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Products of Toeplitz Operator on the Weighted Bergman Space of the Unit Ball

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Abstract We consider the question for what kind of square integrable holomorphic functions f, g on the unit ball the densely defined products $T_fT_{\bar{g}}$ are invertible and Fredholm on the weighted Bergman space of the unit ball. We furthermore obtain necessary and sufficient conditions for bounded Haplitz products $H_fT_{\bar{g}}$, where $f\in L^2(B_n,\mathrm{d}v_\alpha)$ and g is a square integrable holomorphic function.

Keywords Toeplitz operator; Bergman space; unit ball; Hankel operator.

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1. Introduction

Let C^n be the n-dimensional complex Euclidean space. For any two points $z=(z_1,\dots,z_n)$ and $\omega=(\omega_1,\dots,\omega_n)$ in C^n we write $\langle z,\omega\rangle=z_1\overline{\omega_1}+\cdots+z_n\overline{\omega_n}$ and $|z|^2=|z_1|^2+\cdots+|z_n|^2$. The set $B_n=\{z\in B_n:|z|<1\}$ is the open unit ball in C^n.

For $-1<\alpha<\infty$, we denote by $\mathrm{d}v_\alpha$ the measure on B_n defined by $\mathrm{d}v_\alpha(z)=(1-|z|^2)^\alpha\mathrm{d}v(z)$. For any multi-index $m=(m_1,m_2,\dots,m_n)$, where each m_k is a nonnegative integer, we will write $|m|=m_1+m_2+\cdots+m_n$ and $z^m=z_1^{m_1}z_2^{m_2}\dots z_n^{m_n}$ for $z=(z_1,z_2,\dots,z_n)$. In addition, scalar multiplication and conjugation are defined in [1].

For $z\in B_n$, let φ_z be the automorphism of B_n, such that $\varphi_z(0)=z$ and $(\varphi_z)^{-1}=\varphi_z$. The weighted Bergman space $A^2_\alpha(B_n)$ is the closed subspace of $L^2(B_n,\mathrm{d}v_\alpha)$ consisting of holomorphic functions. The reproducing kernel in $A^2_\alpha(B_n)$ is given by

$$K_z^{(\alpha)}(\omega)=\frac{1}{(1-\langle\omega,z\rangle)^{n+\alpha+1}},$$

where $z,\omega\in B_n$. Let $\langle\cdot,\cdot\rangle_\alpha$ denote the inner product in $A^2_\alpha(B_n)$. Then $\langle h,K_z^{(\alpha)}\rangle_\alpha=h(z)$, for every $h\in A^2_\alpha(B_n)$ and $z\in B_n$. Let $\|\cdot\|$ and $\|\cdot\|_{2+\varepsilon}$ denote the norm in $L^2(B_n,\mathrm{d}v_\alpha)$ and $L^{2+\varepsilon}(B_n,\mathrm{d}v_\alpha)$, respectively. Then using the reproducing property of $K_z^{(\alpha)}$, we have

$$\|K_z^{(\alpha)}\|^2=\langle K_z^{(\alpha)},K_z^{(\alpha)}\rangle_\alpha=\frac{1}{(1-|z|^2)^{n+\alpha+1}},$$

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thus the normalized reproducing kernel is given by

$$k_z^{(\alpha)}(\omega) = \frac{(1-|z|^2)^{\frac{n+\alpha+1}{2}}}{(1-\langle \omega, z\rangle)^{n+\alpha+1}},$$

for $z, \omega \in B_n$. For $z \in B_n$ the function φ_z has real Jacobian equal to

$$|\varphi_z'(\omega)|^2 = \frac{(1-|z|^2)^{n+1}}{|1-\langle z, \omega\rangle|^{2n+2}}.$$

Thus we have the change of variable formula which will be very important for us later on.

$$\int_{B_n} f(\varphi_z(\omega))|k_z^{(\alpha)}(\omega)|^2 \mathrm{d}v_\alpha(\omega) = \int_{B_n} f(\omega)\mathrm{d}v_\alpha(\omega),$$

for every $f \in L^1(B_n, \mathrm{d}v_\alpha)$.

