*Journal of Mathematical Research with Applications*
May, 2012, Vol. 32, No. 3, pp. 355–365
DOI:10.3770/j.issn:2095-2651.2012.03.010
Http://jmre.dlut.edu.cn

# Risk Evaluation and Capital Allocation Based on TVaR and EVaR with Copula

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**Abstract** In this paper, the expressions of tail value of risk (TVaR) and exponential tail value of risk (EVaR) for the total risk portfolio are given, which are splitted into two cases: the bivariate case and the multivariate case according to the number of the insurances. Then the risk contributions of the insurances portfolio and the credit portfolio are also obtained. Further more, for clarifying the above results, a numerical example is given.

**Keywords** capital allocation; tail value of risk; exponential tail value of risk; copula.

**MR(2010) Subject Classification** 91B30; 91B32; 91G40

## 1. Introduction

Recently, there are more awareness and relevant research on risk measurement and the capital allocation. Specially insurance company should consider the loss of claim and default loss of credit portfolio. So it is important to consolidate financial reserves and investments. Measure of risk is a useful tool to evaluate the capital amount that has to be allocated to the risk portfolio. Thus the choice of risk measure principle is the key point. Artzner et al. [1, 2] proposed the definition of coherent risk measure, however the well-known VaR introduced by Morgan is not coherent, and then the tail value of risk (TVaR), which is coherent, was introduced. When applied to continuous random variables, the TVaR is the same as the conditional tail expectation. While they are different when applied to the discrete random variables. Acerbi et al. [3], Acerbi and Tasche [4] highlighted the difference between their definitions and properties.

In most literature on capital allocation, continuous situations are widely studied. Tasche [5] firstly introduced the top down capital allocation principle. The capital which is allocated to each risk is expressed in terms of the CTE of the aggregate risk to the portfolio. It has been used to provide several closed formula and approximations of the CTE and the CTE-based allocations for different multivariate continuous distributions. Panjer [6] solved the case of multivariate normal distribution. Furman and Landsman [7] extended the distribution to a multivariate elliptical

Received November 1, 2010; Accepted January 13, 2011
Supported by the National Natural Sciences Foundation of China (Grant No. 61175041), the Funds of Doctoral Programme of China (Grant No. 2010041110036), the Fundamental Research Funds for the Central Universities (Grant Nos. DUT12LK16; DUT12LK29) and the Funds for Frontier Interdisciplines of DUT (Grant Nos. DUT12JS05; DUT10JS06).
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distribution and the case of multivariate Pareto distribution was considered by Chiragiev and Landsman [8].

In most paper mentioned above, the dependence between different random variables is constructed by the multivariate distribution. There is a copula that equals the joint distribution function by Sklar's theorem [9], and this tool is more effective and flexible to represent the dependence between the random variables. Based on this method, more marginal distributions situation can be introduced. Bargés [10] solved the case of the element distribution of portfolio, which is exponential distribution and the dependence structure is described by Farlie-Gumbel-Morgenstern copula. By using the TVaR as defined in Acerbi et al. [3] for its coherence property and top down approach of the capital allocation in Acerbi and Tasche [4], he got closed form of expressions for the TVaR and then the TVaR-based contribution of one risk over the aggregation of all risks.

In the above mentioned papers, the risk portfolio is only constituted by insurances. While in the present paper the total risk portfolio is constituted by the claim risk portfolio and the default risk portfolio. Since the discrete default risk variable leads to that the distribution of aggregate loss is discrete as well, we propose using the TVaR to obtain the aggregate risk for the total risk portfolio and then the TVaR-based capital allocation to get the contribution of claim risk portfolio and the default risk portfolio. In terms of loss distribution of credit portfolio, Fray and McNeil [11] introduced the popular models that are used to deal with the dependent case from the conceptual aspect. In this paper, we assume the elements of the credit portfolio are independent and also independent of insurances claim loss.

The rest of the paper is organized as follows. In Section 2, we give the introduction of the claim amount distribution and portfolio credit loss distribution. Risk of measure and TVaR-based allocation will be introduced in Section 3. The expression of TVaR, EVaR for the aggregate risk portfolio and TVaR-based allocation to the insurances portfolio and credit portfolio are given in Section 4. For clarifying the results, a numerical example with the form of tables is given in the last section.

## 2. The distributions of claim amount and credit portfolio default loss

In this section, we mainly present the related results based on two aspects: claim amount distribution and credit portfolio default loss distribution. For the clarity, firstly we introduce the assumptions in this paper.

