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Acyclic Edge Coloring of Planar Graphs without Adjacent Triangles

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Abstract An acyclic edge coloring of a graph G is a proper edge coloring such that there are no bichromatic cycles. The *acyclic edge chromatic number* of a graph G is the minimum number k such that there exists an acyclic edge coloring using k colors and is denoted by $\chi'_a(G)$. In this paper we prove that $\chi'_a(G) \le \Delta(G)+5$ for planar graphs G without adjacent triangles.

Keywords acyclic edge coloring; acyclic edge chromatic number; planar graph.

MR(2010) Subject Classification 05C15

1. Introduction

All graphs considered in this paper are finite and simple. For any graph G, we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For undefined concepts we refer the readers to [1].

A proper edge coloring C is called an acyclic edge coloring if there are no bichromatic cycles in the graph G. The acyclic edge k-coloring of a graph G is that there exists an acyclic edge coloring using k colors. The acyclic edge chromatic number of a graph G is the minimum number k such that there exists an acyclic edge coloring using k colors and is denoted by $\chi'_a(G)$. In this paper, we use B to denote the color set of coloring.

In 2001, Alon et al. [2] gave the Acyclic Edge Coloring Conjecture (AECC for short). For any graphs G, $\chi'_a(G) \le \Delta(G)+2$.

For any graphs G, Alon et al. [3] proved that $\chi'_a(G) \le 64\Delta(G)$. Molloy and Reed [4] proved that $\chi'_a(G) \le 16\Delta(G)$. Basavaraju and Chandran [5] proved that $\chi'_a(G) \le \Delta(G)+3$ for graphs G with maximum degree 4.

For planar graph G, Hou et al. [6] proved that $\chi'_a(G) \le \max\{2\Delta(G)-2, \Delta(G)+22\}$. Fiedorowicz et al. [7] proved that $\chi'_a(G) \le \Delta(G)+6$ for a planar graph G without cycles of length three. Borowiecki and Fiedorowicz [8] proved that $\chi'_a(G) \le \Delta(G)+15$ for a planar graph G without cycles of length four. Basavaraju and Chandran [9] proved that $\chi'_a(G) \le \Delta(G)+12$ for

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any planar graphs G. Wang et al. [10] proved that $\chi'_a(G) \le \Delta(G)$ if there exists a pair$(k,m) \in \{(3,11),(4,8),(5,7),(8,6)\}$ such that planar graph G satisfies $\Delta \ge k$ and $g(G) \ge m$. In this paper we prove that $\chi'_a(G) \le \Delta(G)+5$ for planar graph G without adjacent triangles.

Let G be a graph. A k-vertex of G is a vertex of degree k. Similarly, a k^+-vertex of G is a vertex of degree at least k. A face of degree 3 will be called a triangle. Furthermore, a triangle is called a good triangle if the boundary vertices of it have at least two 5^+-vertices. A triangle is called a bad triangle if the boundary vertices of it have exactly two 4-vertices and a 5^+-vertex. A 4-vertex is called a bad 4-vertex if it is adjacent to a 4-vertex. For $v \in V(G)$, we denote by $l_k(v)(l_{k^+}(v))$ the number of k-vertices (k^+-vertices) adjacent to v. Furthermore, we denote by $l_t(v)$ the number of triangles incident to v, and by $l_{\overline{4}}(v)$ (or $l_{\overline{t}}(v)$) the number of bad 4-vertices adjacent to v (or bad triangles incident to v).

