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Generalized Local Time of the Indefinite Wiener Integral: White Noise Approach

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Abstract In this paper, the generalized local time of the indefinite Wiener integral X_t is discussed through white noise approach, which means to regard the local time as a Hida distribution. Moreover, similar result is also obtained in case of two independent Brownian motions by using the similar approach.

Keywords local time; Hida distribution; white noise approach.

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1. Introduction

Let $B(u)$ be a Brownian motion. The indefinite Wiener integral X_t is defined as follows $X_t = \int_0^t f(u)\mathrm{d}B(u)$. The object of study in this paper will be the generalized local time of indefinite Wiener integral, which is formally defined as

$$L_T(s,t) = \int_0^T \int_0^T \delta(X_t - X_s)\mathrm{d}s\mathrm{d}t,$$

where $\delta(X_t - X_s)$ is called the Donsker's delta function. Moreover, for two independent Brownian motions $B^{(1)}$ and $B^{(2)}$, the similar result is also discussed.

In recent years, local times of Brownian motion (BM) and fractional Brownian motion (FBM) have been studied by several authors, e.g., see [1–3]. In [1], authors discussed the intersection local time of two independent BMs in $(S)^*$. They gave the chaos expansion of local time and proved it was square integrable through the white noise approach. Drumond et al. [2] discussed the local time for FBM as generalized white noise functionals, and for any dimension $d \geq 1$ expansions of self-intersection local times were given. On the other hand, Liang in [4] considered the generalized local time of the indefinite Skorohod integral by using the technique of the Itô-Skorohod integral and Malliavin calculus.

In this paper, motivated by [1, 4], we discuss the generalized local time of indefinite Wiener integral through white noise approach. The paper is organized as follows. In Section 2, we

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provide some background materials from white noise analysis. In Sections 3 and 4, we present the main results and their demonstrations.

2. White noise analysis

In this section we briefly recall some notions and facts in white noise analysis, and refer to [1, 5] for details.

The starting point of white noise analysis is the real Gelfand triple $S(R) \subset L^2(R, R^d) \subset S^*(R)$ where $S(R), S^*(R)$ are the Schwartz spaces of test functions and tempered distributions, respectively.

Let $(L^2) \equiv L^2(S^*(R), d\mu)$ be the Hilbert space μ-square integrable functionals on $S^*(R)$. Then by the Wiener-Itô-Segal isomorphism theorem, for each $\Phi \in (L^2)$ this implies the chaos expansion $\Phi(\omega) = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, F_n \rangle$. The second Gelfand triple is: $(S) \subset (L^2) \subset (S)^*$. Elements of (S) (resp. $(S)^*$) are called Hida testing (resp. generalized) functionals. For $f \in S(R)$, S-transform is defined to be the bilinear dual product on $(S) \times (S)^*$ by $S\Phi(f) = \ll \Phi, : \exp\langle \cdot, f \rangle : \gg$.

Lemma 2.1 ([1,5]) *Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space, and Φ_λ be a mapping defined on Ω with values in $(S)^*$. We assume S-transform of Φ:*

(1) is a μ-measurable function of λ for $f \in S(R)$;

(2) obeys a U-functional estimate

$$| S\Phi_\lambda(zf) | \leq C_1(\lambda) \exp\{C_2(\lambda) \mid z \mid^2 \mid A^p f \mid_2^2\}$$

for some fixed p and for $C_1 \in L^1(\mu)$, $C_2 \in L^\infty(\mu)$. Then Φ_λ is Bochner-integrable in the Hilbert spaces $(S)_{-q}$ for q large enough and

$$\int_\Omega \Phi_\lambda \mathrm{d}\mu(\lambda) \in (S)^*, \quad S(\int_\Omega \Phi_\lambda \mathrm{d}\mu(\lambda))(f) = \int_\Omega (S\Phi_\lambda)(f) \mathrm{d}\mu(\lambda).$$

3. The generalized intersection local time of X_t and X_s

In this section we will study the generalized local time L_T of indefinite integral $X_t = \int_0^t f(u) \mathrm{d}B(u)$, which is formally defined by the following expression

$$L_T(s,t) = \int_0^T \int_0^T \delta(X_t - X_s) \mathrm{d}s \mathrm{d}t$$

where δ is a Dirac delta function and f is the square integral function of $L^2[0,T]$. We always approximate the Dirac delta function by the heat kernel $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\{-\frac{x^2}{2\varepsilon}\}$.

