Journal of Mathematical Research with Applications
May, 2012, Vol. 32, No. 3, pp. 346–354
DOI:10.3770/j.issn:2095-2651.2012.03.009
Http://jmre.dlut.edu.cn

Minimal Surfaces in a Unit Sphere

Zhonghua HOU*, **Dan YANG**
School of Mathematical Science, Dalian University of Technology, Liaoning 116024*, P. R. China*

Abstract Let M be a closed surface with positive Gauss curvature minimally immersed in a standard Euclidean unit sphere S^n. In this paper, we choose a local orthonormal frame field on M, under which the shape operators have very convenient form. We also give some applications of this kind of frame field.

Keywords minimal surface; Gauss curvature; normal scalar curvature.

MR(2010) Subject Classification 53C42; 53A10

1. Introduction

Let M be a minimal immersion of surface into a standard Euclidean unit sphere S^n. We denote by K and K_N the Gauss curvature and the normal scalar curvature, respectively.

In [1], Calabi showed that if M is a sphere with constant Gauss curvature K and the immersion is full, then the set of possible values of K is discrete, namely $K = K(s) = 2/(s(s+1))$ and the immersion is congruent to the s-th standard minimal immersion. It is well known that the choosing of the frame field plays an important role in studying the surface. In [2], Chern chose a local orthonormal frame field on the two-sphere which is minimally immersed in S^n. Using the frame field, Chern obtained an equality concerning the local invariants and showed that if the Gauss curvature K is constant, then $K = K(s)$. Later, Kenmotsu [3, 4] also obtained an equality for the compact minimal surface immersed in S^n by choosing the frame field, which generalized Chern's result. In [5], Itoh chose a local orthonormal frame field on the compact surface M which is minimally immersed in S^n and proved that if the normal scalar curvature K_N is non-zero constant and the square of the second curvature k_2 is less than $K_N/4$, then M is a generalized Veronese surface, where k_2 is the square of the second curvature. Later, there are some important results concerning minimal surfaces such as [6–14].

In this paper, we choose a local orthonormal frame field on the closed surface with positive Gauss curvature which is minimally immersed in a unit sphere, which is simple and convenient to solve some problems of surfaces. We also give some applications of this kind of frame field.

2. Preliminaries

Received April 14, 2010; Accepted October 3, 2010
Supported by the Fundamental Research Funds for the Central Universities.
* Corresponding author
E-mail address: zhhou@dlut.edu.cn (Zhonghua HOU); danyang82@yahoo.cn (Dan YANG)

Let M be a closed minimal surface immersed in the unit sphere S^n. From now on, we identify M with its immersed image, agree on the following index ranges:

$$1 \leq i, j, k, \cdots \leq 2; \quad 3 \leq \alpha, \beta, \gamma, \cdots \leq n; \quad 1 \leq A, B, C, \cdots \leq n,$$

and use the Einstein convention.

Take a local orthonormal frame field $\{e_A\}_{A=1}^n$ in TS^n along M such that $\{e_i\}_{i=1}^2$ lies in the tangent bundle $T(M)$ and $\{e_\alpha\}_{\alpha=3}^n$ in the normal bundle $N(M)$. Let$\{\omega_A\}_{A=1}^n$ be the dual coframe field of $\{e_A\}_{A=1}^n$. Denote by $(\omega_{AB})_{A,B=1}^n$ the Riemannian connection form matrix associated with $\{\omega_A\}_{A=1}^n$. Then $(\omega_{ij})_{i,j=1}^2$ defines a Riemannian connection in $T(M)$ and $(\omega_{\alpha\beta})_{\alpha,\beta=3}^n$ defines a normal connection in $N(M)$. It follows that the second fundamental form of M can be expressed as

$$\sigma = \sum_{i,\alpha} \omega_i \otimes \omega_{i\alpha} \otimes e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

where

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j; \quad h_{ij}^\alpha = h_{ji}^\alpha, \quad i, j = 1, 2; \quad \alpha = 3, \ldots, n.$$

And the mean curvature vector field of M is expressed as

$$\xi = \frac{1}{2} \sum_{\alpha=3}^n (h_{11}^\alpha + h_{22}^\alpha) e_\alpha.$$

Then M is minimal if and only if $\xi = 0$.