Let P_α be the orthogonal projection from $L^2(B_n, \mathrm{d}v_\alpha)$ onto $A^2_\alpha(B_n)$. P_α is an integral operator represented by

$$(P_\alpha g)(z) = \langle g, K_z^{(\alpha)}\rangle_\alpha = \int_{B_n} g(\omega)\frac{1}{(1-\langle z, \omega\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega),$$

for $g \in L^2(B_n, \mathrm{d}v_\alpha)$ and $z \in B_n$. It is clear that the above integral formula extends the domain of P_α to $L^1(B_n, \mathrm{d}v_\alpha)$. Given $f \in L^2(B_n, \mathrm{d}v_\alpha)$, we define the Toeplitz operator $T_f : A^2_\alpha(B_n) \to A^2_\alpha(B_n)$ by

$$T_f g = P_\alpha(fg).$$

T_f is called the Toeplitz operator with symbol f. So, if $g \in A^2_\alpha(B_n)$, we define $T_{\bar{g}}$ by the formula

$$(T_{\bar{g}}h)(z) = \int_{B_n} \frac{\overline{g(\omega)}h(\omega)}{(1-\langle z, \omega\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega),$$

for $h \in A^2_\alpha(B_n)$ and $z \in B_n$. If $f \in A^2_\alpha(B_n)$, it is clear that $T_f T_{\bar{g}} h$ is the holomorphic function $fT_{\bar{g}}h$. For $f \in L^2(B_n, \mathrm{d}v_\alpha)$, the Hankel operator $H_f : A^2_\alpha(B_n) \to (A^2_\alpha(B_n))^\perp$ is defined by

$$H_f(g) = fg - P_\alpha(fg).$$

Suppose f and g are in $L^2(B_n, \mathrm{d}v_\alpha)$. Consider the operator $f \otimes g$ on $L^2(B_n, \mathrm{d}v_\alpha)$ defined by

$$(f \otimes g)h = \langle h, g\rangle_\alpha f,$$

for $h \in L^2(B_n, \mathrm{d}v_\alpha)$. It is easy to prove that the norm of $f \otimes g$ is $\|f \otimes g\| = \|f\|\|g\|$. If T and S are bounded linear operators, then $T(f \otimes g)S^* = (Tf) \otimes (Sg)$. We observe that the Taylor expansion of the function $(1-z)^{n+\alpha+1}$ around 0,

$$(1-z)^{n+\alpha+1} = \sum_{k=0}^\infty C_{n,\alpha,k} z^k,$$

where

$$C_{n,\alpha,k} = (-1)^k \frac{(n+\alpha+1)(n+\alpha)\dots(n+\alpha+2-k)}{k!}, \quad k = 1, 2, \dots, \ C_{n,\alpha,0} = 1,$$

is absolutely convergent on the closed unit disk in C for $\alpha > -1$.

Let Q_0 be the integral operator on $L^2(B_n, \mathrm{d}v_\alpha)$ defined by

$$Q_0 u(z) = \int_{B_n} \frac{u(\omega)}{|1 - \langle z, \omega\rangle|^{n+\alpha+1}} \mathrm{d}v_\alpha(\omega), \quad z \in B_n.$$

It is clear that Q_0 is L^p bounded [2].

It turns out that the algebra property of Toeplitz and Hankel operator is closely related to the behavior of the symbol on Bergman space of the unit ball. To make this precise, we need to introduce the Berezin transform. If f is in $L^1(B_n, \mathrm{d}v_\alpha)$, the Berezin transform $\tilde{f}$ of f is the function on B_n defined by

$$\tilde{f}(z) = \int_{B_n} f(\omega)|k_z^{(\alpha)}(\omega)|^2 \mathrm{d}v_\alpha(\omega)$$

and it is easy to see that

$$\widetilde{f \circ \varphi_z}(\omega) = \widetilde{f}(\varphi_z(\omega))$$

for $z \in B_n$.

Definition 1.1 *Let H be Hilbert space, and $A : H \to H$ be a bounded operator. Then A is said to be left semi-Fredholm if there is a bounded operator B such that $I - BA$ is compact operator. Analogously, A is right semi-Fredholm if there is such a bounded operator B such that $I - AB$ is compact operator. A is a Fredholm operator if it is both left and right semi-Fredholm.*

Fredholm operators and their properties were described in [3].

Toeplitz and Hankel operators are one of the most widely studied classes of concrete operators. The study of their properties on the Hardy and Bergman spaces has generated an extensive list of results in the operator theory and in the theory of function spaces. Fredholmness and Invertibility of Toeplitz operator are the most important algebra property of Toeplitz operator. In [4], the authors proved that under certain conditions on the group generated by the Fourier support of the symbol, a Toeplitz operator is Fredholm if and only if it is invertible. In [5], the author discuss the Fredholm properties of some Toeplitz operators on Dirichlet spaces. In [6], the authors discussed invertibility and Fredholm property of Toeplitz products $T_f T_{\bar{g}}$ for analytic f and g on the Bergman space and the Hardy space. In [7], Cruz-Uribe characterized the outer functions f and g for which the Toeplitz product $T_f T_{\bar{g}}$ is bounded and ivertible on Hardy space. In [8], K. Stroethoff and Zheng obtained a necessary condition on boundedness of Hankel products $H_f H_g^*$ and proved that the necessary condition is very close to being sufficient, as shown for Toeplitz products on the Bergman space of the unit disk. Many interesting questions concerning the algebra property of Toeplitz and Hankel operator on the Hardy space or the Bergman space still remain open.