Theorem 3.1 *For each $t > s > 0$, the Bochner integral*

$$\delta(X_t - X_s) = \frac{1}{2\pi} \int_R \exp\{i\lambda(X_t - X_s\} \mathrm{d}\lambda = \frac{1}{2\pi} \int_R \exp\{i\lambda \int_s^t f(u) \mathrm{d}B(u)\} \mathrm{d}\lambda$$

is a generalized white noise functional.

Proof To show this result, we need apply Lemma 2.1 to the S-transform of the integral with respect to Lebesgue measure on $[0,T]$. First suppose f is a step function $f(u)=\sum_{j=1}^{n}a_j\mathbf{I}_{[t_{j-1},t_j)}(u)$ where $t_0=s$ and $t_n=t$. We only need prove the result is true for the following equality

$$\delta(X_t-X_s)=\frac{1}{2\pi}\int_R \exp\{i\lambda\sum_{j=1}^{n}a_j(B(t_j)-B(t_{j-1}))\}\mathrm{d}\lambda.$$

In fact, since Brownian motion has independent increments, i.e., for any $s\le t_1<t_2<\cdots<t_n=t$ the random variables $B(t_1),B(t_2)-B(t_1),\ldots,B(t_n)-B(t_{n-1})$ are independent, by the definition of S-transform, we have

$$\begin{aligned}
&S(\exp\{i\lambda\sum_{j=1}^{n}a_j(B(t_j)-B(t_{j-1}))\})(g)\\
&=E(e^{i\lambda a_1\langle\omega+g,\mathbf{I}_{[t_0,t_1)}\rangle})E(e^{i\lambda a_2\langle\omega+g,\mathbf{I}_{[t_1,t_2)}\rangle})\cdots E(e^{i\lambda a_n\langle\omega+g,\mathbf{I}_{[t_{n-1},t_n)}\rangle})\\
&=\exp\{-\frac{|\lambda|^2}{2}\sum_{j=1}^{n}a_j^2(t_j-t_{j-1})\}\exp\{i\lambda\sum_{j=1}^{n}a_j\int_{t_{j-1}}^{t_j}g(x)\mathrm{d}x\}
\end{aligned}$$

for $g\in S(R)$. The measurability condition is obvious. Now we prove the bound condition. For $z\in\mathbb{C}$ and $g\in S(R)$, by Schwartz equality we have

$$\begin{aligned}
&|\, S(\exp\{i\lambda(B(t_j)-B(t_{j-1}))\})(zg)\,|\\
&\le\exp\{-\frac{1}{4}\,|\,\lambda\,|^2\,a_j^2(t_j-t_{j-1})\}\exp\{-\frac{1}{4}\,|\,\lambda\,|^2\,a_j^2(t_j-t_{j-1})+|\,z\,||\,\lambda\,|\int_{t_{j-1}}^{t_j}g(x)\mathrm{d}x\}\\
&\le\exp\{-\frac{1}{4}\,|\,\lambda\,|^2\,a_j^2(t_j-t_{j-1})\}\exp\{\frac{|\,z\,|^2}{t_j-t_{j-1}}(\int_{t_{j-1}}^{t_j}g(x)\mathrm{d}x)^2\}\\
&\le\exp\{-\frac{1}{4}\,|\,\lambda\,|^2\,a_j^2(t_j-t_{j-1})\}\exp\{|\,z\,|^2\|\,g(x)\,\|_{L^2}^2\},
\end{aligned}$$

where, as a function of λ, the first factor is integral on R and the second factor is a constant. Hence

$$|\, S(\exp\{i\lambda\sum_{j=1}^{n}a_j(B(t_j)-B(t_{j-1}))\})(zg)\,|\le\exp\{-\frac{1}{4}C_3\,|\,\lambda\,|^2\,(t-s)\}\exp\{n\,|\,z\,|^2\|\,g\,\|_{L^2}^2\},$$

where $C_3=\min_{1\le j\le n}\{a_j^2\}$. By Lemma 2.1, the result is obtained.