Let $L^\alpha = (h_{ij}^\alpha)_{2\times 2}$. We denote the square of the norm of the second fundamental form S and the normal scalar curvature K_N by

$$S = \sum_\alpha \sum_{i,j} (h_{ij}^\alpha)^2, \quad K_N = \sum_{\alpha,\beta} \sum_{i,j} (R_{\alpha\beta ij})^2.$$

The Riemannian curvature tensor $\{R_{ijkl}\}$ and the normal curvature tensor $\{R_{\alpha\beta kl}\}$ are expressed as

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \quad R_{\alpha\beta kl} = \sum_m (h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta).$$

The first and the second order covariant derivatives of $\{h_{ij}^\alpha\}$, say$\{h_{ijk}^\alpha\}$ and $\{h_{ijkl}^\alpha\}$ are defined respectively as follows.

$$\nabla h_{ij}^\alpha = \sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_m (h_{mj}^\alpha \omega_{mi} + h_{im}^\alpha \omega_{mj}) + \sum_\beta h_{ij}^\beta \omega_{\beta\alpha},$$

$$\nabla h_{ijk}^\alpha = \sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_m (h_{mjk}^\alpha \omega_{mi} + h_{imk}^\alpha \omega_{mj} + h_{ijm}^\alpha \omega_{mk}) + \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}.$$

Then we have the following Codazzi equation

$$h_{ijk}^\alpha - h_{ikj}^\alpha = 0, \tag{1}$$

and the Ricci's formula

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_p (h_{pj}^\alpha R_{pikl} + h_{ip}^\alpha R_{pjkl}) + \sum_\beta h_{ij}^\beta R_{\beta\alpha kl}. \tag{2}$$

The Laplacian of the second fundamental form $\{h^\alpha_{ij}\}$ is defined by $\Delta h^\alpha_{ij}=\sum_m h^\alpha_{ijmm}$. Then it follows from (1) and (2) that

$$\Delta h^\alpha_{ij}=\sum_m h^\alpha_{mmij}+\sum_{p,m}(h^\alpha_{pi}R_{pmjm}+h^\alpha_{mp}R_{pijm})+\sum_{m,\delta}h^\delta_{mi}R_{\delta\alpha jm}. \tag{3}$$

3. The choosing of the frame field

It is well known that the choice of the frame field plays a very important role in the study of submanifolds. In this section, we would like to choose an orthonormal frame field on the closed surfaces minimally immersed in a unit sphere. Firstly, we give the following Proposition.

Proposition 1 *Suppose that M is a surface immersed in the unit sphere S^n. Then there exits a local orthonormal frame filed $\{e_\alpha\}_{\alpha=3}^n$ normal to M such that the shape operator L^α with respect to e_α has the following forms.*

$$L^3=\begin{pmatrix}\lambda^3 & b\\ b & \nu^3\end{pmatrix};\quad L^\beta=\begin{pmatrix}\lambda^\beta & 0\\ 0 & \nu^\beta\end{pmatrix},\quad 4\le\beta\le n. \tag{4}$$

Proof Let $\{e_1,\dots,e_n\}$ be an orthonormal frame field on M so that e_1,e_2 are tangent to M and $e_3,\dots,e_n$ are normal to M. If the normal bundle of M is flat, then the shape operator L^α with respect to e_α can be diagonalized simultaneously. Otherwise, at least one of h^β_{12} $(\beta\ge3)$ is not zero identically. We choose a unit normal vector field $\widetilde{e}_3=e/|e|$ where $e=\sum_{\beta=3}^n h^\beta_{12}e_\beta$, and take an orthogonal transformation in the normal space $N_x(M)$:

$$\begin{pmatrix}\widetilde{e}_3\\ \widetilde{e}_4\\ \vdots\\ \widetilde{e}_n\end{pmatrix}=\begin{pmatrix}h^3_{12}|e|^{-1} & h^4_{12}|e|^{-1} & \cdots & h^n_{12}|e|^{-1}\\ a_{43} & a_{44} & \cdots & a_{4n}\\ \vdots & \vdots & & \vdots\\ a_{n3} & a_{n4} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix}e_3\\ e_4\\ \vdots\\ e_n\end{pmatrix}.$$

