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On Complex Oscillation Theory of Solutions of Some Higher Order Linear Differential Equations

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Abstract In this paper, we shall use Nevanlinna theory of meromorphic functions to investigate the complex oscillation theory of solutions of some higher order linear differential equation. Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and $f^{(k)} + A(z)f = 0$ has a solution f with $\lambda(f) < \rho(A)$, and suppose that $A_1 = A + h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then $g^{(k)} + A_1(z)g = 0$ does not have a solution g with $\lambda(g) < \infty$.

Keywords complex differential equations; entire function; the growth of order; the exponent of convergence of the zeros.

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1. Introduction and main results

We shall assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna theory of meromorphic functions [1–3], such as $T(r, f)$, $N(r, f)$, $m(r, f)$ and $S(r, f) = o(T(r, f))$ outside a set of finite measure. In addition, for a meromorphic function f in the complex plane $\mathbb{C}$, we shall use the notation $\rho(f)$ and $\lambda(f)$ to denote its order and the exponent of convergence of the zeros, respectively. They are defined as follows:

$$\rho(f) = \overline{\lim}_{r\to\infty}\frac{\log^+ T(r, f)}{\log r}, \quad \lambda(f) = \overline{\lim}_{r\to\infty}\frac{\log^+ N(r, \frac{1}{f})}{\log r}.$$

Suppose that $k \in \mathbb{N}$ and A is an entire function, and suppose that f_j $(j = 1, 2, \ldots, k)$ are solutions of

$$f^{(k)} + A(z)f = 0. \tag{1}$$

Hille proved that any solution of (1) is entire [4]. In recent years, a lot of works has been done in the connection between the growth of order ρ of A and the exponent of convergence λ of f_j $(j = 1, 2, \ldots, k)$, such as [5–9]. In particular it was shown in [6, 9] that if $k = 2$ and A is transcendental of order at most $\frac{1}{2}$, then (1) cannot have two linearly independent solutions f_1

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and f_2, each with $\lambda(f_j)$ finite, and a comparable result was proved for higher order equations in [10]. On the other hand, it is possible to have one solution f of (1) with no zeros at all, even for coefficients of very small growth. To see this, set $f = e^B$ where B is an entire function. Then f solves (1) with $k = 2$ and $-A = \frac{f''}{f} = B'' + (B')^2$, as well as similar equations of higher order obtained by computing $\frac{f^{(k)}}{f}$ in terms of B. In [5], Alotaibi proved that small perturbations of such equations lead to the exponent of convergence of zeros of solution is at least the order of growth of the coefficient A.

Theorem 1 ([5]) *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (1) has a solution f with $\lambda(f) < \rho(A)$, and suppose that*

$$A_1 = A + h, \tag{2}$$

where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then

$$g^{(k)} + A_1(z)g = 0 \tag{3}$$

does not have a solution g with $\lambda(g) < \rho(A)$.

In this paper, our main result shows, however, that small perturbations of such equations lead to solutions whose zeros must have infinite exponent of convergence and include a result of Alotaibi. The main result is the following:

Theorem 2 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (1) has a solution f with $\lambda(f) < \rho(A)$, and suppose that A_1 satisfies (2) and $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then the equation (3) does not have a solution g with $\lambda(g) < \infty$.*

The paper is organized as follows. In Section 2, we shall state and prove some lemmas which will be used in the proof of Theorem 2. In Section 3, we shall prove the Theorem 2.

2. Some lemmas

For the proof of Theorem 2, we need the following definition and lemmas.

Definition 1 ([2]) *Let $B(z_n, r_n)$ be open discs in the complex plane. We say that the countable union $\bigcup B(z_n, r_n)$ is an $R-$ set if $z_n \to \infty$ and $\sum r_n < \infty$.*

Lemma 1 ([2]) *Suppose that f is a meromorphic function of finite order. Then there exists a positive integer N such that*

$$|\frac{f'(z)}{f(z)}| = O(|z|^N)$$

holds for large z outside of an $R-$ set.

Before stating the following lemmas, for $E \subset [0, \infty)$, we define the Lebesgue measure of E by $mes(E)$ and the logarithmic measure of $E \subset [1, \infty)$ by $m_l(E) = \int_E \frac{\mathrm{d}t}{t}$, and define the upper and lower logarithmic density of $E \subset [1,\infty)$ respectively by

$$\overline{\mathrm{logdens}}E = \overline{\lim_{r\to\infty}} \frac{m_l(E\bigcap[1,r])}{\log r},$$

and

$$\underline{\mathrm{logdens}}E = \underline{\lim}_{r\to\infty} \frac{m_l(E\bigcap[1,r])}{\log r}.$$

The logarithmic density gives us an idea how big the set E is. The proof of our theorem highly depends on the following lemma.

