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Some Endpoint Estimates for the Multiplier Operator

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Abstract In this paper, the author proves that Multiplier operator is bounded on $\mathrm{BMO}(R^n)$, $\mathrm{LMO}(R^n)$ and $\mathrm{CBMO}^{p,\lambda}(R^n)$ respectively if some concellation conditions are satisfied.

Keywords multiplier operator; BMO space; LMO space; λ-central BMO space.

MR(2010) Subject Classification 42B25; 42B30

1. Introduction

We first recall some basic concepts and results on multiplier. Let $m(x)$ be a bounded function on R^n and consider the multiplier operator T defined initially for a function f in the Schwartz space $\mathcal{S}$ by $\widehat{Tf}(x) = m(x)\widehat{f}(x)$, where $\widehat{f}$ is the Fourier transform of f. For $s \geq 1$ and ℓ a positive real number, we say $m \in M(s,\ell)$ if

$$\sup_{R>0}\Big(R^{s|\alpha|-n}\int_{R<|x|<2R}|D^\alpha m(x)|^s\mathrm{d}x\Big)^{1/s} < \infty, \quad \text{for all } |\alpha| \leq \ell. \tag{1.1}$$

The condition (1.1) has been known to be related to multiplier theorems of Hörmander-Mikhlin [1]. Using interpolation methods, Calderón and Torchinsky [2] considered the condition $m \in M(s,\ell)$ for $s \geq 2$ and $\ell > n/s$. Kurtz and Wheeden [3] obtained the following weighted norm inequality for multiplier operator when $m \in M(s,\ell)$ with $1 < s \leq 2$ and $n/s < \ell \leq n$. Now, we consider the multipliers. Following [3], we select an approximation to the identity

$$\sum_{j=-\infty}^{+\infty}\phi(2^{-j}x) = 1, \quad x \neq 0,$$

where ϕ is an infinitely differentiable, nonnegative function supported on $\frac{1}{2} < |x| < 2$. For $m \in M(s,l)$, let $m_j(x) = m(x)\phi(2^{-j}x)$, so that

$$m(x) = \sum_{j=-\infty}^{+\infty}m_j(x), \quad x \neq 0.$$

We have that $m_j \in L^1 \bigcap L^\infty$. Define $k_j(x) = (m_j)^\vee(x)$, where $g^\vee$ denotes the inverse Fourier

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transform of g, and let

$$m^N(x) = \sum_{j=-N}^{N} m_j(x), \quad K_N(x) = (m^N)^\vee(x) = \sum_{j=-N}^{N} k_j(x).$$

It follows that $\|m^N\|_\infty \le C$ uniformly in N and that $m^N(x) \to m(x)$, $x \ne 0$, as $N \to \infty$. Now define $T_N f(x) = (m^N \hat{f})^\vee(x)$, so that $T_N f = f * K_N$ for $f \in \mathcal{S}$. The following lemma shows how conditions on m can be interpreted as the conditions on K_N.

Lemma 1.1 ([3]) *Let $1 < s \le 2$, $m \in M(s, \ell)$ for a positive integer ℓ, and let K_N be defined as above. If d is an integer such that $0 < d \le \ell$, $1 < t \le s$, $n/t < d < n/t + 1$, and $1 \le p \le t'$, then for all $|y| \le \frac{R}{2}$,*

$$\Big(\int_{R<|x|<2R} |K_N(x-y) - K_N(x)|^p \mathrm{d}x\Big)^{1/p} \le CR^{-d+n/p-n/t'}|y|^{d-n/t},$$

$$\Big(\int_{R<|x|} |K_N(x-y) - K_N(x)|^p \mathrm{d}x\Big)^{1/p} \le CR^{-d+n/p-n/t'}|y|^{d-n/t},$$

and if $\ell > \max\{n/p', n/s\}$, then

$$\Big(\int_{R<|x|<2R} |K_N(x)|^p \mathrm{d}x\Big)^{1/p} \le CR^{n/p-n},$$

with C above independent of N, R and y.