Let $\widetilde{L}^\alpha=(\widetilde{h}^\alpha_{ij})$ be the shape operator with respect to $\widetilde{e}_\alpha$, $3\le\alpha\le n$. It follows that

$$\begin{cases}h^3_{ij}=h^3_{12}|e|^{-1}\widetilde{h}^3_{ij}+a_{43}\widetilde{h}^4_{ij}+\cdots+a_{n3}\widetilde{h}^n_{ij},\\ h^4_{ij}=h^4_{12}|e|^{-1}\widetilde{h}^3_{ij}+a_{44}\widetilde{h}^4_{ij}+\cdots+a_{n4}\widetilde{h}^n_{ij},\\ \cdots\ \cdots\ \cdots\\ h^n_{ij}=h^n_{12}|e|^{-1}\widetilde{h}^3_{ij}+a_{n4}\widetilde{h}^4_{ij}+\cdots+a_{nn}\widetilde{h}^n_{ij}.\end{cases} \tag{*}$$

Put $i=1$ and $j=2$ in $(*)$. Then it is easy to check that $(*)$ has unique solution

$$b:=\widetilde{h}^3_{12}=|e|\neq0,\quad \widetilde{h}^\beta_{12}=0,\quad 4\le\beta\le n.$$

Therefore $\{e_1,e_2;\widetilde{e}_\beta\}_{\beta=3}^n$ is the desired local orthonormal frame field. $\square$

When M is a surface which is minimally immersed in the unit sphere S^n, we take the ortonormal frame field on the surface M according to Proposition 1 such that the shape operators have the form (4) with

$$\lambda^\alpha+\nu^\alpha=0,\quad 3\le\alpha\le n. \tag{5}$$

Denote

$$\overline{S}:=\sum_{(i,j,\beta>3)}(h^\beta_{ij})^2=2\sum_{(\beta>3)}(\lambda^\beta)^2,\quad S_3:=\sum_{(i,j)}(h^3_{ij})^2=2(\lambda^3)^2+2b^2.$$

Then

$$S = \overline{S} + S_3 = 2 \sum_{(\beta>3)} (\lambda^\beta)^2 + 2(\lambda^3)^2 + 2b^2.$$

According to (5) we denote

$$\lambda_i^\alpha := h_{11i}^\alpha = -h_{22i}^\alpha, \quad \lambda_{ij}^\alpha := h_{11ij}^\alpha = -h_{22ij}^\alpha$$

for all $i, j = 1, 2$ and $\alpha = 3, \dots, n$.

The Riemannian curvature tensor, the normal curvature tensor and the first covariant differentials of the normal curvature tensor become

$$R_{ijkl} = (1 - S/2)\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad R_{3\beta12} = -2b\lambda^\beta, \quad R_{\gamma\beta12} = 0, \tag{6}$$

$$R_{3\beta12,k} = 2(\lambda^3 h_{12k}^\beta - \lambda^\beta h_{12k}^3 - b\lambda_k^\beta), \quad R_{\beta\gamma12,k} = 2(\lambda^\beta h_{12k}^\gamma - \lambda^\gamma h_{12k}^\beta), \tag{7}$$

where $4 \le \beta, \gamma \le n$. Next we will study the covariant differentials of S. It is not difficult to check that

$$S_k = 2\sum h_{ij}^\alpha h_{ijk}^\alpha = 4\sum_{\beta=4}^n \lambda^\beta \lambda_k^\beta + 4(\lambda^3\lambda_k^3 + bh_{12k}^3),$$

for all $k = 1, 2$. It follows that

$$\frac{1}{4}S_1 = \sum_{\beta=4}^n \lambda^\beta\lambda_1^\beta + \lambda^3\lambda_1^3 + b\lambda_2^3, \quad \frac{1}{4}S_2 = \sum_{\beta=4}^n \lambda^\beta\lambda_2^\beta + \lambda^3\lambda_2^3 - b\lambda_1^3. \tag{8}$$

It follows from Ricci formula (2) that

$$\begin{aligned} &\lambda_{12}^3 - \lambda_{21}^3 = -(2 - S - \overline{S})b, \quad \lambda_{11}^3 + \lambda_{22}^3 = (2 - S)\lambda^3, \\ &\lambda_{11}^\beta + \lambda_{22}^\beta = (2 - S - 2b^2)\lambda^\beta, \quad \lambda_{12}^\beta - \lambda_{21}^\beta = -2b\lambda^3\lambda^\beta, \end{aligned} \tag{9}$$

where $\beta \ge 4$. Using the frame field introduced by Proposition 1 and the moving frame method, we obtain the following Lemma.