In this paper we consider the question for what kind of square integrable holomorphic functions f, g on the unit ball the densely defined products $T_f T_{\bar{g}}$ are invertible and Fredholm on the weighted Bergman space of the unit ball. We furthermore obtain necessary and sufficient conditions for bounded Haplitz products $H_f T_{\bar{g}}$, where $f \in L^2(B_n, \mathrm{d}v_\alpha)$ and g is a square integrable holomorphic function.

Throughout the paper, we will use the letter c to denote a generic positive constant that can change its value at each occurrence.

2. Invertibility and Fredholmness of Toeplitz products

In this section we will give necessary and sufficient condition for inverse Toeplitz operator and Fredholm Toeplitz operator with square integrable holomorphic symbols. Before doing this, we need the following results.

Lemma 2.1 ([9]) *Let $-1<\alpha<\infty$, and f and g be in $A^2_\alpha(B_n)$. If $T_fT_{\bar g}$ is bounded on $A^2_\alpha(B_n)$, then*

$$\sup_{z\in B_n}\widetilde{|f|^2}(z)\widetilde{|g|^2}(z)<\infty.$$

Lemma 2.2 *Suppose that $f\in A^2_\alpha(B_n)$ is zero free. Let b denote any zero-free bounded holomorphic function and $F=f/b$. Then there exists a constant c such that*

$$\widetilde{|F|^{2+\varepsilon}}(z)\le c\widetilde{|f|^{2+\varepsilon}}(z)$$

for $\varepsilon\ge 0$.

Proof Choose $0<R<1$ such that $|b(\omega)|>\frac{1}{\sqrt2}$, for all $R<|\omega|<1$. Suppose $z\in B_n$. Then

$$\begin{aligned}\widetilde{|f|^{2+\varepsilon}}(z)&=\int_{B_n}|f(\varphi_z(\omega))|^{2+\varepsilon}\mathrm{d}v_\alpha(\omega)\\&=\int_{B_n}|b(\varphi_z(\omega))|^{2+\varepsilon}|F(\varphi_z(\omega))|^{2+\varepsilon}\mathrm{d}v_\alpha(\omega)\\&\ge\frac{1}{2^{1+\frac{\varepsilon}{2}}}\int_{R<|\varphi_z(\omega)|<1}|F(\varphi_z(\omega))|^{2+\varepsilon}\mathrm{d}v_\alpha(\omega)\\&=\frac{1}{2^{1+\frac{\varepsilon}{2}}}\int_{R<|\omega|<1}|F(\omega)|^{2+\varepsilon}|k_z^{(\alpha)}(\omega)|^2\mathrm{d}v_\alpha(\omega).\end{aligned}$$

Now, if h is holomorphic on B_n, then

$$\int_{B_n}|h(z)|^{2+\varepsilon}\mathrm{d}v_\alpha(z)\le\frac{1}{1-R^{(2+\varepsilon)|m|+2n}}\int_{R<|z|<1}|h(z)|^{2+\varepsilon}\mathrm{d}v_\alpha(z).$$

It suffices to prove above equality for monomials $h(z)=z^m$.

$$\begin{aligned}\int_{R<|z|<1}|z^m|^{2+\varepsilon}\mathrm{d}v_\alpha(z)&=\int_{B_n}|z^m|^{2+\varepsilon}\mathrm{d}v_\alpha(z)-\int_{|z|\le R}|z^m|^{2+\varepsilon}\mathrm{d}v_\alpha(z)\\&=\int_{B_n}|z^m|^{2+\varepsilon}\mathrm{d}v_\alpha(z)-\int_{|z|<R}|z^m|^{2+\varepsilon}\mathrm{d}v_\alpha(z)\\&\ge\int_{B_n}|z^m|^2\mathrm{d}v_\alpha(z)-R^{(2+\varepsilon)|m|+2n}\int_{B_n}|z^m|^2\mathrm{d}v_\alpha(z)\\&=(1-R^{(2+\varepsilon)|m|+2n})\int_{B_n}|z^m|^{2+\varepsilon}\mathrm{d}v_\alpha(z).\end{aligned}$$

