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Property (ω) and Its Perturbations

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Abstract In the note, we establish for a bounded linear operator defined on a Hilbert space the necessary and sufficient conditions for the stability of property (ω) by means of the variant of the essential approximate point spectrum and the induced spectrum of consistency in Fredholm and index. In addition, the stability of property (ω) for $H(P)$ operators is considered.

Keywords property (ω); perturbation; consistency in Fredholm and index.

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1. Introduction

Weyl [1] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This "Weyl's theorem" has been considered by many authors. Variants have been discussed by Harte and Lee [2] and Rakočevič [3, 4]. In this note, using a subset of the spectrum derived from "consistent in Fredholm and index", we study the perturbations of a new variant of Weyl's theorem called property (ω), and show that how property (ω) holds under perturbations by power finite rank operators, by nilpotent operators and Riesz operators.

Throughout this paper, H will denote an infinite-dimensional complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on H. For a bounded linear $T \in B(H)$ on Hilbert space the spectrum $\sigma(T)$ collects the complex numbers λ for which $T - \lambda I$ fails to be invertible, equivalently is either not one to one or not onto, let $\rho(T) = \mathbb{C}\backslash\sigma(T)$. We shall denote by $n(T)$ the dimension of the kernel $N(T)$ of $T \in B(H)$, and by $d(T)$ the codimension of the range $R(T)$. We recall that an operator $T \in B(H)$ is called upper semi-Fredholm if $n(T) < \infty$ and $R(T)$ is closed, while $T \in B(H)$ is called lower semi-Fredholm if $d(T) < \infty$. If both the deficiency indices $n(T)$ and $d(T)$ are finite, T is a Fredholm operator. If T is upper (lower) semi-Fredholm, the index of T is denoted by $\mathrm{ind}(T) = n(T) - d(T)$. The operator T is Weyl if it is Fredholm of index zero. Recall that a bounded operator T is said bounded below if it is injective and has

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closed range. Define

$$SF_+^-(H) = \{T \in B(H),\ T \text{ is upper semi-Fredholm and } \operatorname{ind}(T) \leq 0\}.$$

The classes of operators defined above generate the following spectra. The Weyl essential approximate point spectrum $\sigma_{wa}(T)$ and the approximate point spectrum $\sigma_a(T)$ are defined by:

$$\sigma_{wa}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(H)\};$$

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

Note that $\sigma_{wa}(T)$ is the intersection of all approximate point spectra $\sigma_a(T+K)$ of compact perturbations K of T. Write $\operatorname{iso} G$ for the set of all isolated points of $G \subseteq \mathbb{C}$, $\operatorname{acc} G = G \backslash \operatorname{iso} G$. It is known that if K is finite rank operator commuting with T, then $\lambda \in \operatorname{acc} \sigma_a(T)$ if and only if $\lambda \in \operatorname{acc} \sigma_a(T+K)$, for a proof see Theorem 3.2 in [5].

For an operator T the ascent is defined as $p = p(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, while the descent is defined as $q = q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, the infimum over the empty set is taken ∞. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$. Moreover, $0 < p(T - \lambda I) = q(T - \lambda I) < \infty$ precisely when λ is a polar point of the resolvent set of T. The class of all upper semi-Browder operators, and the class of all Browder operators are defined: $B_+(H) = \{T \in B(H) : T \text{ is upper semi-Fredholm with } p(T) < \infty\}$, $BR(H) = \{T \in B(H) : T \text{ is Fredholm with } p(T) = q(T) < \infty\}$. The Browder spectrum of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin BR(H)\}$.

Recall that $T \in B(H)$ is said to be a Riesz operator if $T - \lambda I$ is Fredholm for all $\lambda \in \mathbb{C} \backslash \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. The proof of the following result may be found in Rakočevič [6].

Theorem 1.1 *Let $T \in B(H)$, and $K \in B(H)$ be a Riesz operator commuting with T. Then*

(1) $T \in SF_+^-(H) \Leftrightarrow T + K \in SF_+^-(H)$;

(2) $T \in B_+(H) \Leftrightarrow T + K \in B_+(H)$;

(3) $T \in BR(H) \Leftrightarrow T + K \in BR(H)$.