Using this lemma, Kurtz and Wheeden [3] showed that the kernels K_N satisfy, uniformly in N, the Hörmander condition, so that T_N is bounded on L^p, uniformly in N, for $1 < p < \infty$. For $f \in \mathcal{S}$, we have $Tf(x) = (m\hat{f})^\vee(x)$. It follows that $\|Tf - T_N f\|_\infty \le \|(m - m^N)\hat{f}\|_1 \to 0$, since m^N converges pointwise and boundedly to m. Then, applying Fatou's lemma, we get $\|Tf\|_p \le C\|f\|_p$ for $1 < p < \infty$ and $f \in \mathcal{S}$.

We use p' to denote the index conjugate to p, that is, $1/p + 1/p' = 1$, $p \ge 1$.

Lemma 1.2 ([3]) *If $1 < s \le 2$, $n/s < l \le n$, $m \in M(s, l)$, and $\omega^{n/l} \in A_p$, then*

(1) For $1 < p < \infty$, $\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega}$ for a constant C independent of f.

(2) $\omega(\{x \in R^n : |Tf(x)| > \lambda\}) \le C\lambda^{-1}\|f\|_{1,\omega}$ for a constant C independent of f and $\lambda > 0$.

Lemma 1.3 ([3]) *If $1 < s \le 2$, $m \in M(s, n)$, and $\omega \in A_p$, then we have*

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega}, \quad 1 < p < \infty.$$

In 2010, Lin [7] obtained the boundedness of the Calderón-Zygmund type operator on BMO spaces, LMO spaces and λ-central BMO spaces. The conclusion essentially depends on the cancellation condition of the kernel. A natural question is: if we add a cancellation condition to the multiplier operator Tf, whether the multiplier operator Tf has the same results? Actually, the answer is affirmative. From [5], we know that if a multiplier is homogeneous of degree zero, the associated multiplier operator is a Calderón-Zygmund operator. But in this note, the homogeneous is not needed. So our theorems cannot be contained by the results in [7]. Before stating our results, we are going to give definitions of BMO, LMO and the center BMO spaces

$\mathrm{CBMO}^{p,\lambda}(R^n)$.

Definition 1.1 ([6]) *LMO is a subspace of BMO, equipped with the semi-norm*

$$[f]_{\mathrm{LMO}} = \sup_{0<r<1} \frac{1+|\ln r|}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\mathrm{d}x + \sup_{r\geq 1} \frac{1}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\mathrm{d}x,$$

where B_r denotes the ball in R^n with radius r.

For $1 \leq p < \infty$, define

$$[f]_{\mathrm{LMO}^p} = \sup_{0<r<\frac{1}{2}} (1+|\ln r|)\Big(\frac{1}{|B_r|}\int_{B_r} |f(x) - f_{B_r}|^p \mathrm{d}x\Big)^{1/p}.$$

Definition 1.2 ([9]) *Let $\lambda < 1/n$ and $1 < p < \infty$. A function $f \in L^p_{\mathrm{loc}}(R^n)$ is said to belong to the λ-central bounded mean oscillation space $\mathrm{CBMO}^{p,\lambda}(R^n)$ if*

$$\|f\|_{\mathrm{CBMO}^{p,\lambda}} = \sup_{r>0} \Big(\frac{1}{|B(0,r)|^{1+\lambda p}} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^p \mathrm{d}x\Big)^{1/p} < \infty. \tag{1.2}$$

We can formulate our results as follows:

Theorem 1.1 *Let $1 < s \leq 2$, $n/s < l \leq n$, $m \in M(s,l)$, and T be defined as above, and $T_N 1 = 0$ for any real positive integer number N. Suppose $f \in \mathrm{BMO}$ such that $T_N f(x)$ exists a.e. in R^n. Then $Tf \in \mathrm{BMO}$ and*

$$\|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}},$$

where $C > 0$ is independent of f.