Lemma 1 *Let M be a minimal surface immersed in a unit sphere S^n. Then*

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + (2 - S)S - 4b^2\overline{S}. \tag{10}$$

Proof It follows from (3) that

$$\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,p,m,\alpha} (h_{ij}^\alpha h_{pi}^\alpha R_{pmjm} + h_{ij}^\alpha h_{mp}^\alpha R_{pijm}) + \sum_{i,j,m,\alpha,\delta} h_{ij}^\alpha h_{mi}^\delta R_{\delta\alpha jm}.$$

According to (6) we have

$$\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = (2 - S)\sum_{i,j,\alpha} (h_{ij}^\alpha)^2 + \sum_{\beta=4}^n 4b\lambda^\beta R_{3\beta12} = (2 - S)S - 4b^2\overline{S}. \tag{11}$$

It follows that

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + (2 - S)S - 4b^2\overline{S}.$$

We complete the proof of Lemma 1. □

We assume that the normal bundle of M immersed in S^n is nonflat. In this case we have that $b \neq 0$. We can establish the following theorem.

Theorem 1 *Let M be a closed surface minimally immersed in $S^n(1)$ with nonflat normal bundle. If the Gauss curvature of M is positive, then we can choose a local orthonormal frame filed $\{e_\alpha\}_{\alpha=3}^n$ normal to M such that the shape operator L^α with respect to e_α has the following forms:*

$$L^3 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad L^4 = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad L^\beta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 5 \leq \beta \leq n.$$

Furthermore

$$b = \frac{\sqrt{S}}{2}, \quad \lambda_1^3 = -\lambda_2^4 = -\frac{1}{4\sqrt{S}} S_2, \quad \lambda_2^3 = \lambda_1^4 = \frac{1}{4\sqrt{S}} S_1.$$

Proof Let M be a surface minimally immersed in S^n. We take the orthonormal frame field $\{e_1, e_2, e_3, \dots, e_n\}$ on M according to Proposition 1 such that the shape operators L^α with respect to e_α have the form

$$L^3 = \begin{pmatrix} \lambda^3 & b \\ b & -\lambda^3 \end{pmatrix}; \quad L^\beta = \begin{pmatrix} \lambda^\beta & 0 \\ 0 & -\lambda^\beta \end{pmatrix}, \quad 4 \leq \beta \leq n. \tag{12}$$

It follows from (3) and the simple computation that

$$\begin{aligned}\sum_{i,j,k,\alpha} (h_{ijk}^\alpha \Delta h_{ij}^\alpha)_k = &\sum_{i,j,\alpha} (\Delta h_{ij}^\alpha)^2 + \sum_{i,j,k,l,p,\alpha} (h_{ijk}^\alpha h_{pik}^\alpha R_{pljl} + h_{ijk}^\alpha h_{lpk}^\alpha R_{pijl}) + \\ &\sum_{i,j,k,l,p,\alpha} (h_{ijk}^\alpha h_{pi}^\alpha R_{pljl,k} + h_{ijk}^\alpha h_{lp}^\alpha R_{pijl,k}) + \\ &\sum_{i,j,k,l,\alpha,\delta} h_{ijk}^\alpha h_{lik}^\delta R_{\delta\alpha jl} + \sum_{i,j,k,l,\alpha,\delta} h_{ijk}^\alpha h_{li}^\delta R_{\delta\alpha jl,k}.\end{aligned} \tag{13}$$