Lemma 2 ([11]) *Let f be an entire function with $\rho(f) = \rho < \frac{1}{2}$ and suppose that $m(r)$ is defined as*

$$m(r) = \inf_{|z|=r} \log|f(z)|.$$

If $\sigma < \rho$, then the set $\{r : m(r) > r^{\sigma}\}$ has positive upper logarithmic density.

Moreover, we are going to use the following lemma, which gives an asymptotic representation for the logarithmic derivative of a solutions of (1) with few zeros. The first of these is a special case of a result from [10].

Lemma 3 *Let A be a transcendental entire function of finite order, and let E_1 be a subset of $[1,\infty)$ of infinite logarithmic measure with the property that for each $r \in E_1$ there exists an arc*

$$a_r = \{re^{it} : 0 \le \alpha_r \le t \le \beta_r \le 2\pi\}$$

of the circle $S(0,r)$ such that

$$\lim_{r\to\infty, r\in E_1} \frac{\min\{\log|A(z)| : z \in a_r\}}{\log r} = +\infty.$$

Let $k \ge 2$ and let f be a solution of (1) with $\lambda(f) < \infty$. Then there exists a subset $E_2 \subset [1,\infty)$ of finite measure, such that for large $r \in E_0 = E_1 - E_2$, we have

$$\frac{f'(z)}{f(z)} = c_r A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2})$$

holds for all $z \in a_r$, here the constant c_r satisfies $c_r^k = -1$ and may depend on r but not, for a given $r \in E_0$, on z, and the branch of $A(z)^{\frac{1}{k}}$ is analytic on a_r (including in the case where a_r is the whole circle $S(0,r)$).

We note that E_2 has finite measure and so finite logarithmic measure, and so E_0 has infinite logarithmic measure. Moreover, we exclude the case $k = 1$ because for $k = 1$ the general solution of (1) is

$$f = C\exp\Big(-\int_0^z A(t)\mathrm{d}t\Big), C \in \mathbb{C}.$$

We will employ the following well-known representation for higher order logarithmic derivatives [1].

Lemma 4 *Let f be an analytic function, and let $F = \frac{f'}{f}$. Then for $k \in \mathbb{N}$ we have*

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2}F^{k-2}F' + P_{k-2}(F),$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree at most $k-2$ when $k > 2$.

Remark 1 By using Lemma 4, we shall see that it is possible to have a solution f of (1) with no zeros. Let $F = \frac{f'}{f}$ and $f = e^B$ where B is an entire function with $F = B'$. It is clear to see that $f = e^B$, which has no zeros, solves (1).

Lemma 5 *Suppose that A is a transcendental entire function with $\rho(A) = \rho < \frac{1}{2}$ in complex plane $\mathbb{C}$. Suppose that f is an entire function with $\lambda(f) < \rho$. Then there exists a set $E_3 \subset [1, \infty)$ with $\overline{\text{logdens}} E_3 > 0$, such that for $\sigma < \rho$, we have*

$$\inf_{|z|=r \in E_3} \log |A(z)| > r^{\sigma}$$

and

$$\lim_{r \to \infty, r \in E_3} \frac{n(r, \frac{1}{f}) \log r}{T(r, A)} = 0$$

hold.

Proof By using Lemma 2, for any $\lambda(f) < \sigma < \rho$, there exists a set $E_0 \subset [1, \infty)$ with $\overline{\text{logdens}} E_0 > 0$, where

$$E_0 = \{r > 1 : \inf_{|z|=r} \log |A(z)| > r^{\sigma}\}. \tag{4}$$

Since $\lambda(f) < \sigma$, for any given $0 < \varepsilon < \frac{\sigma - \lambda(f)}{2}$, there exists $r_0 > 1$, such that

$$n(r, \frac{1}{f}) < r^{\lambda(f)+\varepsilon} \tag{5}$$

holds for all $r > r_0$. Set $E_3 = E_0 \bigcap [r_0, \infty)$, we claim $\overline{\text{logdens}} E_3 > 0$. In fact,

$$[r_0, \infty) = ([r_0, \infty) \bigcap E_0) \bigcup ([r_0, \infty) - E_0).$$

Thus,

$$\begin{aligned} \overline{\text{logdens}} E_3 &\geq \underline{\text{logdens}} E_3 = \underline{\text{logdens}} [r_0, \infty) - \underline{\text{logdens}} ([r_0, \infty) - E_0) \\ &\geq \underline{\text{logdens}} [r_0, \infty) - (1 - \overline{\text{logdens}} E_0) = \overline{\text{logdens}} E_0 > 0. \end{aligned}$$