### 2.1. The loss distribution for insurances portfolio

In this subsection, we introduce the cumulative density function (cdf) of total claim amount $S_1$ for the insurances portfolio. We consider the portfolio including N insurances.

$$S_1 = X_1 + X_2 + \cdots + X_N,$$

where the claim amount of the $i$-th insurance is denoted by $X_i$. If $X_i$ $(i = 1, 2, \ldots, N)$ are independent, the distribution of $S_1$ can be obtained by convolution operation. Actually we usually suppose $X_i$ obey normal distribution or gamma distribution in order to simplify the convolution operation. That is because the distribution function of the sums of the sequence of $n$ independent random variables which obey $N\ (\mu, \delta^2)$ is $N\ (n\mu, n\delta^2)$. The gamma distribution has the similar property. In this paper, we introduce an effective tool-copula to present the dependent structure of $X_i$ $(i = 1, 2, \ldots, N)$. The following contexts present the assumptions and the result on the claim amount distribution.

(i) Suppose $X_i$ $(i = 1, 2, \ldots, N)$ are exponentially random variables. The cdfs and probability density functions (pdf) of $X_i$ $(i = 1, 2, \ldots, N)$ are respectively given by

$$F_{X_i}(x_i) = 1 - e^{-\lambda_i x_i},\ f_{X_i}(x_i) = \lambda_i e^{-\lambda_i x_i},\quad i = 1, 2, \ldots, N,$$

where we suppose the expectation satisfies $1 \le \frac{1}{\lambda_i} \le M_0$ $(i = 1, 2, \ldots, N)$, i.e., $\delta = \frac{1}{M_0} \le \lambda_i \le 1$.

(ii) A dependence structure for $(X_1, X_2, \ldots, X_N)$ based on the FGM copula was introduced in Yeo and Valdez [12] and Gatfaoui [13, 14]. The multivariate FGM copula is defined by

$$C(u_1, u_2, \ldots, u_N) = u_1 u_2 \cdots u_N \times \Big(1 + \sum_{k=2}^{N} \sum_{1 \le j_1 \le j_2 \le \cdots j_k \le N} \theta_{j_1 j_2 \cdots j_k} \bar{u}_{j1} \bar{u}_{j2} \cdots \bar{u}_{jk}\Big),$$

where $\bar{u}_{(\cdot)} = 1 - u_{(\cdot)}$ (For the details see Nelsen [9], p. 108). For $u_i \in [0, 1]$, $i = 1, 2, \ldots, N$, and the dependence parameter $\theta_{j_1 j_2 \cdots j_k} \in [-1, 1]$, at the same time we have here $2^N - N - 1$ copula parameters.

(iii) Based on above assumptions, the pdf of $S_1$ can be obtained (for the dails see Bargès [10]) and it is

$$\begin{aligned} f_{S_1}(s_1) =& h(s_1; \lambda_1, \ldots, \lambda_N) + \sum_{k=2}^{N} \sum_{1 \le j_1 \le j_2 \le \cdots j_k \le N} \theta_{j_1 j_2 \cdots j_k} \times \\ & \Big(\sum_{l=0}^{k} \sum_{(a_1 \cdots a_k) \in A_{l,k}} (-1)^l h(s_1; 2^{a_1}\lambda_{j_1}, 2^{a_2}\lambda_{j_2}, \ldots, 2^{a_k}\lambda_{j_k}, \lambda_{i_{k+1}}, \ldots, \lambda_{i_N})\Big), \end{aligned} \tag{1}$$

where

$$h(s_1; \lambda_1, \ldots, \lambda_N) = \sum_{i=1}^{N} \Big(\prod_{j=1, j \ne i}^{N} \frac{\lambda_j}{\lambda_j - \lambda_i}\Big) \lambda_i e^{-\lambda_i s_1},$$

$$A_{0,k} = \{(1, 1, \ldots, 1)_{1\times k}\}, A_{1,k} = \{(1, 1, \ldots, 0)_{1\times k}, \ldots, (0, 1, \ldots, 1)_{1\times k}\},$$

$$A_{2,k} = \{(1, 1, \ldots, 0, 0)_{1\times k}, \ldots, (0, 0, 1, \ldots, 1)_{1\times k}\}, \ldots, A_{k,k} = \{(0, 0, \ldots, 0)_{1\times k}\}.$$