Applying the above estimate to the function

$$h(z)=F(z)(k_z^{(\alpha)})^{\frac{2}{2+\varepsilon}},$$

we get that

$$\int_{R<|z|<1} |F(\omega)|^{2+\varepsilon}|k_z^{(\alpha)}(\omega)|^2 \mathrm{d}v_\alpha(\omega) \geq (1-R^{(2+\varepsilon)|m|+2n})\int_{B_n} |F(\omega)|^{2+\varepsilon}|k_z^{(\alpha)}(\omega)|^2 \mathrm{d}v_\alpha(z)$$
$$= (1-R^{(2+\varepsilon)|m|+2n})\widetilde{|F|^{2+\varepsilon}}(z).$$

Thus

$$\widetilde{|f|^{2+\varepsilon}}(z) \geq \frac{1}{2^{1+\frac{\varepsilon}{2}}}(1-R^{(2+\varepsilon)|m|+2n})\widetilde{|F|^{2+\varepsilon}}(z),$$

so that

$$\widetilde{|F|^{2+\varepsilon}}(z) \leq c\widetilde{|f|^{2+\varepsilon}}(z)$$

with $c = \frac{2^{1+\frac{\varepsilon}{2}}}{1-R^{(2+\varepsilon)|m|+2n}}$, for all $z \in B_n$. □

Next, we will completely characterize the bounded invertible Toeplitz products and Fredholm Toeplitz products.

Lemma 2.3 ([9]) *Let $-1 < \alpha < \infty$, and f and g be in $A_\alpha^2(B_n)$. For $\varepsilon > 0$,*

$$\sup_{z\in B_n} \widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|g|^{2+\varepsilon}}(z) < \infty,$$

then $T_f T_{\bar{g}}$ is bounded on $A_\alpha^2(B_n)$.

Theorem 2.1 *Let $f, g \in A_\alpha^2(B_n)$ and $T_f T_{\bar{g}}$ be bounded and invertible on $A_\alpha^2(B_n)$. Then*

$$\sup_{z\in B_n} \widetilde{|f|^2}(z)\widetilde{|g|^2}(z) < \infty$$

and

$$\inf_{z\in B_n} |f(z)||g(z)| > 0.$$

Proof Suppose that $T_f T_{\bar{g}}$ is bounded and invertible on $A_\alpha^2(B_n)$. By Lemma 2.1 there exists a positive number M such that

$$\sup_{z\in B_n} \widetilde{|f|^2}(z)\widetilde{|g|^2}(z) \leq M,$$

for all $z \in B_n$. Since

$$T_f T_{\bar{g}} k_z^{(\alpha)} = \overline{g(z)} f k_z^{(\alpha)},$$

it follows

$$\|T_f T_{\bar{g}} k_z^{(\alpha)}\|^2 = |g(z)|^2 \|f k_z^{(\alpha)}\|^2 = |g(z)|^2 \widetilde{|f|^2}(z).$$

So the invertibility of $T_f T_{\bar{g}}$ yields

$$|g(z)|^2 \widetilde{|f|^2}(z) \geq \delta_1 > 0.$$

Since $T_g T_{\bar{f}} = (T_f T_{\bar{g}})^*$ is bounded and invertible on the $A_\alpha^2(B_n)$, there exists a constant δ_2 such that

$$|f(z)|^2 \widetilde{|g|^2}(z) \geq \delta_2 > 0,$$

for all $z \in B_n$. Let $\delta = \delta_1\delta_2$. Then we have

$$\delta \leq |f(z)|^2|g(z)|^2\widetilde{|f|^2}(z)\widetilde{|g|^2}(z) \leq M|f(z)|^2|g(z)|^2$$

and thus

$$\inf_{z\in B_n}|f(z)||g(z)|\ge\sqrt{\frac{\delta}{M}}.\ \Box$$

Theorem 2.2 *Let $f,g\in A^2_\alpha(B_n)$. For $\varepsilon>0$,*

$$\sup_{z\in B_n}\widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|g|^{2+\varepsilon}}(z)<\infty$$

and

$$\inf_{z\in B_n}|f(z)||g(z)|>0.$$

Then $T_fT_{\bar g}$ is bounded and invertible on $A^2_\alpha(B_n)$.