Theorem 1.2 *Let $1 < s \leq 2$, $n/s < l \leq n$, $m \in M(s,l)$, and T be defined as above, and $T_N 1 = 0$ for any real positive integer number N. Suppose $f \in \mathrm{LMO}$ such that $T_N f(x)$ exists a.e. in R^n. Then $Tf \in \mathrm{LMO}$ and*

$$[Tf]_{\mathrm{LMO}} \leq C[f]_{\mathrm{LMO}},$$

where $C > 0$ is independent of f.

Theorem 1.3 *Let $1 < s \leq 2$, $n/s < l \leq n$, $m \in M(s,l)$, and T be defined as above, and $T_N 1 = 0$ for any real positive integer number N. Suppose $f \in \mathrm{CBMO}^{p,\lambda}$ such that $T_N f(x)$ exists a.e. in R^n, where $q' \leq p < \infty$ and $-1/n < \lambda \leq 0$. Then $Tf \in \mathrm{CBMO}^{p,\lambda}$ and*

$$\|Tf\|_{\mathrm{CBMO}^{p,\lambda}} \leq C\|f\|_{\mathrm{CBMO}^{p,\lambda}},$$

where $C > 0$ is independent of f.

2. Proofs of theorems

Proof of Theorem 1.1 If we prove the following inequality:

$$\|T_N f\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}.$$

Then, by applying Fatou's lemma, we will get

$$\|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}.$$

Therefore, in order to prove Theorem 1.1, we only need to establish the (BMO,BMO)-boundedness of T_N.

For any ball $B = B(x_0, r) \subset R^n$, $r > 0$, write

$$\begin{aligned} f(x) &= f_{2B} + (f(x) - f_{2B})\chi_{8B}(x) + (f(x) - f_{2B})\chi_{(8B)^c}(x) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

The assumption $T_N 1 = 0$ tells that $T_N f_1 = 0$.

Thanks to Hölder's inequality and the $L^2(R^n)$-boundedness of T_N, we have

$$\begin{aligned} \frac{1}{|B|}\int_B |T_N f_2(x)|\mathrm{d}x &\le \Big(\frac{1}{|B|}\int_B |T_N f_2(x)|^2 \mathrm{d}x\Big)^{1/2} \\ &\le C\Big(\frac{1}{|B|}\int_{R^n} |f_2(x)|^2 \mathrm{d}x\Big)^{1/2} \\ &\le C\Big(\frac{1}{|8B|}\int_{8B} |f(x) - f_{2B}|^2 \mathrm{d}x\Big)^{1/2} \\ &\le C\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Since $T_N f(x)$ and $T_N f_2(x)$ exist a.e. in R^n, there is a point $z \in B$ such that $T_N f_3(z) < \infty$. For $x \in B$, $z \in B$ and $y \in (8B)^c$, we have $|y - z| \ge 2|x - z|$. Choosing t, r such that $n/d < t < \min(s, p)$, $n/t < d < n/t + 1$, $t < r < p$ and by using Lemma 1.1 and Hölder's inequality, we obtain