### 2.2. The loss distribution for insurances portfolio

When the company receives the premiums, it has to consider how to increase the value of the asset by investing, such as buying the default corporate bond. Consider a portfolio of $M$ counterparties and fix some time period $[T, T + \triangle T]$ where $\triangle T$ is typically one year. For $1 < i < M$, let the random variable $Y_i$ be the default indicator for obligor $i$ at time $T + 1$, taking values $\{0,1\}$. 1 represents default and 0 represents non-default. We introduce a sequence of iid exposures $E_i$ $(i=1,2,\ldots,M)$ and entire debt amount $D_i$ $(i=1,2,\ldots,M)$ such as the face value of the discount bone. Under the condition that defaults of different obligator are dependent, the bernoulli mixture models in R. Fray and A. McNeil [11] are widely used in the practice, such as CreditRisk+, CreditMetrics and CreditPortfolioView. In our paper, assume that the entire exposure is lost in the event of default, that is, $E_i=1$ $(i=1,2,\ldots,M)$. Moreover, the default probability and debt amount are identical for all obligators, equal to $p$ and $D$, that is, $P(Y_i=1)=p$ $(i=1,2,\ldots,M)$. And the default probability of different obligator is independent. Over the time period $[T,T+1]$, the default loss is denoted by $S_2$ with $S_2=\sum_{i=1}^{M}D_iE_iY_i$. Based on the above assumption, we have

$$S_2=\sum_{i=1}^{M}DY_i.$$

And the distribution of $S_2$ is binomial distribution, i.e.,

$$P(S_2=Dt)=C_M^t p^t(1-p)^{M-t}.$$

If $M$ is large and $p$ is small, we can use Poisson distribution to estimate the distribution of $S_2$, that is

$$P(S_2=Dt)=\frac{e^{-\mu}\mu^t}{t!},$$

where $\mu=Mp$.

And now we turn to the total loss. We consider the aggregate loss $S$ which is the sum of total claim amount $S_1$ and total default loss $S_2$, that is, $S=S_1+S_2$.

## 3. Risk of measure and TVaR-based allocation

Let $(\Omega,A,p)$ be probability space such that $\Omega$ is the sample space, A the $\sigma$-field of events, and $P$ the probability measure. $S$, which is measurable real-valued, represents a loss random variable such that $\omega\in\Omega$. $S(\omega)>0$ represent a loss. In this part, we show the definitions of some risk of measure.

**Definition 3.1** *Given the confidence level $k\in(0,1)$ and $S$, we define the following:*

*(i) The k-value-at-Risk (k-VaR)*

$$\mathrm{VaR}_k(S)=\inf(s\in R,F_S(s)\geq k).$$

*Actually the VaR is the smallest number $s$ such that the probability that the loss $S$ exceeds $s$ is no larger than $1-k$. Moreover, it is not coherent since it does not satisfy the subadditivity. In other words, let $S_1$ and $S_2$ be two loss random variables.* $\mathrm{VaR}(S_1+S_2)\leq\mathrm{VaR}(S_1)+\mathrm{VaR}(S_2)$ *is not necessary. The example could be found in R. Fray and A. McNeil [11].*

*(ii) The k-Tail-at-Value (k-TVaR)*

$$\mathrm{TVaR}_k(S)=\frac{1}{1-k}\int_k^1\mathrm{VaR}_u(S)\mathrm{d}u \tag{2}$$

$$=\frac{E[SI_{\{S>\mathrm{VaR}_k(s)\}}]+\mathrm{VaR}_k(S)(Pr(S\le \mathrm{VaR}_k(S))-k)}{1-\kappa}. \tag{3}$$

*Similarly to the exponential premium, we introduce another measure of risk.*

*(iii) The k-Exponentially-Value-at-Risk (k-EVaR)*

$$\mathrm{EVaR}_k(S)=\log_{e^\delta}\Big(\frac{\int_k^1 e^{\delta \mathrm{VaR}_u(S)}\mathrm{d}u}{1-k}\Big) \tag{4}$$

$$=\log_{e^\delta}\Big(\frac{E[e^{\delta S}I_{\{S>\mathrm{VaR}_k(s)\}}]+e^{\delta \mathrm{VaR}_k(S)}[Pr(S\le \mathrm{VaR}_k(S))-k]}{1-k}\Big). \tag{5}$$