Let H be a nonempty proper subgraph of G. A coloring C is said to be a partial coloring of G if the coloring C is a coloring of H. Furthermore, an acyclic edge coloring C of H is said to be a partial acyclic edge coloring of G. For $e \in E(G)$, the color α of B is said to candidate for edge e with respect to a partial acyclic edge coloring C of G if none of the adjacent edges of e is colored α. We denote by $R_C(e)$ the set of candidate colors of edge e with respect to the coloring C. An (α,β)-maximal bichromatic path with respect to a partial coloring C of G is a maximal path consisting of edges that are colored using the colors α and β alternatingly. An (α,β,u,v)-maximal bichromatic path is an (α,β)-maximal bichromatic path which starts at the vertex u with an edge colored α and ends at v. An (α,β,uv)-critical path is an (α,β,u,v) maximal bichromatic path which starts out from the vertex u with an edge colored α and ends at the vertex v with an edge colored α. For $uv \in E(G)$, let $H = G - uv$. With respect to an acyclic edge coloring C of H, $C(v)$ denotes the set of colors which are assigned by C to those edges in $E(H)$ incident to v. We denote by $C(uv)$ the color of edge uv with respect to the coloring C. Let $C_{uv} = C(v) - C(uv)$. A multiset is generalized set where a member can appear multiple in the set. If an element x appears t times in the multiset S, then we say that multiplicity of x in S is t, denoted by $D_S(\alpha)$. We denote by $\|S\| = \sum_{\alpha \in S} D_S(\alpha)$ the cardinality of finite multiset. Let S and S' be two multisets. A multiset is said to be the union of S and S', denoted by $S \uplus S'$, if the multiset $S \uplus S'$ has all the members of S and S' and $D_{S\uplus S'}(x) = D_S(x) + D_{S'}(x)$ for any member $x \in S \uplus S'$.

Lemma 1 ([5]) *Given a pair of colors α and β of a proper coloring C of G, there can be at most one maximal (α,β)-bichromatic path containing a particular vertex v, with respect to C.*

Lemma 2([5]) *Let $u,i,j,a,b \in V(G)$, $ui,uj,ab \in E(G)$. Also let $\{\lambda,\xi\} \subseteq B$ such that $\{\lambda,\xi\} \cap \{C(ui),C(uj)\} \neq \emptyset$ and $\{i,j\} \cap \{a,b\} = \emptyset$. Suppose there exists a (λ,ξ,ab)-critical path that contains vertex u with respect to a partial acyclic edge coloring C of G. Let C' be the partial coloring obtained from C by exchanging colors with respect to the edges ui and uj. If C' is proper, then there will not be any (λ,ξ,ab)-critical path in G with respect to the partial coloring C'.*

2. Lemma and the main result

Lemma 3 *Let G be a planar graph without adjacent triangles and $\delta(G) \geq 2$. Let $v \in V(G)$ and $d(v) = l$. If the neighbors of v are $v_1, \dots, v_l$, where $d(v_1) \leq \cdots \leq d(v_l)$, then G contains at least one of the following configurations:*

(A1) $l = 3$, $d(v_1) \leq 6$;

(A2) $l = 4$, $d(v_1) \leq 4$, $d(v_2) \leq 5$;

(A3) $l = 2$.

Proof We use the discharging method to prove the lemma. Suppose that the lemma is false and let G be a counterexample. We fix a plane embedding of G. Thus G contains none of the configurations (A1)–(A3). By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, using $\sum_{v \in V(G)} d(v) = 2|E(G)|$ and $\sum_{f \in F(G)} d(f) = 2|E(G)|$, we rewrite Euler's formula into the following new form:

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8.$$

Suppose that $w(x)$ denotes the weight function defined on $x \in V(G) \cup F(G)$ by $w(x) = d(x) - 4$. By some rules, we will get a new weight function $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. However, the total sum of weight is kept fixed. We have

$$0 \leq \sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) = -8,$$

which leads to an obvious contradiction. For $v \in V(G)$, we define the discharging rules as follows:

(R1) Every 7^+-vertex v, sends $\frac{1}{3}$ to each adjacent 3-vertex and $\frac{1}{12}$ to each adjacent bad 4-vertex;

(R2) Every 5^+-vertex v, sends $\frac{1}{2}$ to each incident triangle;

(R3) Every bad 4-vertex v, sends $\frac{1}{4}$ to each incident bad triangle;

(R4) Every 6-vertex v, sends $\frac{1}{12}$ to each adjacent bad 4-vertex.