Proof Suppose that for $\varepsilon>0$

$$M=\sup_{z\in B_n}\widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|g|^{2+\varepsilon}}(z)<\infty$$

and

$$\eta=\inf_{z\in B_n}|f(z)||g(z)|>0.$$

By Lemma 2.3, we know that $T_fT_{\bar g}$ is bounded on $A^2_\alpha(B_n)$. By the inequality of Cauchy-Schwarz, we have

$$|f(z)|^2|g(z)|^2\le\widetilde{|f|^2}(z)\widetilde{|g|^2}(z)\le\widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|g|^{2+\varepsilon}}(z)$$

and $|f(z)||g(z)|\le\sqrt{M}$ for all $z\in B_n$. So fg is a bounded function on B_n. Note that f and g cannot have zeros in B_n. Since $|g(z)|^{2+\varepsilon}\ge\eta^{2+\varepsilon}|f(z)|^{-2-\varepsilon}$, we have

$$\widetilde{|g|^{2+\varepsilon}}(z)\ge\eta^{2+\varepsilon}\widetilde{|f(z)|^{-2-\varepsilon}}(z).$$

Consequently

$$M\ge\widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|g|^{2+\varepsilon}}(z)\ge\eta^{2+\varepsilon}\widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|f(z)|^{-2-\varepsilon}}(z).$$

By Lemma 2.3, we know that $T_fT_{\overline{f^{-1}}}$ is bounded on $L^2_a(B_n)$. Since fg is bounded on B_n, the operator $T_{\overline{fg}}$ is bounded on $A^2_\alpha(B_n)$. Since $\psi=\frac{1}{\overline{fg}}$ is bounded on B_n, the operator T_ψ is bounded on $A^2_\alpha(B_n)$. Since

$$T_fT_{\bar g}T_\psi=I=T_\psi T_fT_{\bar g},$$

we conclude that $T_fT_{\bar g}$ is invertible on $A^2_\alpha(B_n)$. $\Box$

Next, we will completely characterize the bounded Fredholm Toeplitz products $T_fT_{\bar g}$ on $A^2_\alpha(B_n)$. We have the following result.

Theorem 2.3 *Let $f,g\in A^2_\alpha(B_n)$ and $T_fT_{\bar g}$ be bounded Fredholm operator on $A^2_\alpha(B_n)$. Then*

$$\sup_{z\in B_n}\widetilde{|f|^2}(z)\widetilde{|g|^2}(z)<\infty$$

and there exists a number r with $0<r<1$ such that

$$\inf_{r<|z|<1}|f(z)||g(z)|>0.$$

Proof By Lemma 2.1, there exists a constant M such that

$$\sup_{z\in B_n} \widetilde{|f|^2}(z)\widetilde{|g|^2}(z) \le M.$$

If $T_f T_{\bar g}$ is Fredholm, then there exists a bounded operator S and a compact operator A such that

$$ST_f T_{\bar g} = I + A.$$

Using the fact $T_f T_{\bar g} k_z^{(\alpha)} = \overline{g(z)} f k_z^{(\alpha)}$, we have

$$\begin{aligned}\|S\||g(z)|\widetilde{|f|^2}(z)^{\frac{1}{2}} &= \|S\|\|T_f T_{\bar g} k_z^{(\alpha)}\| \ge \|ST_f T_{\bar g} k_z^{(\alpha)}\| \\ &\ge \|k_z^{(\alpha)}\| - \|Ak_z^{(\alpha)}\| = 1 - \|Ak_z^{(\alpha)}\|.\end{aligned}$$

Since A is compact, $\|Ak_z^{(\alpha)}\| \to 0$ as $z \to 1^-$, so there exists $0 < r_1 < 1$ such that $\|Ak_z^{(\alpha)}\| < \frac{1}{2}$ for all $r_1 < |z| < 1$. The above inequality shows that

$$|g(z)|^2 \widetilde{|f|^2}(z) \ge M_1 = \frac{1}{2}\|S\|^{-1},$$

for all $r_1 < |z| < 1$.

Since $T_g T_{\bar f} = (T_f T_{\bar g})^*$ is also Fredholm, there is a positive constant M_2 and a number r_2 with $0 < r_2 < 1$ such that

$$|f(z)|^2 \widetilde{|g|^2}(z) \ge M_2,$$

for all $r_2 < |z| < 1$. Thus

$$M_1 M_2 \le |f(z)|^2 |g(z)|^2 \widetilde{|f|^2}(z)\widetilde{|g|^2}(z) \le M|f(z)|^2|g(z)|^2$$

and hence

$$\inf_{r<|z|<1} |f(z)||g(z)| \ge \sqrt{\frac{M_1 M_2}{M}} > 0$$

with $r = \max\{r_1, r_2\}$. $\square$

The following theorem gives a sufficient condition slightly stronger than necessary condition for Fredholm Toeplitz operator.