*Among the TVaR of $S$ which represents the capital amount at risk, the capital amount allocated to $S_1$ and $S_2$ are attractive. They are also called the TVaR contribution of the $S_1$ and $S_2$ to the $S$ and denoted $\mathrm{TVaR}_k(S_i;S)$ $(i=1,2)$. We give the expressions of $\mathrm{TVaR}_k(S_i;S)$ $(i=1,2)$ as follows.*

**Definition 3.2** *TVaR-based allocation is defined by*

$$\mathrm{TVaR}_k(S_i;S)=\frac{E[S_i\times I_{\{S>\mathrm{VaR}_k(s)\}}]+\beta_S E[S_i\times I_{\{S=\mathrm{VaR}_k(S)\}}]}{1-k},\quad i=1,2 \tag{6}$$

*where*

$$\beta_S=\begin{cases}\dfrac{Pr(S\le \mathrm{VaR}_k(S))-k}{Pr(S=\mathrm{VaR}_k(S))}, & Pr(S=\mathrm{VaR}_k(S))\ne 0;\\ 0, & Pr(S=\mathrm{VaR}_k(S))=0.\end{cases}$$

*Using the additivity of expectation, we can get*

$$\mathrm{TVaR}_k(S)=\sum_{i=1}^{2}\mathrm{TVaR}_k(S_i;S).$$

## 4. The main results on TVaR, EVaR and TVaR-based capital allocation

For obtaining the distribution function of $S$ to get the risk of measure introduced in the Section 3, we suppose $S_1$ and $S_2$ are independent, therefore,

$$F_S(s)=P(S_1+S_2\le s)=\sum_{t=0}^{M}P(S_1\le s-tD)P(S_2=tD),$$

$$f_S(s)=\sum_{t=0}^{M}f_{S_1}(s-tD)\pi_t, \tag{7}$$

where we denote $\pi_t:=P(S_2=tD)=C_M^t p^t(1-p)(M-t)$.

### 4.1. TVaR and EVaR

In the subsection, we will present the expressions of TVaR and EVaR and they are splitted into two cases according to the number of the insurance portfolio $N$: the bivariate case and the multivariate case.

#### 4.1.1. The bivariate case

Under the situation of two insurances and the assumption mentioned in the Section 2, we have

$$f_{S_1}(s_1) = (1+\theta)h(s_1;\lambda_1;\lambda_2) - \theta h(s_1;\lambda_1;2\lambda_2) - \theta h(s_1;2\lambda_1;\lambda_2) + \theta h(s_1;2\lambda_1;2\lambda_2),$$

where

$$h(s_1;\lambda_1;\lambda_2) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 s_1} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 s_1}.$$

The TVaR and EVaR of the total aggregate risk $S = S_1 + S_2$ are given in the following propositions.

**Proposition 1** *Let $X_1, X_2$ be two exponentially distributed random variables with joint cdf defined by a bivariate FGM copula as follows*

$$F_{X_1,X_2}(x_1,x_2) = C_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2)),$$

*where $\Theta \in [-1,1]$. At the level of $k$, at the term of $S = S_1 + S_2$, we have,*

$$\begin{aligned}\mathrm{TVaR}_k(S) =& \frac{1}{1-k}\sum_{t=0}^{M}[(1+\theta)\varphi_1(t;\mathrm{VaR}_k(S);\lambda_1;\lambda_2) - \theta\varphi_1(t;\mathrm{VaR}_k(S);2\lambda_1;\lambda_2) - \\ & \theta\varphi_1(t;\mathrm{VaR}_k(S);\lambda_1;2\lambda_2) + \theta\varphi_1(t;\mathrm{VaR}_k(S);2\lambda_1;2\lambda_2)],\\ \varphi_1(t;x;\lambda_1;\lambda_2) =& \frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1(tD-x)_-}((x\vee tD)+\frac{1}{\lambda_1}) + \frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2(tD-x)_-}((x\vee tD)+\frac{1}{\lambda_2}).\end{aligned}$$