Next suppose $f\in L^2[0,T]$. By [6], we can choose a sequence $\{f_n\}_{n=1}^{\infty}$ of step functions converging to f in $L^2[0,T]$. By the dominate convergence theorem, we obtain

$$\lim_{n\to\infty}\delta(X_t-X_s)=\lim_{n\to\infty}\frac{1}{2\pi}\int_R\exp\{i\lambda\int_s^t f_n(u)\mathrm{d}B(u)\}\mathrm{d}\lambda.$$

By the first part proof, the result is also proved in the case of $f\in L^2[0,T]$. □

We are now ready to state our main result on the generalized intersection local time L_T as well as on its subtracted counterpart $L_T^{(N)}$.

Theorem 3.2 *For $t>s>0$, the truncated generalized intersection local time of X_t and X_s given by*

$$L_T^{(N)}(s,t)=\int_0^T\int_0^T\delta^{(N)}(X_t-X_s)\mathrm{d}s\mathrm{d}t$$

is a Hida distribution, where

$$\delta^{(N)}(X_t - X_s) \equiv \frac{1}{2\pi}\int_{\mathbb{R}} \exp_N\{i\lambda(X_t - X_s)\}\mathrm{d}\lambda,\ \ \exp_N(x) \equiv \sum_{n=N}^{\infty}\frac{x^n}{n!}.$$

Proof Let f be a step function $f(u) = \sum_{j=1}^n a_j \mathbf{I}_{[t_{j-1},t_j)}(u)$ where $t_0 = s$ and $t_n = t$. By Theorem 3.1, it is easy to see that

$$S(\delta^{(N)}(X_t - X_s))(g) = \frac{1}{(2\pi\sum_{j=1}^n a_j^2(t_j - t_{j-1}))^{\frac{1}{2}}}\exp_N\{-\frac{(\sum_{j=1}^n a_j\int_{t_{j-1}}^{t_j} g(x)\mathrm{d}x)^2}{2\sum_{j=1}^n a_j^2(t_j - t_{j-1})}\}$$

for all $g \in S(R)$. Hence for $z \in \mathbb{C}$, it follows that

$$\begin{aligned}
&\mid S(\delta^{(N)}(X_t - X_s))(zg) \mid \\
&\le \frac{1}{(2\pi\min_{1\le j\le n}\{a_j^2\}\sum_{j=1}^n(t_j - t_{j-1}))^{\frac{1}{2}}}(t-s)^N\exp_N\{C_4 \mid z \mid^2 \{\inf_{s\le x\le t} \mid g(x) \mid\}^2\} \\
&= C_5(t-s)^{N-\frac{1}{2}}\exp\{C_4 \mid z \mid^2 \{\inf_{s\le x\le t} \mid g(x) \mid\}^2\}
\end{aligned}$$

for suitable constants C_4 and C_5, where $(t-s)^{N-\frac{1}{2}}$ is integral on $[0,T]\times[0,T]$ for all positive integers N. In fact, we have

$$\begin{aligned}
&\mid \exp_N\{-\frac{(\sum_{j=1}^n a_j\int_{t_{j-1}}^{t_j} zg(x)\mathrm{d}x)^2}{2\sum_{j=1}^n a_j^2(t_j - t_{j-1})}\} \mid \\
&\le \exp_N\{\frac{\{\sum_{j=1}^n a_j^2\}\{\sum_{j=1}^n(t_j - t_{j-1})^2\} \mid z \mid^2 \{\inf_{s\le x\le t} \mid g(x) \mid\}^2}{2\min_{1\le j\le n}\{a_j^2\}\sum_{j=1}^n(t_j - t_{j-1})}\} \\
&\le \exp_N\{C_4(t-s) \mid z \mid^2 \{\inf_{s\le x\le t} \mid g(x) \mid\}^2\} \\
&\le (t-s)^N\exp\{C_4 \mid z \mid^2 \{\inf_{s\le x\le t} \mid g(x) \mid\}^2\}. \quad \square
\end{aligned}$$

From the proof of Theorem 3.2, when we take $f(u) = \mathbf{I}_{[0,t]}(u)$, the intersection local time of X_t and X_s is the intersection local time of B_t and B_s. Hence the following corollary is obtained.

Corollary 3.3 *For $t > s > 0$, the intersection local time of B_t and B_s given by*

$$L_T(s,t) = \frac{1}{2\pi}\int_0^T\int_0^T\int_R \exp\{i\lambda(B_t - B_s)\}\mathrm{d}\lambda\mathrm{d}s\mathrm{d}t$$

is a Hida distribution.