Theorem 2.4 *Let $f, g \in A_\alpha^2(B_n)$. For $\varepsilon > 0$,*

$$\sup_{z\in B_n} \widetilde{|f|^{2+\varepsilon}}(z)\widetilde{|g|^{2+\varepsilon}}(z) < \infty$$

and there exists a number r with $0 < r < 1$ such that

$$\inf_{r<|z|<1} |f(z)||g(z)| > 0.$$

Then $T_f T_{\bar g}$ is bounded Fredholm operator on $A_\alpha^2(B_n)$.

Proof By Lemma 2.3, we know that $T_f T_{\bar g}$ is bounded operator on $A_\alpha^2(B_n)$. Suppose that

$$|f(z)||g(z)| \ge \delta > 0,$$

for all $0 < r < |z| < 1$. The above inequality implies that f and g have no zeros in the annulus $\{z : r < |z| < 1\}$. Let b_1 and b_2 denote any zero-free bounded holomorphic function, respectively.

Then $F=f/b_1$ and $G=g/b_2$ are zero free. By the above analysis, we have

$$|F(z)||G(z)|\geq \delta|b_1||b_2|,$$

for all $r<|z|<1$. The function on the right is positive and continuous on annulus $\{z:\frac{1+r}{2}\leq |z|\leq 1\}$, thus has a positive minimum. So putting $\rho=\frac{1+r}{2}$, we have

$$|F(z)||G(z)|\geq \eta_1,$$

for all $\rho<|z|<1$. Note that

$$\eta_2=\inf_{|z|\leq\rho}|F(z)||G(z)|>0.$$

If we take $\eta=\min(\eta_1,\eta_2)$, then

$$|F(z)||G(z)|\geq \eta,$$

for all $z\in B_n$ and the Toeplitz operator $T_{\frac{1}{F\bar{G}}}$ is bounded on $A^2_\alpha(B_n)$. By Lemma 2.2 and Theorem 2.5, we have $T_FT_{\bar{G}}$ is bounded and invertible. By the inverse operator theorem, we have $(T_FT_{\bar{G}})^{-1}$ is bounded too. Since $T_{\overline{b_2}}$ is Fredholm, there are bounded and compact operators R_2 and K_2 such that $T_{\overline{b_2}}R_2=I+K_2$. It follows that

$$T_fT_{\bar{g}}R_2=T_{b_1}T_FT_{\bar{G}}+T_{b_1}T_FT_{\bar{G}}K_2,$$

thus

$$T_fT_{\bar{g}}R_2(T_FT_{\bar{G}})^{-1}=T_{b_1}+T_{b_1}T_FT_{\bar{G}}K_2(T_FT_{\bar{G}})^{-1}.$$

Since T_{b_1} is Fredholm, there are bounded and compact operators R_1 and K_1 such that $T_{b_1}R_1=I+K_1$. Then

$$T_fT_{\bar{g}}R_2(T_FT_{\bar{G}})^{-1}R_1=I+K_1+T_{b_1}T_FT_{\bar{G}}K_2(T_FT_{\bar{G}})^{-1}R_1.$$

Hence $T_fT_{\bar{g}}$ is right-Fredholm operator. Similarly $T_fT_{\bar{g}}$ is left-Fredholm operator. So that $T_fT_{\bar{g}}$ is Fredholm operator. □

3. Boundedness of Haplitz product

In this section we will give a necessary and sufficient condition for bounded Haplitz product $H_fT_{\bar{g}}$ with $f\in L^2(B_n,\mathrm{d}v_\alpha)$ and g being square integrable holomorphic function. Before doing this, we need the following results.

Lemma 3.1 ([9])

$$\begin{aligned}\langle F,G\rangle_\alpha =&\frac{1}{\Gamma(2m+1)}\sum_{|\gamma|=m}\int_{B_n}D^\gamma F(z)\overline{D^\gamma G(z)}(1-|z|^2)^{2m}\mathrm{d}v_\alpha(z)+\\&\sum_{j=1}^{2m-1}a'_j\sum_{|\gamma|=m}\int_{B_n}D^\gamma F(z)\overline{D^\gamma G(z)}(1-|z|^2)^{2m+j}\mathrm{d}v_\alpha(z)+\\&\sum_{j=1}^{m}b'_j\int_{B_n}F(z)\overline{G(z)}(1-|z|^2)^{2m+j-1}\mathrm{d}v_\alpha(z),\end{aligned}$$

for any $m\in N$ and $F,G\in A^2_\alpha(B_n)$.