**Proof** Suppose the distribution of $S$ is continuous. We get $Pr(S \le \mathrm{VaR}_k(S) - k) = 0$, i.e.,

$$\frac{\mathrm{VaR}_k(S)(Pr(S \le \mathrm{VaR}_k(S)-k))}{1-k} = 0.$$

The TVaR of $S$ takes the form

$$\begin{aligned}\mathrm{TVaR}_k(S) =& \frac{1}{1-k}\int_{\mathrm{VaR}_k(S)}^{\infty} s f_S(s)\mathrm{d}s = \frac{1}{1-k}\sum_{t=0}^{M}\Big[\pi_t\int_{\mathrm{VaR}_k(S)}^{\infty} s f_{S_1}(s-tD)\mathrm{d}s\Big]\\ =& \frac{1}{1-k}\sum_{t=0}^{M}\Big[\pi_t\int_{\mathrm{VaR}_k(S)}^{\infty} s(1+\theta)h(s-tD;\lambda_1;\lambda_2) - \theta h(s-tD;\lambda_1;2\lambda_2) - \\ & \theta h(s-tD;2\lambda_1;\lambda_2) + \theta h(s-tD;2\lambda_1;2\lambda_2)\mathrm{d}s\Big].\end{aligned} \tag{8}$$

Define

$$\begin{aligned}\int_{\mathrm{VaR}_k(S)}^{\infty} sh(s-tD;\lambda_1;\lambda_2)\mathrm{d}s =& \int_{\mathrm{VaR}_k(S)\vee tD}^{\infty} s\Big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1(s-tD)} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2(s-tD)}\Big]\mathrm{d}s\\ =& \frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1(tD-(\mathrm{VaR}_k(S)\vee tD))}[(\mathrm{VaR}_k(S)\vee tD)+\frac{1}{\lambda_1})] + \\ & \frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2(tD-(\mathrm{VaR}_k(S)\vee tD))}[(\mathrm{VaR}_k(S)\vee tD)+\frac{1}{\lambda_2}]\\ =& \varphi_1(t;\mathrm{VaR}_k(S);\lambda_1;\lambda_2),\end{aligned}$$

where $a \vee b = \max(a,b)$. Substituting $\varphi_1$ into (7) gives

$$\mathrm{TVaR}_k(S) = \frac{1}{1-k}\sum_{t=0}^{M}[(1+\theta)\varphi_1(t;\mathrm{VaR}_k(S);\lambda_1;\lambda_2) - \theta\varphi_1(t;\mathrm{VaR}_k(S);2\lambda_1;\lambda_2) -$$

$$\theta\varphi_1(t;\mathrm{VaR}_k(S);\lambda_1;2\lambda_2)+\theta\varphi_1(t;\mathrm{VaR}_k(S);2\lambda_1;2\lambda_2)],$$

where

$$\varphi_1(t;x;\lambda_1;\lambda_2)=\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1(tD-x)_-}((x\vee tD)+\frac{1}{\lambda_1})+\frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2(tD-x)_-}((x\vee tD)+\frac{1}{\lambda_2}).$$

**Proposition 2** Under the assumptions mentioned in Proposition 1, we have

$$\begin{aligned}\mathrm{EVaR}_k(S)=&\log_{e^\delta}\{\frac{1}{1-k}\sum_{t=0}^{M}[(1+\theta)\varphi_2(t;\mathrm{VaR}_k(S);\lambda_1;\lambda_2)-\theta\varphi_2(t;\mathrm{VaR}_k(S);2\lambda_1;\lambda_2)-\\&\theta\varphi_2(t;\mathrm{VaR}_k(S);\lambda_1;2\lambda_2)+\theta\varphi_2(t;\mathrm{VaR}_k(S);2\lambda_1;2\lambda_2)]\},\end{aligned}$$

where

$$\varphi_2(t;x;\lambda_1;\lambda_2)=\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 tD}[\frac{1}{\lambda_1-\delta}e^{(\delta-\lambda_1)(x\vee tD)}]+\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_2 tD}[\frac{1}{\lambda_2-\delta}e^{(\delta-\lambda_2)(x\vee tD)}].$$

**Proof** Basically, the proof is analogous to the prior proposition. According to the definition of EVaR (5), if we have the related integral result to the $\varphi_1(t;b;a;\lambda_1;\lambda_2)$ in the proposition which we denote $\varphi_2(t;b;a;\lambda_1;\lambda_2)$, actually

$$\begin{aligned}\varphi_2(t;x;\lambda_1;\lambda_2):=&\int_x^\infty e^s h(s-tD;\lambda_1;\lambda_2)\mathrm{d}s\\=&\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 tD}[\frac{1}{\lambda_1-\delta}e^{(\delta-\lambda_1)(x\vee tD)}]+\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_2 tD}[\frac{1}{\lambda_2-\delta}e^{(\delta-\lambda_2)(x\vee tD)}].\end{aligned}$$

We could obtain the expression of EVaR represented in proposition.