Next we calculate the right hand side of equation (13) one by one. Using (3) and (6), we get

$$\begin{aligned}&\Delta h_{11}^\beta = (2 - S - 2b^2)\lambda^\beta, \quad \Delta h_{12}^\beta = 2b\lambda^3\lambda^\beta, \\ &\Delta h_{12}^3 = (2 - S - \overline{S})b, \quad \Delta h_{11}^3 = (2 - S)\lambda^3,\end{aligned}$$

where $\beta \geq 4$. It follows that

$$\begin{aligned}\sum_{i,j,\alpha} (\Delta h_{ij}^\alpha)^2 =& 2(\Delta h_{11}^3)^2 + 2\sum_{\beta=4}^n (\Delta h_{11}^\beta)^2 + (\Delta h_{12}^3)^2 + 2\sum_{\beta=4}^n (\Delta h_{12}^\beta)^2 \\ =& (2 - S)^2 S + 2(5S - 8)b^2\overline{S}.\end{aligned} \tag{14}$$

Making use of (6), we get

$$\begin{aligned}&\sum_{i,j,k,l,p,\alpha} (h_{ijk}^\alpha h_{pik}^\alpha R_{pljl} + h_{ijk}^\alpha h_{lpk}^\alpha R_{pijl}) + \sum_{i,j,k,l,\alpha,\delta} h_{ijk}^\alpha h_{lik}^\delta R_{\delta\alpha jl} \\ &= (2 - S) \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{\beta=4}^n 8(\lambda_1^3\lambda_2^\beta - \lambda_2^3\lambda_1^\beta) R_{\beta 312} \\ &= (2 - S) \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + 16b \sum_{\beta=4}^n (\lambda_1^3\lambda^\beta\lambda_2^\beta - \lambda_2^3\lambda^\beta\lambda_1^\beta).\end{aligned} \tag{15}$$

By (8), we immediately have

$$\sum_{\beta=4}^{n}\lambda^\beta\lambda_1^\beta=\frac{1}{4}S_1-\lambda^3\lambda_1^3-b\lambda_2^3,\quad \sum_{\beta=4}^{n}\lambda^\beta\lambda_2^\beta=\frac{1}{4}S_2-\lambda^3\lambda_2^3+b\lambda_1^3. \tag{16}$$

Substituting (16) into (15), we get

$$\begin{aligned}&\sum_{i,j,k,l,p,\alpha}(h_{ijk}^\alpha h_{pik}^\alpha R_{pljl}+h_{ijk}^\alpha h_{lpk}^\alpha R_{pijl})+\sum_{i,j,k,l,\alpha,\delta}h_{ijk}^\alpha h_{lik}^\delta R_{\delta\alpha jl}\\&=(2-S)\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2+4b^2\sum_{i,j,k}(h_{ijk}^3)^2+4b(\lambda_1^3S_2-\lambda_2^3S_1).\end{aligned} \tag{17}$$

It follows from the first formula of (7) that

$$\sum_{i,j,k,l,p,\alpha}(h_{ijk}^\alpha h_{pi}^\alpha R_{pljl,k}+h_{ijk}^\alpha h_{lp}^\alpha R_{pijl,k})=-\sum_{i,j,k,\alpha}h_{ij}^\alpha h_{ijk}^\alpha S_k=-\frac{1}{2}|\nabla S|^2. \tag{18}$$

We calculate in detail the last term of equation (13), it is not hard to see that

$$\begin{aligned}\sum_{i,j,k,l,\alpha,\delta}h_{ijk}^\alpha h_{li}^\delta R_{\delta\alpha jl,k}=&\sum_{\gamma=4}^{n}2(\lambda^\gamma\lambda_2^3+b\lambda_1^\gamma-\lambda^3\lambda_2^\gamma)R_{3\gamma12,1}+\\&\sum_{\gamma=4}^{n}2(-\lambda^\gamma\lambda_1^3+b\lambda_2^\gamma+\lambda^3\lambda_1^\gamma)R_{3\gamma12,2}+\\&\sum_{\beta,\gamma=4}^{n}(-2\lambda^\gamma\lambda_2^\beta R_{\gamma\beta12,1}+2\lambda^\gamma\lambda_1^\beta R_{\gamma\beta12,2}).\end{aligned} \tag{19}$$