Lemma 3.2 ([9]) *Let $f \in L^2(B_n, \mathrm{d}v_\alpha)$. Then*

$$|(T_{\bar{f}}h)(z)| \le \frac{1}{(1-|z|^2)^{\frac{n+\alpha+1}{2}}}\|h\|\widetilde{|f|^2}(z)^{\frac{1}{2}},$$

and

$$|(H_{\bar{f}}^* u)(z)| \le \frac{1}{(1-|z|^2)^{\frac{n+\alpha+1}{2}}}\|u\|\|f\circ\varphi_z - P_\alpha(f\circ\varphi_z)\|,$$

for all $h \in A_\alpha^2(B_n)$, $u \in (A_\alpha^2(B_n))^\perp$ and $z \in B_n$.

Lemma 3.3 ([10]) *On $A_\alpha^2(B_n)$, we have*

$$k_z^{(\alpha)} \otimes k_z^{(\alpha)} = \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T_{\varphi_z^\gamma} T_{\overline{\varphi}_z^\gamma}.$$

Adopting the same method as in Lemma 4.5 in [9], we have

Lemma 3.4 *Let $\varepsilon > 0$, $\delta = \frac{2+\varepsilon}{1+\varepsilon}$ and $f \in L^2(B_n, \mathrm{d}v_\alpha)$. For all $u \in (A_\alpha^2(B_n))^\perp$, $h \in A_\alpha^2(B_n)$ and multi-index γ with $|\gamma| = m \ge \frac{n+\alpha+1}{2}$, we have*

$$|(D^\gamma T_{\bar{f}}h)(z)| \le c\frac{\widetilde{|f|^2}(z)^{\frac{1}{2}}}{(1-|z|^2)^m}[Q_0(|u|^\delta)(z)]^{\frac{1}{\delta}},$$

and

$$|(D^\gamma H_{\bar{f}}^*)u(z)| \le c\frac{\|f\circ\varphi_z - P(f\circ\varphi_z)\|_{2+\varepsilon}}{(1-|z|^2)^m}[Q_0(|u|^\delta)(z)]^{\frac{1}{\delta}},$$

for all $z \in B_n$.

Theorem 3.1 *Suppose $f \in L^2(B_n, \mathrm{d}v_\alpha)$, $g \in A_\alpha^2(B_n)$,*

$$\sup_{z\in B_n} \|f\circ\varphi_z - P(f\circ\varphi_z)\|_{2+\varepsilon}\widetilde{|g|^2}(z)^{\frac{1}{2}} < \infty.$$

Then $H_f T_{\bar{g}}$ is bounded.

Proof Suppose $u \in A_\alpha^2(B_n)$ and $v \in (A_\alpha^2(B_n))^\perp$. By Lemma 3.1, we have

$$\begin{aligned}\langle H_f T_{\bar{g}}u, v\rangle_\alpha =& \langle T_{\bar{g}}u, H_{\bar{f}}^* v\rangle_\alpha \\ =& \frac{\Gamma(1)}{\Gamma(2m+1)} \sum_{|\gamma|=m} \int_{B_n} D^\gamma(T_{\bar{g}}u)(z)\overline{D^\gamma(H_{\bar{f}}^* v)(z)}(1-|z|^2)^{2m}\mathrm{d}v_\alpha(z) + \\ & \sum_{j=1}^{2m-1} a_j' \sum_{|\gamma|=m} \int_{B_n} D^\gamma(T_{\bar{g}}u)(z)\overline{D^\gamma(H_{\bar{f}}^* v)(z)}(1-|z|^2)^{2m+j}\mathrm{d}v_\alpha(z) + \\ & \sum_{j=1}^{m} b_j' \int_{B_n} (T_{\bar{g}}u)(z)\overline{(H_{\bar{f}}^* v)(z)}(1-|z|^2)^{2m+j-1}\mathrm{d}v_\alpha(z).\end{aligned}$$

By Hölder's inequality,

$$\Big(\int_{B_n} |f|^2(\omega)\mathrm{d}v_\alpha(\omega)\Big)^{\frac{1}{2}} \le \Big(\int_{B_n} |f|^{2+\varepsilon}(\omega)\mathrm{d}v_\alpha(\omega)\Big)^{\frac{1}{2+\varepsilon}}.$$

By Lemma 3.2, we get

$$|(H_{\bar{f}}^* v)(z)(T_{\bar{g}}u)(z)| \le \frac{1}{(1-|z|^2)^{n+\alpha+1}}\|f\circ\varphi_z - P(f\circ\varphi_z)\|\widetilde{|g|^2}(z)^{\frac{1}{2}}\|u\|\|v\|$$

and

$$\Big|\int_{B_n}(H_f^*v)(z)(T_{\bar{g}}u)(z)(1-|z|^2)^q\mathrm{d}v_\alpha(z)\Big|\le c\|u\|\|v\|,$$

for all $q\ge n+\alpha+1$. So if we choose a large m, such that $2m\ge n+\alpha+1$, then each of the terms

$$\int_{B_n}(H_f^*v)(z)(H_g^*u)(z)(1-|z|^2)^{2m+j-1}\mathrm{d}v_\alpha(z)$$

is bounded by $M\|u\|\|v\|$, for $j=1,\dots,m$.