$$\begin{aligned} &\frac{1}{|B|}\int_B |T_N f_3(x) - T_N f_3(z)|\mathrm{d}x \\ &\quad\le \frac{1}{|B|}\int_B \int_{(8B)^c} |K_N(x-y) - K_N(z-y)||f(y) - f_{2B}|\mathrm{d}y\mathrm{d}x \\ &\quad\le \frac{1}{|B|}\int_B \sum_{j=1}^{\infty} \int_{2^j|x-z|\le|y-z|<2^{j+1}|x-z|} |K_N(x-y) - K_N(z-y)||f(y) - f_{2B}|\mathrm{d}y\mathrm{d}x \\ &\quad\le \frac{1}{|B|}\int_B \sum_{j=1}^{\infty} \Big(\int_{2^j|x-z|\le|y-z|<2^{j+1}|x-z|} |K_N(x-y) - K_N(z-y)|^{r'}\mathrm{d}y\Big)^{\frac{1}{r'}} \times \\ &\qquad \Big(\int_{2^j|x-z|\le|y-z|<2^{j+1}|x-z|} |f(y) - f_{2B}|^r \mathrm{d}y\Big)^{\frac{1}{r}}\mathrm{d}x \\ &\quad\le C\sum_{j=1}^{\infty} (2^j|x-z|)^{-d+n/r'-n/t'}|x-z|^{d-n/t}\Big(\int_{2^{j+1}B} |f(y) - f_{2B}|^r \mathrm{d}y\Big)^{\frac{1}{r}} \\ &\quad\le C\|f\|_{\mathrm{BMO}} \sum_{j=1}^{\infty} j2^{j(-d+n/t)} \le C\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{|B|}\int_B |T_N f(x) - (T_N f)_B|\mathrm{d}x &\le \frac{2}{|B|}\int_B |T_N f(x) - T_N f_3(z)|\mathrm{d}x \\ &\le \frac{2}{|B|}\int_B |T_N f_2(x)|\mathrm{d}x + \frac{2}{|B|}\int_B |T_N f_3(x) - T_N f_3(z)|\mathrm{d}x \\ &\le C\|f\|_{\mathrm{BMO}}, \end{aligned}$$

which concludes the proof of Theorem 1.1. □

Lemma 2.2 ([8]) *If $f \in \mathrm{LMO}$, then for any $1 \leq p < \infty$, there exists a constant $C > 0$ depending only on n and p such that $[f]_{\mathrm{LMO}^p} \leq C[f]_{\mathrm{LMO}}$.*

Next we present the proof of Theorem 1.2.

Proof of Theorem 1.2 For any ball $B = B(x_0, r) \subset R^n$ with $r \geq 1$, by BMO-boundedness of T in Theorem 1.1, we have

$$\frac{1}{|B|}\int_B |Tf(x) - (Tf)_B|\mathrm{d}x \leq \|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}} \leq C[f]_{\mathrm{LMO}}.$$

Therefore it suffices to prove that, for any ball $B = B(x_0, r) \subset R^n$ with $0 < r < 1$, the following inequality holds:

$$\frac{1+|\ln r|}{|B|}\int_B |Tf(x) - (Tf)_B|\mathrm{d}x \leq C[f]_{\mathrm{LMO}}. \tag{2.1}$$

Similarly, if the following inequaliity

$$\frac{1+|\ln r|}{|B|}\int_B |T_N f(x) - (T_N f)_B|\mathrm{d}x \leq C[f]_{\mathrm{LMO}}$$

is proved, we will establish the (LMO, LMO)-boundedness of T. We consider two cases respectively.

(i) $1/16 \leq r < 1$. The BMO-boundedness of T_N also implies that

$$\begin{aligned}\frac{1+|\ln r|}{|B|}\int_B |T_N f(x) - (T_N f)_B|\mathrm{d}x &= \frac{1+|\ln \frac{1}{r}|}{|B|}\int_B |T_N f(x) - (T_N f)_B|\mathrm{d}x\\ &\leq C\frac{1}{|B|}\int_B |T_N f(x) - (T_N f)_B|\mathrm{d}x \leq C\|T_N f\|_{\mathrm{BMO}}\\ &\leq C\|f\|_{\mathrm{BMO}} \leq C[f]_{\mathrm{LMO}}.\end{aligned}$$

(ii) $0 < r < 1/16$. Write

$$f(x) = f_{8B} + (f(x) - f_{8B})\chi_{8B}(x) + (f(x) - f_{8B})\chi_{(8B)^c}(x) = f_1(x) + f_2(x) + f_3(x).$$