Now we begin to prove the non-negativity of new weight $w'(x)$ for every $x \in V(G) \cup F(G)$. Suppose that $v \in V(G)$. Since G contains no (A3), we have $\delta(G) \geq 3$.

If $d(v) = 3$, then $w(v) = -1$. Since G contains no (A1), v is adjacent to all vertices which are 7^+-vertices. By rule (R1), we have $w'(v) = d(v) - 4 + \frac{1}{3} \cdot l_{7^+}(v) = -1 + \frac{1}{3} \cdot 3 = 0$.

If $d(v) = 4$, then $w(v) = 0$. If v is not a bad 4-vertex, we have $w'(v) = w(v) = 0$. If v is a bad 4-vertex, since G contains no (A1) and (A2), by definition of the bad 4-vertex, it is easy to find that v is exactly adjacent to three 6^+-vertices and incident to at most one bad triangle. By rules (R1), (R3) and (R4), we have $w'(v) = d(v) - 4 + \frac{1}{12} \cdot l_{6^+}(v) - \frac{1}{4} \cdot l_{\overline{t}}(v) \geq \frac{1}{12} \cdot 3 - \frac{1}{4} = 0$.

If $d(v) = 5$, then $w(v) = 1$. Since G contains no adjacent triangles, v is incident to at most two triangles. By rule (R2), we have $w'(v) = d(v) - 4 - \frac{1}{2} \cdot l_t(v) \geq 1 - \frac{1}{2} \cdot 2 = 0$.

If $d(v) = 6$, then $w(v) = 2$. Since G contains no adjacent triangles, v is incident to at most three triangles. By rules (R2) and (R4), we have $w'(v) = d(v) - 4 - \frac{1}{2} \cdot l_t(v) - \frac{1}{12} \cdot l_{\overline{4}}(v) \geq 2 - \frac{1}{2} \cdot 3 - \frac{1}{12} \cdot 6 = 0$.

If $d(v)=7$, then $w(v)=3$. Since G contains no adjacent triangles, v is incident to at most three triangles. We have $l_3(v)+l_t(v)\le d(v)$ and 3-vertex is adjacent to 7^+-vertices since G contains no (A1). By rules (R1) and (R2), we have $w'(v)=d(v)-4-\frac{1}{3}\cdot l_3(v)-\frac{1}{12}\cdot l_{\overline{4}}(v)-\frac{1}{2}\cdot l_t(v)\ge 3-\frac{1}{3}\cdot 4-\frac{1}{2}\cdot 3=\frac{1}{6}$.

If $d(v)\ge 8$, then $w(v)=d(v)-4$. Since G contains no adjacent triangles and (A1), $l_t(v)\le\lfloor\frac{d(v)}{2}\rfloor$ and $l_3(v)+l_t(v)\le d(v)$. By rules (R1) and (R2), we have $w'(v)=d(v)-4-\frac{1}{3}\cdot l_3(v)-\frac{1}{12}\cdot l_{\overline{4}}(v)-\frac{1}{2}\cdot l_t(v)\ge d(v)-4-\frac{1}{2}\cdot d(v)=\frac{d(v)}{2}-4\ge 0$.

Suppose that $f\in F(G)$. If $d(f)=3$, then $w(f)=-1$. Since G contains no (A1) and (A2), a triangle of G is either a good triangle or a bad triangle. If f is a good triangle, by rule (R2), we have $w'(f)=d(f)-4+\frac{1}{2}\cdot l_{5^+}(v)\ge -1+\frac{1}{2}\cdot 2=0$. If f is a bad triangle, by rules (R2) and (R3), we have $w'(f)=d(f)-4+\frac{1}{2}\cdot l_{5^+}(v)+\frac{1}{4}\cdot l_{\overline{4}}(v)=-1+\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2=0$.

If $d(f)\ge 4$, then $w'(f)=w(f)\ge 0$.