By using (4), (5) and $T(r, A) \leq \log^+ M(r, A) \leq 3T(2r, A)$, for any $r \in E_3$, we obtain

$$\frac{n(r, \frac{1}{f}) \log r}{T(r, A)} \leq \frac{r^{\lambda(f)+\varepsilon} \log r}{(\frac{r}{2})^{\sigma}}.$$

So,

$$\lim_{r \to \infty, r \in E_3} \frac{n(r, \frac{1}{f}) \log r}{T(r, A)} = 0. \quad \Box$$

3. Proof of the Theorem 2

In this section, we give the proof of Theorem 2.

Proof Suppose that the equation (1) has a solution f with $\lambda(f) < \rho(A)$, and suppose that the equation (3) has a solution g with $\lambda(g) < \infty$. We can let

$$f = Pe^U, \tag{6}$$

$$g = Qe^V, \tag{7}$$

where P, Q, U and V are entire functions which satisfy $\rho(P) = \lambda(f) < \infty$ and $\rho(Q) = \lambda(g) < \infty$, and $\max\{\rho(U), \rho(V)\} \le \rho(A)$ (see [8]). Let

$$F = \frac{f'}{f} = \frac{P'}{P} + U', \quad G = \frac{g'}{g} = \frac{Q'}{Q} + V'. \tag{8}$$

Applying Lemma 4, we obtain

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2} F^{k-2}F' + P_{k-2}(F), \tag{9}$$

$$\frac{g^{(k)}}{g} = G^k + \frac{k(k-1)}{2} G^{k-2}G' + P_{k-2}(G), \tag{10}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \le 2$ and has degree at most $k-2$ when $k > 2$.

Pick τ, σ, such that

$$\max\{\lambda(f), \rho(h)\} < \tau < \sigma < \rho(A) < \frac{1}{2}. \tag{11}$$

By using Lemma 5, there exits a set $E_1 \subset [1, \infty)$ with $\overline{\text{logdens}} E_1 > 0$, such that

$$\inf_{|z|=r} \log |A(z)| > r^\sigma \tag{12}$$

holds for all $r \in E_1$. Let $E_2 \subset [1, \infty)$ be a subset of finite measure so that, for some $M_1 \in \mathbb{N}$,

$$|\frac{A'(z)}{A(z)}| + |\frac{P'(z)}{P(z)}| + |\frac{Q'(z)}{Q(z)}| \le r^{M_1}, |z| = r \ge 1, \quad r \notin E_2. \tag{13}$$

Such E_2 and M_1 exist by Lemma 1. For large $|z| = r \in E_1$ we also have, using (2), (11) and (12),

$$\log |A_1(z)| > \frac{r^\sigma}{2}. \tag{14}$$

The next step is to estimate $\frac{f'(z)}{f(z)}$ and $\frac{g'(z)}{g(z)}$ in terms of $A(z)$. We apply Lemma 3 to equation (1) and equation (3) by choosing a_r to be the whole circle $|z| = r \in E_1$. This is possible since (12) and (14) imply that Lemma 3 holds. Hence for large $r \in E_0 = E_1 - E_3$, where $E_2 \subset E_3$ and E_3 has finite measure, the following is true. We have, by Lemma 3,

$$\frac{f'(z)}{f(z)} = cA(z)^{\frac{1}{k}} - \frac{k-1}{2k} \frac{A'(z)}{A(z)} + O(r^{-2}), |z| = r, \quad c^k = -1, \tag{15}$$

and

$$\frac{g'(z)}{g(z)} = dA_1(z)^{\frac{1}{k}} - \frac{k-1}{2k} \frac{A_1'(z)}{A_1(z)} + O(r^{-2}), |z| = r, \quad d^k = -1. \tag{16}$$

Here c, d may depend on r, but not on z.