### 4.1.2. The multivariate case

Now supposing that there are $N$ different exponential claim losses variables joined by a multivariate FGM copula, we have the pdf of $S_2$ (see (1)). Therefore, in terms of TVaR and EVaR for the multivariate case, the results can be got by substituting function $\varphi_1$ for $\varphi_3$ and $\varphi_2$ for $\varphi_4$, where

$$\begin{aligned}\varphi_3(t;x;\lambda_1;\lambda_2;\ldots;\lambda_N):=&\int_x^\infty sh(s-tD;\lambda_1;\lambda_2;\ldots;\lambda_N)\mathrm{d}s\\=&\sum_{i=1}^{N}\Big[\prod_{j=1,j\neq i}^{N}\frac{\lambda_j}{\lambda_i(\lambda_i-\lambda_j)}e^{\lambda_i(tD-x)_-}((x\vee tD)+\frac{1}{\lambda_i})\Big],\\\varphi_4(t;x;\lambda_1;\lambda_2;\ldots;\lambda_N):=&\int_x^\infty e^{\delta s}h(s-tD;\lambda_1;\lambda_2:\ldots;\lambda_N)\mathrm{d}s\\=&\sum_{i=1}^{N}\Big[\prod_{j=1,j\neq i}^{N}\frac{\lambda_i\lambda_j}{\lambda_i-\lambda_j}\frac{\lambda_i e^{\lambda_i tD}e^{(\delta-\lambda_i)(x\vee tD)}}{\lambda_i-\delta}\Big].\end{aligned}$$

Thus

$$\begin{aligned}&\mathrm{TVaR}_k(S)\\&=\frac{1}{1-k}\sum_{t=0}^{M}\Big[\varphi_3(t;\mathrm{VaR}_k(S);\lambda_1,\ldots,\lambda_N)+\sum_{k=2}^{N}\sum_{1\le j_1\le j_2\le\cdots j_k\le N}\theta_{j_1j_2\cdots j_k}\times\end{aligned}$$

$$\Big(\sum_{l=0}^{k}\sum_{(a_1\cdots a_k)\in A_{l,k}}(-1)^l\varphi_3(t;\mathrm{VaR}_k(S);2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_N})\Big)\Big],$$

$$\begin{aligned}&\mathrm{EVaR}_k(S)\\&=\log_{e^\delta}\Big\{\frac{1}{1-k}\sum_{t=0}^{M}\Big[\varphi_4(t;\mathrm{VaR}_k(S);\lambda_1,\ldots,\lambda_N)+\sum_{k=2}^{N}\sum_{1\le j_1\le j_2\le\cdots j_k\le N}\theta_{j_1j_2\cdots j_k}\times\\&\quad\Big(\sum_{l=0}^{k}\sum_{(a_1\cdots a_k)\in A_{l,k}}(-1)^l\varphi_4(t;\mathrm{VaR}_k(S);2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_N})\Big)\Big]\Big\}.\end{aligned}$$

### 4.2. TVaR-based capital allocation

The risk of measure presents the capital amount that has to be allocated to the total risk portfolio which includes $N$ insurances portfolio and credit portfolio for consolidating the capital, such as TVaR-based capital. Then this subsection presents the expressions for the contributions to the $N$ insurances portfolio and the credit portfolio by using the criteria (6). The related results are given in the following propositions.

**Proposition 3** *On the above assumptions, the attribution to $S_2$ over the TVaR-based capital is:*

$$\begin{aligned}&\mathrm{TVaR}_k(S_2;S)\\&=\frac{1}{1-k}\Big\{\sum_{t=0}^{M}tD\pi_t\Big[\varphi_5(t;\mathrm{VaR}_k(S);\lambda_1,\ldots,\lambda_N)+\sum_{k=2}^{N}\sum_{1\le j_1\le j_2\le\cdots j_k\le N}\theta_{j_1j_2\cdots j_k}\times\\&\quad\Big(\sum_{l=0}^{k}\sum_{(a_1\cdots a_k)\in A_{l,k}}(-1)^l\varphi_5(t;\mathrm{VaR}_k(S);2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_N})\Big)\Big]\Big\}\end{aligned}$$

*where*

$$\varphi_5(t;\mathrm{VaR}_k(S);\lambda_1,\ldots,\lambda_N)=\sum_{i=1}^{N}\Big(\prod_{j=1,j\ne i}^{N}\frac{\lambda_j}{\lambda_j-\lambda_i}\Big)e^{-\lambda_i(\mathrm{VaR}_k(S)-tD)_+}$$

