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On Retractable S-Acts

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Abstract In this paper we introduce a class of right S-acts called retractable S-acts which are right S-acts with homomorphisms into their all subacts. We also give some classifications of monoids by comparing such acts with flatness properties.

Keywords retractable; S-acts; monoid.

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1. Introduction

Khuri in [4] introduced the notion of retractable modules, and then some excellent papers have appeared investigating this subject. Many papers have been written describing classes of acts relating to projectivity and injectivity. In this paper we introduce retractable right S-acts. In Section 1, we investigate general properties of retractable acts. In Section 2, we give a classification of monoids when retractable acts imply flatness properties. Finally, in Section 3, when flatness properties imply retractable acts are studied.

Throughout this paper S will denote a monoid. The reader is referred to [3] for basic results and definitions relating to semigroups, acts and flatness properties which are used here. A right S-act A_S is called retractable if for any subact B_S of A_S, $\mathrm{Hom}(A_S, B_S) \neq \emptyset$. It is clear that for a right S-act A_S to be retractable it is enough that $\mathrm{Hom}(A_S, aS) \neq \emptyset$ for each $a \in A_S$. Obviously, every simple or θ-simple right S-act is retractable. Since for each $s \in S$ there exists $\lambda_s : S \to sS$ by $\lambda_s(t) = st$, S is retractable. The following lemmas include some general properties of retractable acts.

Lemma 1.1 *The following hold for a monoid S.*

(i) S and Θ are retractable.

(ii) Every subact of a retractable right S-act is retractable.

(iii) A retract of a retractable S-act is retractable.

(iv) Let $\{A_i\}_{i\in I}$ be a family of retractable S-acts such that $\mathrm{Hom}(A_i, A_j) \neq \emptyset$ for each $i, j \in I$. Then $\coprod_{i\in I} A_i$ is retractable.

(v) Every projective or free S-act is retractable.

(vi) If A_S is retractable, then $\coprod_I^B A_S$ is retractable for any subact B_S of A_S.

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Lemma 1.2 *Let A_S be a right S-act such that* $\mathrm{Hom}(A_S, S) \neq \emptyset$. *Then A_S is retractable.*

Proof Suppose that $f : A_S \to S$ is a homomorphism. For each $a \in A_S$, we get $\pi f : A_S \to aS$, and A_S is retractable. □

The converse of the previous lemma is not true. For example Θ is retractable, but for a monoid S with no left zero $\mathrm{Hom}(\Theta_S, S) = \emptyset$. Moreover, by the previous lemma, for each right S-act A_S, $A_S \coprod S$ is retractable.

Two following examples show that retractable S-acts are not preserved under product and factor.

Example 1.3 Let $S = (\mathbb{N}, \cdot)$, and P be the collection of prime numbers. If G is a group, G is a right S-act by the action $g.n = g^n$ for $g \in G, n \in \mathbb{N}$. For any $p \in P$, $(\mathbb{Z}_p\ , +)$ as a right S-act is retractable since it is θ-simple. Now, take $A = \prod_{p \in P} \mathbb{Z}_p$. Consider $(\bar{1}, \bar{1}, \ldots) \in A$. There is no homomorphism $f : A_S \to (\bar{1}, \bar{1}, \ldots)S$. Otherwise, $f(\bar{0}, \bar{0}, \ldots) = (\bar{1}, \bar{1}, \ldots)k$ for some $k \in \mathbb{N}$, and then $(\bar{1}, \bar{1}, \ldots)kn = f(\bar{0}, \bar{0}, \ldots)n = f(\bar{0}, \bar{0}, \ldots) = (\bar{1}, \bar{1}, \ldots)k$ for each $n \in \mathbb{N}$. Let $n = 2$. Thus $k \cong 2k\ (p)$ for each $p \in P$, leading to a contradiction.

Example 1.4 A factor of a retractable S-act need not be retractable. Let S be a monoid without left zero. Then $S \coprod S$ is retractable but $S \coprod \Theta$ is not.

In general a coproduct of a family of retractable S-acts need not be retractable. For a group S with $|S| > 1$, S and Θ are retractable, but $S \coprod \Theta$ is not retractable.

Proposition 1.5 *The following are equivalent for a monoid S.*

(i) Every right S-act is retractable.

(ii) S contains a left zero.

(iii) Every coproduct of a family of retractable right S-acts is retractable.

(iv) Every factor of a retractable right S-act is retractable.

(v) Let $\{A_i\}_{i \in I}$ be a family of retractable S-acts. If $\prod_{i \in I} A_i$ is retractable, then so is each A_i.