4. The generalized collision local time of $X_t^{(1)}$ and $X_s^{(2)}$

In the section we will discuss the generalized local time L_T of indefinite integral $X_t^{(1)} = \int_0^t f_1(u)\mathrm{d}B^{(1)}(u)$ and $X_s^{(2)} = \int_0^s f_2(v)\mathrm{d}B^{(2)}(v)$, where $B^{(1)}$ and $B^{(2)}$ are two independent Brownian motions and f_1, f_2 are all in $L^2[0,T]$.

Theorem 4.1 *For each $t, s > 0$, the Bochner integral*

$$\delta(X_t^{(1)} - X_s^{(2)}) = \frac{1}{2\pi}\int_R \exp\{i\lambda(X_t^{(1)} - X_s^{(2)})\}\mathrm{d}\lambda$$

is a generalized white noise functional.

Proof Suppose f_1 and f_2 are all step functions

$$f_1(u)=\sum_{l=1}^{n}a_l\mathbf{I}_{[t_{l-1},t_l)}(u),\ \ f_2(v)=\sum_{k=1}^{m}b_k\mathbf{I}_{[s_{k-1},s_k)}(v),$$

where $t_0=0$, $t_n=t$, $0\le u<t\le T=1$ and $s_0=0$, $s_m=s$, $0\le v<s\le T=1$. Since $B^{(1)}$ and $B^{(2)}$ are two independent Brownian motions, $B^{(1)}(t_l)-B^{(1)}(t_{l-1})$ and $B^{(2)}(s_k)-B^{(2)}(s_{k-1})$ are also independent. By the definition of S-transform, we have

$$\begin{aligned}
&S(\exp\{i\lambda(X_t^{(1)}-X_s^{(2)})\})(g)\\
&=\prod_{l=1}^{n}E(e^{i\lambda a_l\langle\omega_1+g,\mathbf{I}_{[t_{l-1},t_l)}\rangle})\prod_{k=1}^{m}E(e^{-i\lambda b_k\langle\omega_2+g,\mathbf{I}_{[s_{k-1},s_k)}\rangle})\\
&=\{\exp\{-\frac{|\lambda|^2}{2}\sum_{l=1}^{n}a_l^2(t_l-t_{l-1})\}\exp\{i\lambda\sum_{l=1}^{n}a_l\int_{t_{l-1}}^{t_l}g(x)\mathrm{d}x\}\}\cdot\\
&\quad\{\exp\{-\frac{|\lambda|^2}{2}\sum_{k=1}^{m}b_k^2(s_k-s_{k-1})\}\exp\{-i\lambda\sum_{k=1}^{m}b_k\int_{s_{k-1}}^{s_k}g(x)\mathrm{d}x\}\}
\end{aligned}$$

for $g\in S(R)$. The measurability condition is obvious. Similarly to the proof of Theorem 3.1, for $z\in\mathbb{C}$ and $g\in S(R)$, we have

$$\begin{aligned}
&|\ S(\{\exp i\lambda(X_t^{(1)}-X_s^{(2)})\})(zg)\ |\\
&\le\exp\{-\frac{1}{4}|\lambda|^2(\sum_{l=1}^{n}a_l^2(t_l-t_{l-1})+\sum_{k=1}^{m}b_k^2(s_k-s_{k-1}))\}\exp\{(n+m)\,|z|^2\|\,g(x)\,\|_{L^2}^2\},
\end{aligned}$$

where, as a function of λ, the first factor is integral on R and the second factor is a constant, which implies that $\delta(X_t^{(1)}-X_s^{(2)})$ is a Hida distribution. □

Theorem 4.2 *For $t,s>0$, the truncated generalized collision local time of $X_t^{(1)}$ and $X_s^{(2)}$ given by*

$$L_T^{(N)}(s,t)=\int_0^T\int_0^T\delta^{(N)}(X_t^{(1)}-X_s^{(2)})\mathrm{d}s\mathrm{d}t$$

is a Hida distribution, where

$$\delta^{(N)}(X_t^{(1)}-X_s^{(2)})\equiv\frac{1}{2\pi}\int_{\mathbb{R}}\exp_N\{i\lambda(X_t^{(1)}-X_s^{(2)})\}\mathrm{d}\lambda,\ \ \exp_N(x)\equiv\sum_{n=N}^{\infty}\frac{x^n}{n!}.$$