The hypothesis $T_N 1 = 0$ says that $T_N f_1 = 0$. Noting that $0 \leq 8r < 1/2$ and employing Hölder's inequality, the (L^2, L^2)-boundedness of T_N, Lemma 2.2 and the following inequality

$$(1+|a+b|)^{-1} \leq (1+|a|)^{-1}(1+|b|), \quad \text{for any} \quad a, b \in R, \tag{2.2}$$

we obtain

$$\begin{aligned}\frac{1}{|B|}\int_B |T_N f_2(x)|\mathrm{d}x &\leq \Big(\frac{1}{|B|}\int_B |T_N f_2(x)|^2\mathrm{d}x\Big)^{1/2}\\ &\leq C\Big(\frac{1}{|8B|}\int_{8B} |f(x) - f_{8B}|^2\mathrm{d}x\Big)^{1/2}\\ &= C\frac{1+|\ln 8r|}{1+|\ln 8r|}\Big(\frac{1}{|8B|}\int_{8B} |f(x) - f_{8B}|^2\mathrm{d}x\Big)^{1/2}\\ &\leq C[f]_{\mathrm{LMO}^2}(1+|\ln 8r|)^{-1}\\ &\leq C[f]_{\mathrm{LMO}}(1+\ln 8)(1+|\ln r|)^{-1} \leq C\|f\|_{\mathrm{LMO}}(1+|\ln r|)^{-1}.\end{aligned}$$

Since $T_N f(x)$ and $T_N f_2(x)$ exist a.e. in R^n, there is a point $z \in B$ such that $T_N f_3(z) < \infty$. Since $r^{-1} > 16$, there exists a $k \in N$ with $k \geq 4$ such that $2^k < r^{-1} \leq 2^{k+1}$, namely $k \sim |\ln r|$. For $x \in B$, $z \in B$ and $y \in (8B)^c$, we have $|y - z| \geq 2|x - z|$. Choosing t, r such that $n/d < t < \min(s, p)$, $n/t < d < n/t + 1$, $t < r < p$ and by using Lemma 1.1 and Hölder's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{|B|}\int_B |T_N f_3(x) - T_N f_3(z)|\mathrm{d}x \\
&\quad \leq \frac{1}{|B|}\int_B \int_{(8B)^c} |K_N(x-y) - K_N(z-y)||f(y) - f_{8B}|\mathrm{d}y\mathrm{d}x \\
&\quad \leq \frac{1}{|B|}\int_B \sum_{j=1}^{\infty} \int_{2^j|x-z|\leq|y-z|<2^{j+1}|x-z|} |K_N(x-y) - K_N(z-y)||f(y) - f_{8B}|\mathrm{d}y\mathrm{d}x \\
&\quad \leq \frac{1}{|B|}\int_B \sum_{j=1}^{\infty} \Big(\int_{2^j|x-z|\leq|y-z|<2^{j+1}|x-z|} |K_N(x-y) - K_N(z-y)|^{r'}\mathrm{d}y\Big)^{\frac{1}{r'}} \times \\
&\qquad \Big(\int_{2^j|x-z|\leq|y-z|<2^{j+1}|x-z|} |f(y) - f_{8B}|^r\mathrm{d}y\Big)^{\frac{1}{r}}\mathrm{d}x \\
&\quad \leq C\sum_{j=1}^{\infty} (2^j|x-z|)^{-d+n/r'-n/t'}|x-z|^{d-n/t}\Big(\int_{2^{j+3}B} |f(y) - f_{8B}|^r\mathrm{d}y\Big)^{\frac{1}{r}} \\
&\quad \leq C\sum_{j=1}^{\infty} 2^{j(-d+n/t)}\Big[\Big(\frac{1}{|2^jB|}\int_{2^{j+3}B} |f(y) - f_{2^{j+3}B}|^r\mathrm{d}y\Big)^{\frac{1}{r}} + |f_{2^{j+3}B} - f_{8B}|\Big] \\
&\leq C\sum_{j=1}^{k-4} 2^{j(-d+n/t)}\Big[\Big(\frac{1}{|2^jB|}\int_{2^{j+3}B} |f(y) - f_{2^{j+3}B}|^r\mathrm{d}y\Big)^{\frac{1}{r}} + \sum_{i=3}^{j+2} |f_{2^{i+1}B} - f_{2^iB}|\Big] + \\
&\qquad \sum_{j=k-3}^{\infty} 2^{j(-d+n/t)}\Big[\Big(\frac{1}{|2^jB|}\int_{2^{j+3}B} |f(y) - f_{2^{j+3}B}|^r\mathrm{d}y\Big)^{\frac{1}{r}} + \sum_{i=3}^{j+2} |f_{2^{i+1}B} - f_{2^iB}|\Big] \\
&\quad = \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