By (7), we have

$$\begin{aligned}&R_{3\gamma12,1}=2(-b\lambda_1^\gamma+\lambda^3\lambda_2^\gamma-\lambda^\gamma\lambda_2^3),\quad R_{\gamma\beta12,1}=2(\lambda^\gamma\lambda_2^\beta-\lambda^\beta\lambda_2^\gamma),\\&R_{3\gamma12,2}=2(-b\lambda_2^\gamma-\lambda^3\lambda_1^\gamma+\lambda^\gamma\lambda_1^3),\quad R_{\gamma\beta12,2}=2(\lambda^\beta\lambda_1^\gamma-\lambda^\gamma\lambda_1^\beta).\end{aligned} \tag{20}$$

Substituting (20) into (19) and using (16), we have

$$\sum_{i,j,k,l,\alpha,\delta}h_{ijk}^\alpha h_{li}^\delta R_{\delta\alpha jl,k}=-\frac{1}{2}S\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2+4b^2\sum_{i,j,k}(h_{ijk}^3)^2+4b(\lambda_1^3S_2-\lambda_2^3S_1)+\frac{1}{4}|\nabla S|^2. \tag{21}$$

Substituting (14), (17),(18) and (21) into (13) we get

$$\begin{aligned}\sum_{i,j,k,\alpha}(h_{ijk}^\alpha\Delta h_{ij}^\alpha)_k=&(2-\frac{3}{2}S)\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2+(2-S)^2S+2(5S-8)b^2\overline{S}+\\&8b^2\sum_{i,j,k}(h_{ijk}^3)^2+8b(\lambda_1^3S_2-\lambda_2^3S_1)-\frac{1}{4}|\nabla S|^2.\end{aligned} \tag{22}$$

It follows from (10) that

$$\sum_{\alpha,i,j,k}(h_{ijk}^\alpha)^2=\frac{1}{2}\Delta S-(2-S)S+4b^2\overline{S}. \tag{23}$$

Substituting (23) into (22) and from $S\Delta S=\frac{1}{2}\Delta S^2-|\nabla S|^2$, we have

$$\sum_{i,j,k,\alpha}(h_{ijk}^\alpha\Delta h_{ij}^\alpha)_k=\frac{1}{2}(2-S)S^2+4(S-2)b^2\overline{S}+\Delta S-\frac{3}{8}\Delta S^2+$$

$$8b^2\sum_{i,j,k}(h^3_{ijk})^2+8b(\lambda^3_1S_2-\lambda^3_2S_1)+\frac{1}{2}|\nabla S|^2. \tag{24}$$

Since

$$\begin{aligned}&8b^2\sum_{i,j,k}(h^3_{ijk})^2+8b(\lambda^3_1S_2-\lambda^3_2S_1)+\frac{1}{2}|\nabla S|^2\\&=32(b\lambda^3_1)^2+32(b\lambda^3_2)^2+8b(\lambda^3_1S_2-\lambda^3_2S_1)+\frac{1}{2}(S_1^2+S_2^2)\\&=2(4b\lambda^3_1+\frac{1}{2}S_2)^2+2(4b\lambda^3_2-\frac{1}{2}S_1)^2,\end{aligned} \tag{25}$$

substituting (25) into (24) gives

$$\begin{aligned}\sum_{i,j,k,\alpha}(h^\alpha_{ijk}\Delta h^\alpha_{ij})_k=&\frac{1}{2}(2-S)S^2+4(S-2)b^2\overline{S}+\Delta S-\frac{3}{8}\Delta S^2+\\&2(4b\lambda^3_1+\frac{1}{2}S_2)^2+2(4b\lambda^3_2-\frac{1}{2}S_1)^2.\end{aligned} \tag{26}$$

Since M is compact, taking integration over M on both sides of (26), we have

$$0=\int_M\Big\{\frac{1}{2}S^2(2-S)+4(S-2)b^2\overline{S}+2(4b\lambda^3_1+\frac{1}{2}S_2)^2+2(4b\lambda^3_2-\frac{1}{2}S_1)^2\Big\},$$

that is

$$\begin{aligned}\int_M(2-S)b^2\overline{S}=&\int_M\Big\{\frac{1}{8}S^2(2-S)+\frac{1}{2}(4b\lambda^3_1+\frac{1}{2}S_2)^2+\frac{1}{2}(4b\lambda^3_2-\frac{1}{2}S_1)^2\Big\}\\\geq&\int_M\frac{1}{8}S^2(2-S),\end{aligned} \tag{27}$$

where the equality holds if and only if $b\lambda^3_1=-\frac{1}{8}S_2$, $b\lambda^3_2=\frac{1}{8}S_1$.