**Proof** By the conditional expectation and independence of $S_1$ and $S_2$, we obtain:

$$E[S_2I_{\{S>\mathrm{VaR}_k(s)\}}]=\sum_{t=0}^{M}tD\pi_nP(S_1>(\mathrm{VaR}_k(S)-tD)).$$

Define

$$\begin{aligned}&\int_{\mathrm{VaR}_k(S)-tD}^{\infty}h(s_1;\lambda_1,\ldots,\lambda_N)\mathrm{d}s_1\\&=\int_{(\mathrm{VaR}_k(S)-nD)_+}^{\infty}h(s_1;\lambda_1,\ldots,\lambda_N)\mathrm{d}s_1\\&=\sum_{i=1}^{N}\Big(\prod_{j=1,j\ne i}^{N}\frac{\lambda_j}{\lambda_j-\lambda_i}\Big)e^{-\lambda_i(\mathrm{VaR}_k(S)-tD)_+}\\&=\varphi_5(t;\mathrm{VaR}_k(S);\lambda_1,\ldots,\lambda_N),\end{aligned}$$

thus

$$
\begin{aligned}
&E[S_2 I_{\{S>\mathrm{VaR}_k(s)\}}]\\
&\quad=\sum_{t=0}^{M} tD\pi_t\Big[\varphi_5(t;\mathrm{VaR}_k(S);\lambda_1,\ldots,\lambda_N)+\sum_{k=2}^{N}\sum_{1\le j_1\le j_2\le\cdots j_k\le N}\theta_{j_1 j_2\cdots j_k}\times\\
&\qquad\Big(\sum_{l=0}^{k}\sum_{(a_1\cdots a_k)\in A_{l,k}}(-1)^l\varphi_5(t;\mathrm{VaR}_k(S);2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_N})\Big)\Big].
\end{aligned}
$$

Because $\beta=0$, i.e., $Pr(S\le \mathrm{VaR}_k(S)-k)=0$, the proposition is got.

**Proposition 4** *The capital attributed to* $S_1$ *can be expressed as*

$$
\begin{aligned}
&\mathrm{TVaR}_k(S_1;S)\\
&\quad=\frac{1}{1-k}\sum_{t=0}^{M}\Big[\varphi_6((\mathrm{VaR}_k(S)-tD);\lambda_1,\ldots,\lambda_N)+\sum_{k=2}^{N}\sum_{1\le j_1\le j_2\le\cdots j_k\le N}\theta_{j_1 j_2\cdots j_k}\times\\
&\qquad\Big(\sum_{l=0}^{k}\sum_{(a_1\cdots a_k)\in A_{l,k}}(-1)^l\varphi_6(\mathrm{VaR}_k(S)-tD;2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_N})\Big)\Big],
\end{aligned}
$$

*where*

$$
\varphi_6(t;x;\lambda_1;\lambda_2;\ldots;\lambda_N)=\sum_{i=1}^{N}\Big[\prod_{j=1,j\ne i}^{N}\frac{\lambda_j}{\lambda_i(\lambda_i-\lambda_j)}e^{-\lambda_i x_+}(x_+ +\frac{1}{\lambda_i})\Big].
$$

**Proof** Because $S_1$ is a continuous random variable, by the conditional expectation and independence of $S_1$ and $S_2$, we obtain:

$$
E[S_1 I_{\{S=\mathrm{VaR}_k(s)\}}]=\sum_{n=0}^{+\infty}E[S_1 I_{\{S_1=\mathrm{VaR}_k(S)-nL\}}]P(S_2=nL)=0,
$$

$$
\frac{1}{1-k}E[S_1 I_{\{S>\mathrm{VaR}_k(s)\}}]=\frac{1}{1-k}\sum_{n=0}^{M}\pi_n\int_{\mathrm{VaR}_k(S)-tD}^{\infty}s_1 f_{s_1}(s_1)\mathrm{d}s_1.
$$