We first estimate I. When $1 \leq j \leq k-4$ and $3 \leq i \leq j+2$, there are $0 < 2^{j+3}r < 1/2$ and $0 < 2^{j+1}r < 1/2$. It follows from Lemma 2.1 and inequality (2.2) that

$$
\begin{aligned}
\mathrm{I} \leq& C\sum_{j=1}^{k-4} 2^{j(-d+n/t)}\Big[\frac{1 + |\ln 2^{j+3}r|}{1 + |\ln 2^{j+3}r|}\Big(\frac{1}{|2^{j+3}B|}\int_{2^{j+3}B} |f(y) - f_{2^{j+1}B}|^r\mathrm{d}y\Big)^{\frac{1}{r}} + \\
&\sum_{i=3}^{j+2} \frac{1 + |\ln 2^{i+1}r|}{1 + |\ln 2^{i+1}r|}\Big(\frac{1}{|2^{i+1}B|}\int_{2^{i+1}B} |f(y) - f_{2^{i+1}B}|\mathrm{d}y\Big)\Big] \\
\leq& C\sum_{j=1}^{k-4} 2^{j(-d+n/t)}\Big(\frac{[f]_{\mathrm{LMO}^r}}{1 + |\ln 2^{j+3}r|} + \sum_{i=3}^{j+2} \frac{[f]_{\mathrm{LMO}}}{1 + |\ln 2^{i+1}r|}\Big) \\
\leq& C[f]_{\mathrm{LMO}} \sum_{j=1}^{k-4} 2^{j(-d+n/t)} \sum_{i=3}^{j+2} \frac{1}{1 + |\ln 2^{i+1}r|}
\end{aligned}
$$

$$
\begin{aligned}
&\le C[f]_{\mathrm{LMO}} \sum_{j=1}^{k-4} 2^{j(-d+n/t)} \sum_{i=3}^{j+2} \frac{i}{1+|\ln r|} \\
&\le C[f]_{\mathrm{LMO}} (1+|\ln r|)^{-1} \sum_{j=1}^{\infty} j^2 2^{j(-d+n/t)} \\
&\le C[f]_{\mathrm{LMO}} (1+|\ln r|)^{-1}.
\end{aligned}
$$

Now we estimate II.

$$
\begin{aligned}
\mathrm{II} \le C & \sum_{j=k-3}^{\infty} 2^{j(-d+n/t)} \Big[\Big(\frac{1}{|2^{j+3}B|} \int_{2^{j+3}B} |f(y) - f_{2^{j+3}B}|^r \mathrm{d}y \Big)^{\frac{1}{r}} + \\
& \sum_{i=3}^{j+2} \frac{1}{|2^i B|} \int_{2^i B} |f(y) - f_{2^{i+1}B}| \mathrm{d}y \Big] \\
& \le C \|f\|_{\mathrm{BMO}} \sum_{j=k-3}^{\infty} j 2^{j(-d+n/t)} \le C[f]_{\mathrm{LMO}} \sum_{j=k-3}^{\infty} \frac{j^2}{k+1} 2^{j(-d+n/t)} \\
& \le C[f]_{\mathrm{LMO}} \sum_{j=1}^{\infty} \frac{j^2}{1+|\ln r|} 2^{j(-d+n/t)} \le C[f]_{\mathrm{LMO}} (1+|\ln r|)^{-1}.
\end{aligned}
$$