On the other hand, we have

$$b^2\overline{S}\leq\frac{1}{2}S_3\overline{S}\leq\frac{1}{8}(S_3+\overline{S})^2=\frac{1}{8}S^2, \tag{28}$$

where the equality holds if and only if

$$\lambda^3=0,\quad \overline{S}=S_3=2b^2.$$

Since the Gauss curvature of M is positive, we have $2-S>0$. Taking integration over M on both sides of (28), we obtain

$$\int_M(2-S)b^2\overline{S}\leq\int_M\frac{1}{8}S^2(2-S). \tag{29}$$

It follows from (27) and (29) that

$$\int_M(2-S)b^2\overline{S}=\int_M\frac{1}{8}S^2(2-S),$$

which implies that the equalities in (27) and (28) hold. Therefore

$$\lambda^3=0,\quad b\lambda^3_1=-\frac{1}{8}S_2,\quad b\lambda^3_2=\frac{1}{8}S_1,\quad \overline{S}=2b^2.$$

This together with the fact that $S=\overline{S}+2b^2$ yields

$$b^2=\frac{1}{4}S,\quad \overline{S}=\frac{1}{2}S,\quad \lambda^3_1=-\frac{1}{4\sqrt{S}}S_2,\quad \lambda^3_2=\frac{1}{4\sqrt{S}}S_1.$$

Therefore we deduce that there must exist a β such that $\lambda^\beta \neq 0$ for $\beta \geq 4$. We choose a unit normal vector field $\overline{e}_4 = e/|e|$ where $e = \sum_{\gamma=4}^n h_{11}^\gamma e_\gamma$. Similarly to the proof of Proposition 1, we can choose a local orthonormal frame filed under which the shape operators have the following forms:

$$L^3 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad L^4 = \begin{pmatrix} \lambda^4 & 0 \\ 0 & -\lambda^4 \end{pmatrix}, \quad L^\beta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 5 \leq \beta \leq n.$$

Furthermore

$$b^2 = (\lambda^4)^2 = S/4, \quad \lambda_1^3 = -\frac{1}{4\sqrt{S}} S_2, \quad \lambda_2^3 = \frac{1}{4\sqrt{S}} S_1. \tag{30}$$

We take covariant differential of h_{11}^4 which is defined globally on M and have

$$h_{11k}^4 \omega_k = dh_{11}^4 + 2h_{12}^4 \omega_{21} + \sum_{\alpha=3}^n h_{11}^\alpha \omega_{\alpha 4} = dh_{11}^4 = \frac{1}{4\sqrt{S}} S_k \omega_k,$$

which imply

$$\lambda_1^4 = \lambda_2^3 = \frac{1}{4\sqrt{S}} S_1, \quad \lambda_2^4 = -\lambda_1^3 = \frac{1}{4\sqrt{S}} S_2. \tag{31}$$

This completes the proof of Theorem 1. $\square$

Let M be a surface immersed in the unit sphere S^n with parallel mean curvature vector ξ. Then the mean curvature $H = |\xi|$. We choose $e_3 = \xi/H$, and the choice of e_4, e_5 are similar to the proof of Theorem 1. Then we can establish the following theorem.

Theorem 2 *Let M be a closed surface immersed in the unit sphere S^n with parallel mean curvature vector and nonflat normal bundle. If the Gauss curvature of M is positive, then we can choose a local orthonormal frame filed $\{e_\alpha\}_{\alpha=3}^n$ normal to M such that the shape operator L^α with respect to e_α has the following forms:*

$$L^3 = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad L^4 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad L^5 = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad L^\beta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $6 \leq \beta \leq n$. Furthermore

$$b = \frac{\sqrt{S - 2H^2}}{2}, \ \lambda_1^4 = -\lambda_2^5 = -\frac{1}{4\sqrt{S - 2H^2}} S_2, \ \lambda_2^4 = \lambda_1^5 = \frac{1}{4\sqrt{S - 2H^2}} S_1.$$

4. The application of the frame field

The orthonormal frame field introduced by Theorem 1 is very simple and convenient to solve some problems on surfaces. As an application, we give the following Proposition.