Next, we apply the binomial theorem to expand $A_1(z)^{\frac{1}{k}}$ and $\frac{A_1'(z)}{A_1(z)}$ in terms of $A(z)^{\frac{1}{k}}$ and $\frac{A'(z)}{A(z)}$. Using (11) and (12), we get for $|z| = r \in E_0$, by suppressing the variable z for brevity,

$$A_1^{\frac{1}{k}} = (A+h)^{\frac{1}{k}} = A^{\frac{1}{k}}(1+\frac{h}{A})^{\frac{1}{k}} = A^{\frac{1}{k}}(1+O(\frac{|h|}{|A|})) \tag{17}$$

and

$$\frac{A_1'}{A_1} = \frac{A'+h'}{A+h} = \frac{A'+h'}{A(1+\frac{h}{A})} = \frac{A'}{A}(1+O(\frac{|h|}{|A|})) + o(\frac{|h|}{|A|}). \tag{18}$$

Using (13), (16), (17) and (18), we deduce that, for $|z| = r \in E_0$,

$$\frac{g'(z)}{g(z)} = dA(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}), \quad d^k = -1. \tag{19}$$

We recall from Lemma 3 that c and d may depend on r but, for given $r \in E_0$, do not depend on z. The following Lemma is then the key to proof of Theorem 2.

Lemma 6 *Suppose that c and d are as in (15) and (19), respectively. Then $c = d$ for all large $r \in E_0$.*

Proof We may write $d = c\omega$ where $\omega^k = 1$. Using (19), we obtain

$$\frac{g'(z)}{g(z)} = c\omega A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}), \quad \omega^k = 1. \tag{20}$$

Multiplying (15) by ω and subtracting (20), we get

$$\omega(\frac{f'(z)}{f(z)} + \frac{k-1}{2k}\frac{A'(z)}{A(z)}) = \frac{g'(z)}{g(z)} + \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}).$$

Integrating around $|z| = r_n$, where $r_n \to \infty$ with $r_n \in E_0$, we then find that

$$\omega(n(r_n, \frac{1}{f}) + \frac{k-1}{2k}n(r_n, \frac{1}{A})) + o(1) = n(r_n, \frac{1}{g}) + \frac{k-1}{2k}n(r_n, \frac{1}{A}). \tag{21}$$

But the right hand side of (21) must be positive since $n(r_n, \frac{1}{g}) \geq 0$ and $n(r_n, \frac{1}{A}) > 0$. This is because if $n(r_n, \frac{1}{A}) = 0$ we get $N(r_n, \frac{1}{A}) = 0$. Since $\inf_{|z|=r_n} \log|A(z)|$ is very big for $r_n \to \infty, r_n \in E_0$, we get

$$m(r_n, \frac{1}{A}) = 0.$$

Hence,

$$T(r_n, \frac{1}{A}) = 0.$$

Using the first fundamental theorem of Nevanlinna theory, we obtain

$$T(r_n, A) = O(1).$$

This contradicts the fact that A is transcendental and proves the claim that $n(r_n, \frac{1}{A}) > 0$. For the same reason, $n(r_n, \frac{1}{f}) + n(r_n, \frac{1}{A})$ is a non-zero positive integer by recalling that $\omega^k = 1$. Hence, ω is a positive rational number. Since $|\omega| = 1$, we get $\omega = 1$ and so $c = d$. □

To complete the proof of theorem 2, we can now use (15), (19) and Lemma 6 to get, as $r \to \infty$ with $r \in E_0$,

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + o(1), \quad |z| = r.$$

Hence,

$$n(r, \frac{1}{f}) = n(r, \frac{1}{g}) \tag{22}$$

holds for lager $r \in E_0$.

Using (8), we get

$$\frac{P'(z)}{P(z)} + U' = \frac{Q'(z)}{Q(z)} + V' + o(1).$$

Using (13), we obtain

$$|U'(z) - V'(z)| \leq 2r^{M_1}$$

holds for $|z| = r$ and large $r \in E_0$. Since U and V are entire, we deduce that $Q_0 = U' - V'$ is a polynomial. Thus (8) becomes

$$F = G + M, \quad M = \frac{P'}{P} - \frac{Q'}{Q} + Q_0. \tag{23}$$

Using (1) and (9), we get

$$F^k + \frac{k(k-1)}{2} F^{k-2} F' + P_{k-2}(F) = -A, \tag{24}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree at most $k-2$ when $k > 2$. Combining (2), (3) and (10), we obtain

$$G^k + \frac{k(k-1)}{2} G^{k-2} G' + P_{k-2}(G) = -A - h. \tag{25}$$

Using (23) and (24), we get

$$(G + M)^k + \frac{k(k-1)}{2}(G+M)^{k-2}(G' + M') + P_{k-2}(G + M) = -A. \tag{26}$$

Combining (25) and (26), by the binomial theorem, we get

$$h = kMG^{k-1} + S_{k-2}(G, M), \tag{27}$$

where $S_{k-2}(G, M)$ is a differential polynomial in G, M and their derivatives, of total degree at most $k-2$ in G and its derivatives.