Hence for each $x\in V(G)\cup F(G)$, we have $w'(x)\ge 0$ and the proof is completed. $\Box$

Theorem 1 *Let G be a planar graph without adjacent triangles. Then $\chi'_a(G)\le\Delta(G)+5$.*

Proof Let G be a counterexample to the theorem with the minimum number of edges. It is obvious that G is a connected graph and $\delta(G)\ge 2$. So G contains at least one of three configurations described in Lemma 3. Let $k=\Delta(G)+5$.

Case 1 G contains a 3-vertex v. Let the neighbors of v be v_1,v_2,v_3, where $d(v_1)\le d(v_2)\le d(v_3)$ and $d(v_1)\le 6$.

Let $H=G-vv_1$. By the minimality of G, H has an acyclic edge coloring C using k colors. Without loss of generality, we can assume that $d(v_1)=6$. Let S_v be a multiset defined as $S_v=C_{vv_2}\uplus C_{vv_3}$.

Suppose that $|C(v)\cap C(v_1)|=0$. Since $|C(v)\cup C(v_1)|\le 2+\Delta(G)-1=\Delta(G)+1$, there exists a color α of B such that $\alpha\in B-C(v)\cup C(v_1)$. Using it to color the edge vv_1, therefore, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction.

Suppose that $|C(v)\cap C(v_1)|=1$. Let $v_1'\in N_H(v_1)$. Without loss of generality, we can assume that $C(vv_3)=C(v_1v_1')=1$. If there exists a color θ of B such that $\theta\in R_C(vv_1)$, using it to color the edge vv_1, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. Otherwise, there exists a $(1,\theta,vv_1)$-critical path with respect to the coloring C. We denote by C_1 the set of candidate colors of the edge vv_1, and one of C_1 and color 1 are the colors of the critical paths with respect to the coloring C. Since $|R_C(vv_1)|=k-6=\Delta(G)-1$, this implies that $C(vv_2)\notin C(v_3)$. If $C(vv_3)\notin C(v_2)$, now we exchange colors of the edges vv_2 and vv_3 to get a coloring C'. It is obvious that the coloring C' is an acyclic edge coloring of H. By Lemma 2, there exists no $(1,\gamma,vv_1)$-critical path for any color $\gamma\in C_1$ with respect to the coloring C'. We color vv_1 with a color α of B such that $\alpha\in R_{C'}(vv_1)$, therefore, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. If $C(vv_3)\in C(v_2)$, since $|B - C(v_1) \cup C(v_2)| \geq k - (5 + \Delta(G) - 1) = 1$, there exists a color α of B such that $\alpha \in B - C(v_1) \cup C(v_2)$. We recolor the edge vv_2 with α to get a coloring C'. It is obvious that the coloring C' is an acyclic edge coloring of H. Otherwise, there exists a $(1, \alpha, vv_2)$-critical path with respect to the coloring C. But, there exists a $(1, \alpha, vv_1)$-critical path with respect to the coloring C, leading to a contradiction by Lemma 1. Now we color vv_1 with $C(vv_2)$. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction.

Suppose that $|C(v) \cap C(v_1)| = 2$. Let $v_1', v_1'' \in N_H(v_1)$, and let $C(vv_2) = C(v_1v_1')$=1, $C(vv_3) = C(v_1v_1'')$=2. Since $|C(v) \cup C(v_1)| = 5$, we have $|R_C(vv_1)| = k - 5 = \Delta(G)$. If there exists a color θ of B such that $\theta \in R_C(vv_1)$, using it to color the edge vv_1, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. Otherwise, there exists a $(1, \theta, vv_1)$-critical path or $(2, \theta, vv_1)$-critical path with respect to the coloring C. Since $\|S_v\| = d(v_2) - 1 + d(v_3) - 1 \leq 2\Delta(G) - 2$, there exists a color α of B such that $\alpha \in R_C(vv_1)$ with multiplicity at most one in S_v. We can assume that the color α is in $C(v_3)$. Since there exists a $(2, \alpha, vv_1)$-critical path with respect to the coloring C, by Lemma 1, there exists no $(2, \alpha, vv_2)$-critical path with respect to the coloring C. Thus we recolor the edge vv_2 with the color α to get a coloring C'. It is obvious that the coloring C' is an acyclic edge coloring of H. We have $|C'(v) \cap C'(v_1)| = 1$, this situation is argued out as above, a contradiction.