Let

$$\Pi=\frac{\Gamma(1)}{\Gamma(2m+1)}\sum_{|\gamma|=m}\int_{B_n}D^\gamma(T_{\bar{g}}u)(z)\overline{D^\gamma(H_f^*v)(z)}(1-|z|^2)^{2m}\mathrm{d}v_\alpha(z)$$

and

$$\prod=\sum_{j=1}^{2m-1}a_j'\sum_{|\gamma|=m}\int_{B_n}D^\gamma(T_{\bar{g}}u)(z)\overline{D^\gamma(H_f^*v)(z)}(1-|z|^2)^{2m+j}\mathrm{d}v_\alpha(z).$$

By Lemma 3.4 and the L^p boundedness of operator Q_0, we have

$$\Pi\le c\sup_{z\in B_n}\|f\circ\varphi_z-P(f\circ\varphi_z)\|_{2+\varepsilon}\widetilde{|g|^2}(z)^{\frac12}\|u\|\|v\|$$

and

$$\prod\le c\sup_{z\in B_n}\|f\circ\varphi_z-P(f\circ\varphi_z)\|_{2+\varepsilon}\widetilde{|g|^2}(z)^{\frac12}\|u\|\|v\|.$$

So the product $H_fT_{\bar{g}}$ is bounded on $(A_\alpha^2(B_n))^\perp$, as desired. $\square$

Theorem 3.2 *Suppose $f\in L^2(B_n,\mathrm{d}v_\alpha)$, $g\in A_\alpha^2(B_n)$, $H_fT_{\bar{g}}$ is bounded, then*

$$\sup_{z\in B_n}\|f\circ\varphi_z-P(f\circ\varphi_z)\|_2\widetilde{|g|^2}(z)^{\frac12}<\infty.$$

Proof By Lemma 3.3, we have

$$\begin{aligned}H_f(k_z^{(\alpha)}\otimes k_z^{(\alpha)})T_{\bar{g}}&=H_f(\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}T_{\varphi_z^\gamma}T_{\overline{\varphi}_z^\gamma})T_{\bar{g}}\\&=\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}H_fT_{\varphi_z^\gamma}T_{\overline{\varphi}_z^\gamma}T_{\bar{g}}\\&=\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}S_{\varphi_z^\gamma}H_fT_{\bar{g}}S_{\overline{\varphi}_z^\gamma}.\end{aligned}$$

Since $H_f(k_z^{(\alpha)}\otimes k_z^{(\alpha)})T_{\bar{g}}=(H_fk_z^{(\alpha)})\otimes(T_{\bar{g}}k_z^{(\alpha)})$, it follows

$$\|H_f(k_z^{(\alpha)}\otimes k_z^{(\alpha)})T_{\bar{g}}\|=\|f\circ\varphi_z-P(f\circ\varphi_z)\|_2\widetilde{|g|^2}(z)^{\frac12}$$

and

$$\begin{aligned}\|f\circ\varphi_z-P(f\circ\varphi_z)\|\widetilde{|g|^2}(z)^{\frac12}&=\|H_f(k_z^\alpha\otimes k_z^{(\alpha)})T_{\bar{g}}\|\\&=\|\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}S_{\varphi_z^\gamma}H_fT_{\bar{g}}S_{\overline{\varphi}_z^\gamma}\|\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=0}^{\infty} |C_{n,\alpha,k}| \| \sum_{|\gamma|=k} \frac{k!}{\gamma!} S_{\varphi_z^\gamma} H_f T_{\bar{g}} S_{\overline{\varphi}_z^\gamma} \| \\
&\le \sum_{k=0}^{\infty} |C_{n,\alpha,k}| \| H_f T_{\bar{g}} \| \\
&\le c \| H_f T_{\bar{g}} \| < \infty,
\end{aligned}
$$

where we use the fact that $\sum_{k=0}^{\infty} |C_{n,\alpha,k}|$ is convergent. $\square$

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