Now we claim that $M \not\equiv 0$. To prove the claim, we may assume that $M \equiv 0$. Using (23), we get $F \equiv G$. Using (24) and (25), we have $h \equiv 0$. This contradicts the hypothesis $h \not\equiv 0$ and completes the proof of the claim.

Dividing (27) by MG^{k-2}, we get

$$kG + \frac{S_{k-2}(G, M)}{MG^{k-2}} = \frac{h}{MG^{k-2}}. \tag{28}$$

Suppose that $|G| > 1$. Now $\frac{S_{k-2}(G,M)}{MG^{k-2}}$ is a sum of terms

$$\frac{1}{MG^{k-2}} M^{j_0}(M')^{j_1} \cdots (M^{(k)})^{j_k} G^{q_0}(G')^{q_1} \cdots (G^{(k)})^{q_k}$$

where $q_0 + q_1 + \cdots q_k \leq k-2$ and hence such a term has modulus at most

$$|M|^{j_0+j_1+\cdots+j_k-1} |\frac{M'}{M}|^{j_1} \cdots |\frac{M^{(k)}}{M}|^{j_k} |G|^{q_0+q_1+\cdots+q_k-k+2} |\frac{G'}{G}|^{q_1} \cdots |\frac{G^{(k)}}{G}|^{q_k}$$

$$\leq |M|^{j_0+j_1+\cdots+j_k-1}|\frac{M'}{M}|^{j_1}\cdots|\frac{M^{(k)}}{M}|^{j_k}|\frac{G'}{G}|^{q_1}\cdots|\frac{G^{(k)}}{G}|^{q_k}. \tag{29}$$

Using (28) and (29), we get

$$\begin{aligned} m(r,G) &\leq c_0 m(r,M) + m(r,\frac{1}{M}) + m(r,h) + S(r,G) + S(r,M) \\ &\leq c_1 T(r,M) + T(r,h) + S(r,G), \end{aligned} \tag{30}$$

where c_j $(j=0,1)$ denote positive constants. Using (8), (11)–(13), (19), (22), (23), (30) and Lemma 6, we deduce that

$$\begin{aligned} m(r,A) &\leq c_2(m(r,G)+\log r + S(r,G)) \\ &\leq c_3(m(r,M)+m(r,\frac{1}{M})+\log r + m(r,h)+S(r,G)+S(r,M)) \\ &\leq c_4 T(r,M) + o(T(r,A)) \leq c_4 N(r,M) + o(T(r,A)) \\ &\leq c_4(N(r,\frac{1}{f}) + N(r,\frac{1}{g})) + o(T(r,A)) \\ &\leq c_4(n(r,\frac{1}{f}) + n(r,\frac{1}{g}))\log r + o(T(r,A)) \\ &\leq 2c_4 n(r,\frac{1}{f})\log r + o(T(r,A)) \\ &= o(T(r,A)) = o(m(r,A)) \end{aligned}$$

holds for large $r \in E_0$, where c_j $(j = 2,3,4)$ denote positive constants. This is evidently a contradiction, and the proof is completed. $\square$

References

[1] W. K. HAYMAN. *Meromorphic Functions.* Clarendon Press, Oxford, 1964.

[2] I. LAINE. *Nevanlinna Theory and Complex Differential Equations.* Walter de Gruyter & Co., Berlin, 1993.

[3] Lo YANG. *Value Distribution Theory.* Translated and revised from the 1982 Chinese original. Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993.

[4] E. HILLE. *Ordinary Differential Equations in the Complex Domain.* Dover Publications, Inc., Mineola, NY, 1997.

[5] A. ALOTAIBI. *On complex oscillation theory.* Results Math., 2005, **47**(3-4): 165–175.

[6] S. BANK, I. LAINE. *On the oscillation theory of $f''+Af=0$ where A is entire.* Trans. Amer. Math. Soc., 1982, **273**(1): 351–363.

[7] S. BANK, J. K. LANGLEY. *Oscillation theory for higher order linear differential equations with entire coefficients.* Complex Variables Theory Appl., 1991, **16**(2-3): 163–175.

[8] J. K. LANGLEY. *Some oscillation theorems for higher order linear differential equations with entire coefficients of small growth.* Results Math., 1991, **20**(1-2): 517–529.

[9] J. ROSSI. *Second order differential equations with transcendental coefficients.* Proc. Amer. Math. Soc., 1986, **97**(1): 61–66.

[10] J. K. LANGLEY. *On entire solutions of linear differential equations with one dominant coefficients.* Analysis, 1995, **15**(2): 187–204.

[11] P. BARRY. *Some theorems related to the $\cos\pi\rho$ theorem.* Proc. London Math. Soc., 1970, **21**(3): 334–360.