Define

$$
\begin{aligned}
\varphi_6(t;x;\lambda_1;\lambda_2;\ldots;\lambda_N):=&\int_x^{\infty}s_1 h(s_1;\lambda_1;\lambda_2;\ldots;\lambda_N)\mathrm{d}s_1\\
=&\sum_{i=1}^{N}\Big[\prod_{j=1,j\ne i}^{N}\frac{\lambda_j}{\lambda_i(\lambda_i-\lambda_j)}e^{-\lambda_i (x)_+}((x)_+ +\frac{1}{\lambda_i})\Big],
\end{aligned}
$$

thus

$$
\begin{aligned}
&\mathrm{TVaR}_k(S_1;S)\\
&\quad=\frac{1}{1-k}\sum_{t=0}^{M}\Big[\varphi_6((VaR_k(S)-tD);\lambda_1,\ldots,\lambda_N)+\sum_{k=2}^{N}\sum_{1\le j_1\le j_2\le\cdots j_k\le N}\theta_{j_1 j_2\cdots j_k}\times\\
&\qquad\Big(\sum_{l=0}^{k}\sum_{(a_1\cdots a_k)\in A_{l,k}}(-1)^l\varphi_6((\mathrm{VaR}_k(S)-tD);2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_N})\Big)\Big].
\end{aligned}
$$

## 5. An numerical example

In this concluding example we show how the risk measure and risk contribution discussed in Section 3 may be calculated. We consider the default portfolio of size $M$=1000 and insurance portfolio of size $N$=2 with parameter $\theta = 0.8$ which implicates a correlation of 0.2 between $X_1$ and $X_2$. In terms of other parameters, $\lambda_1 = 1/2$, $\lambda_1 = 1/3$ and the default probability $p$ equals 0.05. In order to make the expectation loss of two risk portfolio be equal, we consider $D = 0.1$. Then we estimate VaR, TVaR and EVaR at different probability levels. The results are listed in Table 1.

| | $S_1$ | | | $S_2$ | | | $S$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | VaR | TVaR | EVaR | VaR | TVaR | EVaR | VaR | TVaR | EVaR |
| 0.95 | 12.5627 | 15.7272 | 17.5756 | 6.1000 | 6.4579 | 6.4676 | 17.6887 | 20.7826 | 22.6194 |
| 0.97 | 14.4736 | 17.3885 | 19.0187 | 6.3000 | 6.6558 | 6.6645 | 19.5288 | 22.4125 | 24.0206 |
| 0.99 | 18.0508 | 21.2234 | 22.5651 | 6.7000 | 6.9307 | 6.9385 | 23.1233 | 26.2947 | 27.5809 |

Table 1 Estimates of risk measure for different portfolios

From Table 1 we have that with the increase of the value of $k$, the estimates of risk of measure are also increasing. Secondly, we could find that TVaR and EVaR for $S$ are smaller than the sum of TVaR and EVaR for $S_1$ and the risk of measure for $S_2$. From the economical aspect, the risk becomes smaller by diversification. Another result is that the EVaR mostly highlights the tail risk, TVaR more and the VaR is smallest among the three risk measure tools.

| | $S_2'$ | | | $S_2$ | | |
|---|---|---|---|---|---|---|
| $k$ | VaR | TVaR | EVaR | VaR | TVaR | EVaR |
| 0.95 | 6.1000 | 6.4579 | 6.4676 | 6.2000 | 6.5250 | 6.5375 |
| 0.97 | 6.3000 | 6.6558 | 6.6645 | 6.4000 | 6.6801 | 6.6905 |
| 0.99 | 6.7000 | 6.9307 | 6.9395 | 6.7000 | 6.9703 | 6.9764 |

Table 2 Estimates of default portfolio with binomial distribution and poisson distribution

We could find that when dealing with the large default portfolio, the poisson distribution is a nice estimate of the binomial distribution from Table 2. The relative errors are less than 2 percent at different levels and it could simplify the process when getting the expression of TVaR and EVaR.

| $k$ | TVaR($S$) | TVaR($S_1; S$) | TVaR($S_2; S$) |
|---|---|---|---|
| 0.95 | 20.7826 | 6.0821 | 14.7005 |
| 0.97 | 22.4125 | 6.0937 | 16.3188 |
| 0.99 | 26.2947 | 6.1313 | 20.1634 |

Table 3 TVaR-based allocations for different portfolios

At last we show the estimates of TVaR-based allocations for different portfolios by Table 3. Comparing the $\mathrm{TVaR}_k(S_1)$ and $\mathrm{TVaR}_k(S_2)$ in Table 1 with $\mathrm{TVaR}(S_1; S)$ and $\mathrm{TVaR}(S_2; S)$, we could find that the risks become smaller whatever is $S_1$ or $S_2$. In other ways, the two portfolios both decrease their risk by diversification.

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