Proof (i)⇒(ii). Since $S \coprod \Theta$ is retractable, there exists a homomorphism $f : S \coprod \Theta \to S$. Then $z = f(\theta)$ is a left zero of S.

(ii)⇒(i). Suppose that S contains a left zero. So every right S-act contains a zero. Let A_S be an arbitrary right S-act and B_S be a subact of A_S. Therefore, $f : A_S \to B_S$ defined by $f(a) = \theta$ for each $a \in A$ is a homomorphism, where θ is a zero element of B_S, and so $\mathrm{Hom}(A_S, B_S) \neq \emptyset$.

(iii)⇒(ii). Since $S \coprod \Theta$ is retractable, S contains a left zero.

(iv)⇒(ii). Since $S \coprod \Theta$ is a factor of $S \coprod S$, $S \coprod \Theta$ is retractable, and then S contains a left zero.

(v)⇒(i). Let A_S be a right S-act. By Lemma 1.2, $S \coprod A$ is retractable, then by assumption A_S is retractable. □

Now we consider retractable cyclic right S-acts.

Theorem 1.6 *Let ρ be a right congruence on S. Then S/ρ is retractable if and only if for each $s \in S$, there exists $u \in S$ such that $\rho \leq \rho(su)$.*

Proof Necessity. Suppose that S/ρ is retractable, and $s \in S$. There exists a homomorphism $f : S/\rho \to [s]_\rho S$. Let $f([1]_\rho) = [su]_\rho$ for $u \in S$. We show that $\rho \leq \rho(su)$. If $x\rho y$, then $f([x]_\rho) = f([y]_\rho)$. So $[sux]_\rho = [suy]_\rho$ and $sux\rho suy$. Therefore, $x\rho(su)y$.

Sufficiency. Let ρ be a right congruence on S, and $s \in S$. Then there exists $u \in S$ such that $\rho \leq \rho(su)$. Define $f : S/\rho \to [s]_\rho S$ by $f([1]_\rho) = [su]_\rho$. Since $\rho \leq \rho(su)$, f is well-defined. Therefore, S/ρ is retractable. □

Proposition 1.7 *Let K_S be a right ideal of S. Then S/K is retractable if and only if for each $s \in S$, $sS \cap K \neq \emptyset$ or S contains a left zero.*

Proof Necessity. Let S/K be retractable, and $s \in S$. Take $f : S/K \to [s]_\rho S$, and $f([t]_\rho) = [su]_\rho$ for $t \in K$. Thus $[su]_\rho$ is a zero of $[s]_\rho S$, and so $[su]_\rho t = [su]_\rho$ for each $t \in S$. If $su \in K$, $sS \cap K \neq \emptyset$. Now suppose that $z = su \notin K$. Then $zt = z$ for all $t \in S$, and z is a left zero of S.

Sufficiency. If S contains a left zero, by Proposition 1.5 every right S-act is retractable. Now, suppose that for each $s \in S$, $sS \cap K \neq \emptyset$. Define $f : S/K \to [s]_\rho S$, by $f([1]_\rho) = [su]_\rho$ for $su \in K$. Clearly f is well-defined, and thus S/K is retractable. □

In view of the previous proposition we have the following.

Corollary 1.8 *Every Rees factor right S-act is retractable if and only if S is left reversible or S contains a left zero.*

For a commutative monoid S, $\rho \leq \rho s$ for every right congruence ρ on S and $s \in S$. Therefore, we deduce the next corollary.

Corollary 1.9 *If S is a commutative monoid, then every cyclic right S-act is retractable.*

2. When retractability implies flatness properties

In this section we give a classification of monoids when retractable acts imply other properties of acts. For any S-act, we have the following implications:

free ⇒ projective ⇒ strongly flat ⇒ condition (P)⇒ flat ⇒ weakly flat ⇒ principally weakly flat ⇒ torsion free.

We start with torsion free.

Proposition 2.1 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is torsion free.

(ii) Every retractable right S-act with two generating elements is torsion free.

(iii) Any right cancellable element of S is right invertible.

(iv) All right S-acts are torsion free.

Proof (ii)⇒(iii). Let $s \in S$ be a right cancellable element of S. Take $I = sS$. Since, by Lemma

1.1, $S\coprod^I S$ is retractable, $S\coprod^I S$ is torsion free. So $I = S$, and s is right invertible. (iii)$\Leftrightarrow$(iv). It is shown in [5]. $\square$

A right ideal I_S of S satisfies condition (LU) if for every $s \in I$ there exists $x \in I$ such that $xs = s$.

Proposition 2.2 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is principally weakly flat.

(ii) S is a regular monoid.

(iii) Every right S-act is principally weakly flat.