Proof Let f_1 and f_2 be step functions $f_1(u)=\sum_{l=1}^{n}a_l\mathbf{I}_{[t_{l-1},t_l)}(u)$, $f_2(v)=\sum_{k=1}^{m}b_k\mathbf{I}_{[s_{k-1},s_k)}(v)$. By Theorem 4.1, we find that

$$\begin{aligned}
S(\delta^{(N)}(X_t^{(1)}-X_s^{(2)}))(g)=&\frac{1}{(2\pi(\sum_{l=1}^{n}a_l^2(t_l-t_{l-1})+\sum_{k=1}^{m}b_k^2(s_k-s_{k-1})))^{\frac{1}{2}}}\cdot\\
&\exp_N\{-\frac{(\sum_{l=1}^{n}a_l\int_{t_{l-1}}^{t_l}g(x)\mathrm{d}x-\sum_{k=1}^{m}b_k\int_{s_{k-1}}^{s_k}g(x)\mathrm{d}x)^2}{2(\sum_{l=1}^{n}a_l^2(t_l-t_{l-1})+\sum_{k=1}^{m}b_k^2(s_k-s_{k-1}))}\}
\end{aligned}$$

for all $g\in S(R)$. Because

$$|\ \exp_N\{-\frac{(\sum_{l=1}^{n}a_l\int_{t_{l-1}}^{t_l}g(x)\mathrm{d}x-\sum_{k=1}^{m}b_k\int_{s_{k-1}}^{s_k}g(x)\mathrm{d}x)^2}{2(\sum_{l=1}^{n}a_l^2(t_l-t_{l-1})+\sum_{k=1}^{m}b_k^2(s_k-s_{k-1}))}\}\ |$$

$$\le \exp_N\{\frac{2(\sum_{l=1}^n a_l\int_{t_{l-1}}^{t_l} g(x)\mathrm{d}x)(\sum_{k=1}^m \int_{s_{k-1}}^{s_k} g(x)\mathrm{d}x)}{C_6(s+t)}\}$$
$$\le \exp_N\{\frac{2\min_{1\le l\le n}\{a_l\}\inf_{0\le x\le T}\mid\{g(x)\mid\}\sum_{l=1}^n\int_{t_{l-1}}^{t_l}\mathrm{d}x}{C_6(s+t)}\cdot$$
$$\min_{1\le k\le m}\{b_k\}\inf_{0\le x\le T}\{\mid g(x)\mid\}\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\mathrm{d}x\}$$
$$\le \exp_N\{\frac{\min_{1\le l\le n}\{a_l\}\min_{1\le k\le m}\{b_k\}(\inf_{0\le x\le T}\mid g(x)\mid)^2(s+t)^2}{C_6(s+t)}\}$$
$$\le (s+t)^N\exp\{C_7(\inf_{0\le x\le T}\mid g(x)\mid)^2\},$$

where $C_6=\min_{1\le l\le n,1\le k\le m}\{a_l^2,b_k^2\}$ and $C_7=\frac{(\max_{1\le l\le n,1\le k\le m}\{a_l,b_k\})^2}{C_6}$.

Hence, it follows that

$$\mid S(\delta^{(N)}(X_t^{(1)}-X_s^{(2)}))(zg)\mid\le\frac{(s+t)^N}{(2\pi C_6(t+s))^{\frac12}}\exp\{C_7(\inf_{0\le x\le T}\mid g(x)\mid)^2\mid z\mid^2\}$$
$$\le C_8(s+t)^{N-\frac12}\exp_N\{C_7(\inf_{0\le x\le T}\mid g(x)\mid)^2\mid z\mid^2\},$$

where $C_8=(2\pi C_6)^{-\frac12}$ is a constant. And $(s+t)^{N-\frac12}$ is integral on $[0,T]\times[0,T]$ for all positive integers N. $\square$

From the proof of Theorem 4.2, the following corollary is obvious.

Corollary 4.3 *For $t,s>0$, the collision local time of $B_t^{(1)}$ and $B_s^{(2)}$ given by*

$$L_T(s,t)=\frac{1}{2\pi}\int_0^T\int_0^T\int_R\exp\{i\lambda(B_t^{(1)}-B_s^{(2)})\}\mathrm{d}\lambda\mathrm{d}s\mathrm{d}t$$

is a Hida distribution.

Remark 4.4 Comparing with work in [1], we extend the collision local time of Brownian motion to the case of indefinite Wiener integral $X_t^{(1)}$ and $X_s^{(2)}$.

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