Case 2 G contains a 4-vertex v. Let the neighbors of v be v_1, v_2 v_3 and v_4, where $d(v_1) \leq d(v_2) \leq d(v_3) \leq d(v_4)$, $d(v_1) \leq 4$ and $d(v_2) \leq 5$.

Let $H = G - vv_1$. By the minimality of G, H has an acyclic edge coloring C using k colors. Without loss of generality, we can assume that $d(v_1) = 4$ and $d(v_2) = 5$. Let S_v be a multiset defined as $S_v = C_{vv_2} \uplus C_{vv_3} \uplus C_{vv_4}$.

Suppose that $|C(v) \cap C(v_1)| = 0$. Since $|C(v) \cup C(v_1)| \leq 3 + (\Delta(G) - 1) = \Delta(G) + 2$, there exists a color α of B such that $\alpha \in B - C(v) \cup C(v_1)$, using it to color the edge vv_1, therefore, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction.

Suppose that $|C(v) \cap C(v_1)| = 1$. Let $v_1' \in N_H(v_1)$. Without loss of generality, we can assume that $C(vv_4) = C(v_1v_1') = 1$. Thus $|R_C(vv_1)| = k - 5 = \Delta(G)$. So there exists a color α of B such that $\alpha \in R_C(vv_1)$, using it to color the edge vv_1, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction.

Suppose that $|C(v) \cap C(v_1)| = 2$. Let $v_1', v_1'' \in N_H(v_1)$. Without loss of generality, we can assume that $C(vv_3) = C(v_1v_1')$=1, $C(vv_4) = C(v_1v_1'')$=2. Since $|C(v) \cup C(v_1)| = 4$, we have $|R_C(vv_1)| = k - 4 = \Delta(G) + 1$. If there exists a color θ of B such that $\theta \in R_C(vv_1)$, using it to color the edge vv_1, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. Otherwise, there exists a $(1, \theta, vv_1)$-critical path or $(2, \theta, vv_1)$-critical path with respect to the coloring C. We denote by C_2 the set of candidate colors of the edge vv_1, and one of C_2 and color 2 are the colors of the critical paths with respect to the coloring C. If colors 1 and 2 are not in S_v, we exchange colors of the edges vv_3 and

vv_4 to get a coloring C'. It is obvious that the coloring C' is an acyclic edge coloring of H. By Lemma 2, there exist no $(1,\gamma,vv_1)$-critical path for any color $\gamma\in C_1$ and $(2,\xi,vv_1)$-critical path for any color $\xi\in C_2$. If there exists a color θ of B such that $\theta\in C_1$, using it to color the edge vv_1, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. Otherwise, by Lemma 2, there exists no $(1,\theta,vv_1)$-critical path with respect to the coloring C', we have $C_1\subseteq C'(v_1'')$. Thus $(C_1\cup C_2)\subseteq C'(v_1'')$. But $|C_1\cup C_2|\geq\Delta(G)+1$, a contradiction since $|C'(v_1'')|\leq\Delta(G)$. If the color 1 or 2 is in S_v, since $\|S_v\|=d(v_2)-1+d(v_3)-1+d(v_4)-1\leq 4+\Delta(G)-1+\Delta(G)-1=2\Delta(G)+2$ and $|R_C(vv_1)|=\Delta(G)+1$, there exists a color α of $R_C(vv_1)$ such that the color α is in S_v with multiplicity at most one. Without loss of generality, we can assume that $\alpha\in C(v_4)$. Now we can recolor the edge vv_3 with α to get a coloring C'. The coloring C' is an acyclic edge coloring of H since the coloring C' is proper edge coloring and there exists no $(2,\alpha,vv_3)$-critical path with respect to the coloring C (since there exists a $(2,\alpha,vv_1)$-critical path with respect to the coloring C and Lemma 1). Thus $|C'(v)\cap C'(v_1)|=1$, this situation is argued out as above, a contradiction.