Proof (i)$\Rightarrow$(ii). Let $s \in S$ and $I = sS$. Since $S\coprod^I S$ is retractable, $S\coprod^I S$ is principally weakly flat. So I satisfies condition (LU), and $sxs = s$ for some $x \in S$. Thus S is regular. (ii)$\Leftrightarrow$(iii). It is shown in [2]. $\square$

A monoid S satisfies condition (R) if for any elements $s, t \in S$ there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$.

Theorem 2.3 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is weakly flat.

(ii) Every right S-act is weakly flat.

(iii) S is a regular monoid which satisfies condition (R).

Proof Let A_S be an arbitrary right S-act. Since $S\prod A$ is retractable by Lemma 1.2, $S\prod A$ is weakly flat. Moreover, since Θ is retractable, Θ is weakly flat. So S is right reversible, and by Theorem 3.3 of [7], A_S is weakly flat. (ii)$\Leftrightarrow$(iii). It is shown in [1]. $\square$

Similarly to the proof of the previous proposition we can prove the following proposition.

Proposition 2.4 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is flat.

(ii) Every right S-act is flat.

Proposition 2.5 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act satisfies condition (P).

(ii) S is a group.

(iii) Every right S-act satisfies condition (P).

Proof (i)$\Rightarrow$(ii). If I_S is a proper right ideal of S, since $S\coprod^I S$ is retractable, $S\coprod^I S$ satisfies condition (P), a contradiction. Thus S must be a group. (ii)$\Leftrightarrow$(iii). It is shown in [6]. $\square$

Theorem 2.6 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is equalizer flat.

(ii) Every retractable right S-act satisfies condition (E).

(iii) $S = \{1\}$ or $S = \{0, 1\}$.

(iv) Every right S-act satisfies condition (E).

Proof (ii)⇒(iii). Since Θ is retractable, Θ satisfies condition (E), and so S is left collapsible. If S is simple, S is a left collapsible group, $S = \{1\}$.

Now suppose that S is not simple and I_S is a proper right ideal of S. By Zorn's Lemma there exists a maximal right ideal N such that $I \subseteq N$. Then S/N is a θ-simple right S-act. So S/N is retractable, and then it satisfies condition (E). Thus $|N| = 1$, and then $|I| = 1$. Therefore, S is θ-simple. Since S contains a left zero, every right S-act is retractable, and by assumption every right S-act satisfies condition (E). Then $S = \{0, 1\}$. □

In light of the previous theorem and proposition we have the following result.

Proposition 2.7 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is free.

(ii) Every retractable right S-act is projective.

(iii) Every right S-act is strongly flat.

(iv) $S = \{1\}$.

(iv) Every right S-act is free.

Let P be a property of acts that reflects product, i.e., if A_i, $i \in I$, are right S-acts and $\prod_{i\in I} A_i$ has property P, then each A_i has, too. So retractable acts have property P if and only if all right S-acts satisfy property P. Thus we can replace property P with injectivity, weak injectivity, principal weak injectivity, divisibility, absolute (1)-purity.

Proposition 2.8 *The following are equivalent for a monoid S.*

(i) Every retractable right S-act is cofree.

(ii) $S = \{1\}$.

(iii) Every retractable right S-act is cofree.

Proof (i)⇒(ii). By assumption S is cofree. Then there exists a set I such that $|S| = |I^S|$. Then $|I| = 1$, and so $S = \{1\}$.

(iii)⇒(ii). Suppose that $|S| \geq 2$. Take $A = \Theta_S \coprod \Theta_S$ to be cofree. Thus $2 = |A| = |I^S|$ for some set I, which is a contradiction. Therefore $S = \{1\}$. □

3. When flatness properties imply retractability

In this section we consider monoids over which flatness properties of acts imply retractable acts.

As mentioned in Section 1, every free and projective right S-act is retractable. But in general strongly flat right S-act need not be retractable. Let $S = (\mathbb{N}, \max)$. S is left collapsible but contains no left zero. So $S \coprod \Theta$ is strongly flat but not retractable.

In light of Theorem 1.6, the following lemma can be easily proved.

Lemma 3.1 *Every cyclic right S-act which satisfies condition (P) is retractable if and only if for each right reversible submonoid T of S and for each $s \in S$ there exists $u \in S$ such that if $px = qy$ for $p, q \in T$, $x, y \in S$, then there exist $p', q' \in T$ such that $p'sux = q'suy$.*

Lemma 3.2 *The following are equivalent for a monoid S.*

(i) Every right S-act with a minimal generating set that satisfies condition (P) is retractable.

(ii) Every finitely generated right S-act which satisfies condition (P) is retractable.

