gamma(\alpha+q)(1-\int_0^1 g(t)\mathrm{d}t)}{\lambda\Gamma(q)} = \frac{6+3e\Gamma(\frac{31}{6})}{2\Gamma(\frac{5}{3})} = 52.636.$$

Take $a=1$, $b=2$, $c=5$, $d=30$, then $0 < a = 1 < b = 2 < Nc = 2.5 \le \frac{N}{\alpha+q-1}d = 3.6 < d = 30$

It is easy to see that $f \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. Furthermore,

$0 \le f(t,u,v) \le 6700 < \varphi_p(r_1 d) = 7122.09$, for any $(t,u,v) \in [0,1]\times[0,Md]\times[-d,0] = [0,1]\times[0,60]\times[-30,0]$, thus condition (C1) holds;

$f(t,u,v) \ge 3110 > \varphi_p(r_2 b) = 2949.16$, for any $(t,u,v) \in [0,1]\times[b,c]\times[-d,0] = [0,1]\times[2,5]\times[-30,0]$, thus condition (C2) holds;

$0 \le f(t,u,v) \le 372 < \varphi_p(r_3 a) = 381.872$, for any $(t,u,v) \in [0,1]\times[Na,a]\times[-d,0] = [0,1]\times[\frac{1}{2},1]\times[-30,0]$, thus condition (C3) holds.

The conditions of Theorem 3.1 are all satisfied. So by Theorem 3.1, the boundary value problems (4.1) has at least three distinct solutions x_1, x_2, x_3, and moreover,

$$\max_{t\in[0,1]} |x_i'(t)| < d = 30, \ i=1,2,3; \ \min_{t\in[0,1]} |x_1(t)| > b = 2;$$

$$\max_{t\in[0,1]} x_2(t) > a = 1, \text{ with } \min_{t\in[0,1]} x_2(t) < b = 2; \text{ and } \max_{t\in[0,1]} x_3(t) < a = 1. \ \square$$

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