Suppose that $|C(v)\cap C(v_1)|=3$. Let $C(vv_2)$=1, $C(vv_3)$=2 and $C(vv_4)$=3. Thus $|R_C(vv_1)|=k-3=\Delta(G)+2$. If there exists a color θ of B such that $\theta\in R_C(vv_1)$, using it to color the edge vv_1, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. Otherwise, there exists a $(1,\theta,vv_1)$-critical path, $(2,\theta,vv_1)$-critical path or $(3,\theta,vv_1)$-critical path with respect to the coloring C. Since $\|S_v\|=(d(v_2)-1)+(d(v_3)-1)+(d(v_4)-1)\leq 4+\Delta(G)-1+\Delta(G)-1=2\Delta(G)+2$ and $|R_C(vv_1)|=\Delta(G)+2$, there exists a color α of B such that $\alpha\in R_C(vv_1)$ with multiplicity at most one in S_v. We can assume that the color α is in $C(v_4)$. Now we recolor the edge vv_3 with the color α to get a coloring C'. If the coloring C' is an acyclic edge coloring of H, thus $|C'(v)\cap C'(v_1)|=2$, this situation is argued out as above, a contradiction. If the coloring C' is not an acyclic edge coloring of H, there exists a $(3,\alpha,vv_3)$-critical path with respect to the coloring C. But, there exist a $(3,\alpha,vv_1)$-critical path with respect to the coloring C, by Lemma 1, a contradiction.

Now we consider that there exists no vertex v that belongs to configurations (A1) and (A2) as follows.

Case 3 G contains a 2-vertex v. Let the neighbors of v be v_1 and v_2, where $d(v_1)\leq d(v_2)$.

Now we delete all the 2-vertices from G to get a graph G'.

Case 3.1 If $\delta(G')\leq 1$, without loss of generality, we can assume that $\delta(G')=1$. Let $d_{G'}(v')=1$, and let u be the neighbor of v' in G'. Since $\delta(G)\geq 2$ and there exists no vertex v that belongs to configurations (A1) and (A2), we have $d_G(v')\geq 5$. Let x be the neighbor of v' and $d_G(x)=2$, and let $H=G-v'x$. By the minimality of G, H has an acyclic edge coloring C using k colors. Let y be the neighbor of x different from v'. If $|C(x)\cap C(v')|=0$, since $|C(v')\cup C(x)|\leq\Delta(G)-1+1=\Delta(G)$, there exists a color α of B such that $\alpha\in B-C(v')\cup C(x)$, using it to color the edge $v'x$, therefore, there are no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. If $|C(x)\cap C(v')|=1$, let z be 2-vertex in G and the neighbor of v', and let $C(v'z) = C(xy)$. Since $|C(v') \cup C(x) \cup C(z)| \leq (\Delta(G) - 1) + 1 - 1 + 1 = \Delta(G)$, there exists a color α of B such that $\alpha \in B - C(v') \cup C(x) \cup C(z)$, using it to color the edge $v'x$, therefore, there exists no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. If $C(xy) = C(uv')$, since $|C(y)| < k$, there exists a color α of B such that $\alpha \in B - C(y)$. We recolor the edge xy with α to get a coloring C', so it is obvious that the coloring C' is an acyclic edge coloring of H. If $|C'(x) \cap C'(v')| = 0$, this situation is argued out as above, a contradiction. If $|C'(x) \cap C'(v')| = 1$, then there exists a 2-vertex w which is the neighbor of v' in H such that $C'(v'w) = C'(xy)$, this situation is argued out as above, a contradiction.