(iii) For any right reversible submonoid T of S there exists $z \in S$ such that $zt = t$ for each $t \in T$.

(iv) For each cyclic right S-act A_S which satisfies condition (P), $\mathrm{Hom}(A_S, S) \neq \emptyset$.

Proof (ii)⇒(iii). Let T be a right reversible submonoid of S, and $\rho = \rho(T \times T)$. Clearly, $S/\rho \coprod S$ satisfies condition (P), and by assumption $S/\rho \coprod S$ is retractable. Then $\mathrm{Hom}(S/\rho, S) \neq \emptyset$. Let $f : S/\rho \to S$ and $f([1]_\rho) = z$. So for each $t \in T$, since $[t]_\rho = [1]_\rho = T$, $zt = f([t]_\rho) = f([1]_\rho) = z$.

(iii)⇒(iv). Suppose that S/ρ satisfies condition (P). So $\rho = \rho(T \times T) = \{(x, y) | \exists p, q \in T,\ px = qy\}$, $T = [1]_\rho$, and T is right reversible submonoid of S. So there exists $z \in S$ such that $zt = z$ for each $t \in T$. Now, define $f : S/\rho \to S$ by $f([s]_\rho) = zs$. Suppose that $x\rho\, y$. Then there exist $p, q \in T$ such that $px = qy$. Thus $zx = zpx = zqy = zy$, and f is well-defined.

(iv)⇒(i). Suppose that A_S is a right S-act with a minimal generating set $\{a_i | I \in i\}$ which satisfies condition (P). Since A_S satisfies condition (P), $A_S = \coprod_{i \in I} a_i S$. Since $\mathrm{Hom}(a_i S, S) \neq \emptyset$, $\mathrm{Hom}(a_i S, a_j S) \neq \emptyset$ for each $i, j \in I$. Now, by Lemma 1.1 the result follows. □

Suppose that P is a property of acts that is preserved under coproduct and also S has property P. Let A_S ba a right S-act with property P. Then $A_S \coprod S$ has property P. Now, if $A_S \coprod S$ is retractable, then $\mathrm{Hom}(A_S, S) \neq \emptyset$. Therefore we can deduce the next result.

Proposition 3.3 *The following are equivalent for a monoid S.*

(i) Every right S-act which satisfies condition (P) is retractable.

(ii) For each locally cyclic right S-act A_S which satisfies condition (P), $\mathrm{Hom}(A_S, S) \neq \emptyset$.

Proof It suffices to show (ii)⇒(i). Suppose that A_S is a right S-act which satisfies condition (P). So $A_S = \coprod_{i \in I} A_i$ such that each A_i is indecomposable which satisfies condition (P). Thus each A_i is locally cyclic. Since $\mathrm{Hom}(A_i, S) \neq \emptyset$, $\mathrm{Hom}(A_i, A_j) \neq \emptyset$ for each $i, j \in I$. Now, by Lemma 1.1 the result follows. □

Example 3.4 If every cyclic right S-act which satisfies condition (P) is retractable, we cannot deduce that every finitely generated right S-act which satisfies condition (P) is retractable. $S = (\mathbb{N}, .)$ is commutative and by Corollary 1.8 every cyclic right S-act is retractable. But S is right reversible but contains no left zero. So $S \coprod \Theta$ satisfies condition (P) but it is not retractable.

As we know, all flatness properties are preserved under coproduct. So similarly to the argument of Theorem 3.2 and Proposition 3.3, the following two results hold.

Theorem 3.5 *The following are equivalent for a monoid S.*

(i) Every strongly flat right S-act with a minimal generating set is retractable.

(ii) Every strongly flat finitely generated right S-act is retractable.

(iii) For any left collapsible submonoid T of S there exists $z \in S$ such that $zt = t$ for each $t \in T$.

(iv) For each strongly flat cyclic right S-act A_S, Hom$(A_S, S) \neq \emptyset$.

Corollary 3.6 *Every flat (weakly flat, principally weakly flat, condition (E), strongly flat) right S-act is retractable if and only if* $\mathrm{Hom}(A_S, S) \neq \emptyset$ *for each indecomposable flat (weakly flat, principally weakly flat, condition (E), strongly flat) right S-act A_S.*

Proposition 3.7 *The following are equivalent for a monoid S.*

(i) Every torsion free right S-act is retractable.

(ii) Every principally weakly flat right S-act is retractable.

(iii) S contains a left zero.

(iv) Every right S-act is retractable.

Proof (ii)$\Rightarrow$(iii). Since $S\coprod\Theta$ is principally weakly flat, $S\coprod\Theta$ is retractable. Then S contains a left zero. $\square$

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