Proposition 2 *Let M be a closed surface which is minimally immersed in S^n with positive Gauss curvature. If the normal scalar curvature K_N is non-zero constant, then M is a generalized Veronese surface.*

Proof It follows from (6) and Theorem 1 that

$$R_{3412} = -\frac{1}{2} S, \quad R_{3\beta 12} = R_{4\beta 12} = R_{\beta\gamma 12} = 0. \tag{32}$$

Hence we have $K_N=\sum_{\alpha,\beta}\sum_{i,j}R^2_{\alpha\beta ij}=S^2$. Similarly to the proof of [5], we conclude that M is a generalized Veronese surface. □

On the other hand, using the orthonormal frame field introduced by Theorem 1, we can simply give the complete classification of the minimal surface with parallel second fundamental form and non-negative Gauss curvature.

Proposition 3 *Let M be a closed minimal surface immersed in S^n with non-negative Gauss curvature. If M has parallel second fundamental form, then M is totally geodesic, or Veronese surface, or a Clifford minimal surface.*

Proof If the normal bundle of M is flat, since Gauss curvature of M is nonnegative, we have that

$$\frac{1}{2}\Delta S=\sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2+S(2-S)\geq 0. \tag{33}$$

Taking integration over M on both sides of (33), we have that $S=0$ and M is totally geodesic, or $S=2$ and M is a Clifford minimal surface. Otherwise, we choose the frame fields introduced by Theorem 1. Then (10) becomes

$$\frac{1}{2}\Delta S=\sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2+\frac{1}{2}S(3S-4). \tag{34}$$

Since the second fundamental form is parallel, we have that S is constant and $h^\alpha_{ijk}=0$, $\forall i,j,k,\alpha$. Hence from (34) we have that $S=4/3$ and M is a Veronese surface. □

References

[1] E. CALABI. *Minimal immersions of surfaces in Euclidean spheres*. J. Differential Geometry, 1967, **1**: 111–125.

[2] S. S. CHERN. *On the Minimal Immersions of the Two-Sphere in a Space of Constant Curvature*. Princeton Univ. Press, Princeton, N.J., 1970.

[3] K. KENMOTSU. *On compact minimal surfaces with non-negative Gaussian curvature in a space of constant curvature (I)*. Tôhoku Math. J. (2), 1973, **25**: 469–479.

[4] K. KENMOTSU. *On compact minimal surfaces with non-negative Gaussian curvature in a space of constant curvature (II)*. Tôhoku Math. J. (2), 1975, **27**(3): 291–301.

[5] T. ITOH. *Minimal surfaces in a Riemannian manifold of constant curvature*. Kōdai Math. Sem. Rep., 1973, **25**: 202–214.

[6] Xingxiao LI, Anmin LI. *Explicit representations of minimal immersions of 2-spheres into S^{2m}*. Acta Math. Sinica (N.S.), 1997, **13**(2): 175–186.

[7] M. KOZLOWSKI, U. SIMON. *Minimal immersions of 2-manifolds into spheres*. Math. Z., 1984, **186**(3): 377–382.

[8] R. L. BRYANT. *Minimal surfaces of constant curvature in S^n*. Trans. Amer. Math. Soc., 1985, **290**(1): 259–271.

[9] U. SIMON. *Eigenvalues of the Laplacian and Minimal Immersions into Spheres*. Pitman, Boston, MA, 1985.

[10] F. DILLEN. *Minimal immersions of surfaces into spheres*. Arch. Math. (Basel), 1987, **49**(1): 94–96.

[11] T. OKAYASU. *Minimal immersions of curvature pinched 2-manifolds into spheres*. Kodai Math. J., 1987, **10**(1): 116–126.

[12] J. BOLTON, L. M. WOODWARD. *On the Simon conjecture for minimal immersions with S^1-symmetry*. Math. Z., 1988, **200**(1): 111–121.

[13] K. SAKAMOTO. *On the curvature of minimal 2-spheres in spheres*. Math. Z., 1998, **228**(4): 605–627.

[14] W. H. MEEKS. *Proofs of some classical theorems in minimal surface theory*. Indiana Univ. Math. J., 2005, **54**(4): 1031–1045.