Combining the above two estimates, we have

$$
\frac{1}{|B|} \int_B |T_N f_3(x) - T_N f_3(z)| \mathrm{d}x \le C[f]_{\mathrm{LMO}} (1+|\ln r|)^{-1}.
$$

Therefore

$$
\begin{aligned}
& \frac{1+|\ln r|}{|B|} \int_B |T_N f(x) - (T_N f)_B| \mathrm{d}x \\
& \le 2 \frac{1+|\ln r|}{|B|} \int_B |T_N f(x) - T_N f_3(z)| \mathrm{d}x \\
& \le 2 \frac{1+|\ln r|}{|B|} \int_B |T_N f_2(x)| \mathrm{d}x + 2 \frac{1+|\ln r|}{|B|} \int_B |T_N f_3(x) - T_N f_3(z)| \mathrm{d}x \\
& \le C[f]_{\mathrm{LMO}}.
\end{aligned}
$$

Then we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3 If we prove the following inequality: $\|T_N f\|_{\mathrm{CBMO}^{p,\lambda}} \le C\|f\|_{\mathrm{CBMO}^{p,\lambda}}$. Then, by applying Fatou's lemma, we get

$$
\|Tf\|_{\mathrm{CBMO}^{p,\lambda}} \le C\|f\|_{\mathrm{CBMO}^{p,\lambda}}.
$$

Therefore, in order to prove Theorem 1.3, we only need to establish the $(\mathrm{CBMO}^{p,\lambda}, \mathrm{CBMO}^{p,\lambda})$-boundedness of T_N.

For any ball $B = B(0, r) \subset R^n$, with $r > 0$, write

$$
\begin{aligned}
f(x) &= f_{16B} + (f(x) - f_{16B})\chi_{16B}(x) + (f(x) - f_{16B})\chi_{(16B)^c}(x) \\
&= f_1(x) + f_2(x) + f_3(x).
\end{aligned}
$$

It follows from the hypothesis $T_N 1 = 0$ that $T_N f_1 = 0$.

Thanks to Hölder's inequality and the $L^p(R^n)$-boundedness of T_N, we have

$$\Big(\frac{1}{|B|}\int_B |T_Nf_2(x)|^p\mathrm{d}x\Big)^{1/p} \le C\Big(\frac{1}{|B|}\int_{16B}|f(x)-f_{16B}|^p\mathrm{d}x\Big)^{1/p} \le C\|f\|_{\mathrm{CBMO}^{p,\lambda}}|B|^{\lambda}.$$

Since $T_Nf(x)$ and $T_Nf_2(x)$ exist a.e. in R^n, there is a point $z\in 3B\backslash 2B$ such that $T_Nf_3(z)<\infty$. For $x\in B$, $z\in 3B\backslash 2B$ and $y\in(16B)^c$, there is $|y-z|\ge 2|x-z|$. Choosing t,q such that $n/d<t<\min(s,p)$, $n/t<d<n/t+1$, $t<q'<p$ and by using Lemma 1.1 and Hölder's inequality, we obtain