Case 3.2 If $\delta(G') \geq 2$, by Lemma 3, there exists a vertex v' in G' such that v' belongs to one of the configurations (A1)-(A3), say A', and is not already in configuration A' in G. Let $M = \{x | x \in \{v'\} \cup N_{G'}(v'), d_{G'}(x) < d_G(x)\}$, and let u be the minimum degree vertex in M in the graph G'. It is obvious that $d_{G'}(u) \leq 6$. Let $N'(u) = \{x | x \in N_G(u), d_G(x) = 2\}$, and let $N''(u) = N_G(u) - N'(u)$. So, $N''(u) = N_{G'}(u)$. Since $u \in M$, we have $|N'(u)| \neq 0$. Thus there exists a vertex u' in $N'(u)$. Let the neighbors of u' be u and u'' in G, and let $H = G - \{uu'\}$. By the minimality of G, H has an acyclic edge k-coloring C. Let $B''(u) = \{C(ux) | x \in N''(u)\}$.

Suppose that $|C(u) \cap C(u')| = 0$. Since $|B - C(u) \cup C(u')| \geq k - (\Delta(G) - 1) - 1 = 5$, there exists a color α of B such that $\alpha \in B - C(u) \cup C(u')$, using it to color the edge uu', therefore, there exists no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction.

Suppose that $|C(u) \cap C(u')| = 1$. Let u_1 be the neighbor of u different from u' and $C(u'u'') = C(uu_1) = 1$. Furthermore, if $u_1 \in N'(u)$, then $|C(u) \cup C(u') \cup C(u_1)| \leq (\Delta(G) - 1) + 1 - 1 + 1 = \Delta(G)$. So there exists a color α of B such that $\alpha \in B - C(u) \cup C(u') \cup C(u_1)$, using it to color the edge uu', therefore, there exists no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. If $u_1 \in N''(u)$, we need to consider two cases as follows:

(1) If $d_{G'}(u) \leq 5$, since $|B - B''(u) \cup C(u'')| \geq k - (5 + \Delta(G) - 1) = 1$, there exists a color α of B such that $\alpha \in B - B''(u) \cup C(u'')$. We recolor the edge $u'u''$ with α to get a coloring C', it is obvious that the coloring C' is an acyclic edge coloring of H. If $|C'(u) \cap C'(u')| = 0$, this situation is argued out as above, a contradiction. If $|C'(u) \cap C'(u')| = 1$, then there exists a 2-vertex u_2 in $N'(u)$ different from u' such that $C'(uu_2) = C'(u'u'')$, this situation is argued out as above, a contradiction.

(2) If $d_{G'}(u) = 6$, then the vertex u is adjacent to at least one 3-vertex y in G which is not adjacent to 2-vertices. If $u_1 = y$, since $|B - C(u) \cup C(u') \cup C(u_1)| \geq k - (\Delta(G) - 1 + 1 - 1 + 2) = 4$, there exists a color α of B such that $\alpha \in B - C(u) \cup C(u') \cup C(u_1)$, using it to color the edge uu', therefore, there exists no bichromatic cycles. So we can extend the coloring C to an acyclic edge k-coloring of G, a contradiction. If $u_1 \neq y$, since there exists a 3-vertex y which is in $N''(u)$ and is not adjacent to 2-vertices in G, we have $|B - C(u'') \cup (B''(u) \setminus C(uy))| \geq k - (\Delta(G) + 6 - 1 - 1) = 1$. Thus there exists a color α of B such that $\alpha \in B - C(u'') \cup (B''(u) \setminus C(uy))$ with respect to the

coloring C. We recolor the edge $u'u''$ with the color α to get a coloring C'. It is obvious that the coloring is an acyclic edge coloring of H. If $|C'(u)\cap C'(u')|=0$, this situation is argued out as above, a contradiction. If $|C'(u)\cap C'(u')|=1$, then there exists a vertex u_2 which is 2-vertex or 3-vertex not adjacent to 2-vertices in $N_H(u)$ such that $C'(uu_2)=C'(u'u'')$, this situation is argued out as above, a contradiction. $\square$

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