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B |T_Nf_3(x)-T_Nf_3(z)|^p\mathrm{d}x\Big)^{1/p}\\
&\le \Big\{\frac{1}{|B|}\int_B\Big(\int_{(16B)^c}|K_N(x-y)-K_N(z-y)||f(y)-f_{16B}|\mathrm{d}y\Big)^p\mathrm{d}x\Big\}^{1/p}\\
&\le \Big\{\frac{1}{|B|}\int_B\Big(\sum_{j=1}^{\infty}\int_{2^j|x-z|\le|y-z|<2^{j+1}|x-z|}|K_N(x-y)-K_N(z-y)||f(y)-f_{16B}|\mathrm{d}y\Big)^p\mathrm{d}x\Big\}^{1/p}\\
&\le \Big\{\frac{1}{|B|}\int_B\Big[\sum_{j=1}^{\infty}\Big(\int_{2^j|x-z|\le|y-z|<2^{j+1}|x-z|}|K_N(x-y)-K_N(z-y)|^q\mathrm{d}y\Big)^{\frac{1}{q}}\times\\
&\quad\Big(\int_{2^{j+4}B}|f(y)-f_{2B}|^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\Big]^p\mathrm{d}x\Big\}^{1/p}\\
&\le \Big\{\frac{1}{|B|}\int_B\Big[\sum_{j=1}^{\infty}(2^{j+4}|x-z|)^{-d+n/q-n/t'}|x-z|^{d-n/t}\Big(\int_{2^{j+4}B}|f(y)-f_{16B}|^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\Big]^p\mathrm{d}x\Big\}^{1/p}\\
&\le C\sum_{j=1}^{\infty}2^{j(-d+n/t)}\Big(\frac{1}{|2^{j+4}r_B|}\int_{2^{j+4}B}|f(y)-f_{16B}|^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\\
&\le C\sum_{j=1}^{\infty}2^{j(-d+n/t)}\Big(\frac{1}{|2^{j+4}r_B|}\int_{2^{j+4}B}|f(y)-f_{16B}|^{p}\mathrm{d}y\Big)^{\frac{1}{p}}\\
&\le C\sum_{j=1}^{\infty}2^{j(-d+n/t)}\Big[\Big(\frac{1}{|2^{j+4}B|}\int_{2^{j+4}B}|f(y)-f_{2^{j+4}B}|^{p}\mathrm{d}y\Big)^{\frac{1}{p}}+\sum_{i=4}^{j+3}|f_{2^{i+1}B}-f_{2^iB}|\Big]\\
&\le C\sum_{j=1}^{\infty}2^{j(-d+n/t)}\Big[\|f\|_{\mathrm{CBMO}^{p,\lambda}}|2^{j+4}B|^{\lambda}+\sum_{i=4}^{j+3}\Big(\frac{1}{|2^{i+1}B|}\int_{2^{i+1}B}|f(y)-f_{2^{i+1}B}|^{p}\mathrm{d}y\Big)^{\frac{1}{p}}\Big]\\
&\le C\|f\|_{\mathrm{CBMO}^{p,\lambda}}|B|^{\lambda}\sum_{j=1}^{\infty}2^{j(-d+n/t)}\Big[2^{jn\lambda}+\sum_{i=4}^{j+3}2^{in\lambda}\Big]\\
&\le C\|f\|_{\mathrm{CBMO}^{p,\lambda}}|B|^{\lambda}\sum_{j=1}^{\infty}j2^{j(-d+n/t)}2^{jn\lambda}\le C\|f\|_{\mathrm{CBMO}^{p,\lambda}}|B|^{\lambda}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big(\frac{1}{|B|^{1+\lambda p}}\int_B|T_Nf(x)-(T_Nf)_B|^p\mathrm{d}x\Big)^{1/p}\\
&\quad\le C\Big(\frac{1}{|B|^{1+\lambda p}}\int_B|T_Nf(x)-T_Nf_3(z)|^p\mathrm{d}x\Big)^{1/p}
\end{aligned}$$

$$\leq C\Big(\frac{1}{|B|^{1+\lambda p}}\int_B |T_N f_2(x)|^p \mathrm{d}x\Big)^{1/p} + C\Big(\frac{1}{|B|^{1+\lambda p}}\int_B |T_N f_3(x) - T_N f_3(z)|^p \mathrm{d}x\Big)^{1/p}$$
$$\leq C\|f\|_{\mathrm{CBMO}^{p,\lambda}},$$

which completes the proof of Theorem 1.3. □

We also remark that Theorem 1.3 remains true for the inhomogeneous version of λ-central BMO spaces by taking the supremum over $r > 1$ in Definition 1.2 instead of $r > 0$ there.

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