We shall describe a Hilbert space operator $T \in B(H)$ as consistent in Fredholm and index(CFI) provided there is implication, for arbitrary $S \in B(H)$, one of the cases occures: (1) ST and TS are Fredholm and $\operatorname{ind}(ST) = \operatorname{ind}(TS) = \operatorname{ind}(S)$; (2) Both TS and ST are not Fredhlom. We shall write

$$\sigma_{CFI}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ is not CFI}\}.$$

The CFI spectrum does not need to be closed or nonempty [7].

In Section 2, using the CFI spectrum, we study the stability of property (ω), for a bounded operator T acting on a Hilbert space, under perturbations by power finite rank operators, by nilpotent operators and Riesz operators commuting with T. Also, the stability of property (ω) for $H(P)$ operators is considered.

2. Property (ω) and perturbations

A bounded operator $T \in B(H)$ is said to satisfy property (ω) if

$$\sigma_a(T)\backslash\sigma_{aw}(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \pi_0(T) \cap \operatorname{iso}\sigma(T)$; $\pi_0(T) = \{\lambda \in \mathbb{C} : 0 < n(T - \lambda I) < \infty\}$.

An operator $K \in B(H)$ is said to be power finite rank if there exists a positive integer m such that K^m is finite rank.

Property (ω) is fulfilled by a relevant number of Hilbert space operators [8], for example, property (ω) is satisfied by scalar operators, or if the Hilbert adjoint T^* has property $H(p)$. For the stability of property (ω), let us begin with a lemma.

Lemma 2.1 *Let $T \in B(H)$. If $K \in B(H)$ is a power finite rank operator that commutes with T, then*

(1) K is a Riesz operator;

(2) $n(T+K) < \infty \Leftrightarrow n(T) < \infty$;

(3) $\operatorname{iso}\sigma(T+K) \subseteq \operatorname{iso}\sigma(T) \cup \rho(T)$;

(4) $\operatorname{iso}\sigma_a(T+K) \subseteq \operatorname{iso}\sigma_a(T) \cup \rho_a(T)$.

Proof Suppose that there exists a positive integer m such that K^m is finite rank.

(1) For any $\lambda \in \mathbb{C}\backslash\{0\}$, we have $K^m - \lambda^m I = (K - \lambda I)h(K)$, $h(K) = K^{m-1} + \cdots + \lambda^{m-1}I$. Since $K^m - \lambda^m I$ is Browder and $K - \lambda I$ commutes with $h(K)$, $K - \lambda I$ is Fredholm. Hence K is a Riesz operator.

(2) We only need to prove that $n(T) < \infty$ whenever $n(T+K) < \infty$. If $n(T+K) < \infty$, then $n((T+K)^m) < \infty$. As $(T+K)^m = (T^m + \cdots + mTK^{m-1}) + K^m$, let $S = T^m + \cdots + mTK^{m-1}$. Then $n(S) < \infty$. We conclude that $n(T) < \infty$ since $N(T) \subseteq N(S)$.

(3) Assume that $\lambda_0 \in \operatorname{iso}\sigma(T+K)$. If $\lambda_0 \notin \operatorname{iso}\sigma(T) \cup \rho(T)$, then $\lambda_0 \in \operatorname{acc}\sigma(T)$. There exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subseteq \sigma(T)$, $\lambda_n \to \lambda_0$ $(n \to \infty)$, and $T - \lambda_n I$ is not invertible. Also we can get $T + K - \lambda_n I$ is invertible since $\lambda_0 \in \operatorname{iso}\sigma(T+K)$. Then $T - \lambda_n I$ is Browder (Theorem 1.1), and $0 < \dim N(T - \lambda_n I) < \infty$. Let $K_n = K|_N(T - \lambda_n I)$. Then K_n is invertible. In fact, if $K_n x = 0$, where $x \in N(T - \lambda_n I)$, then $(T + K_n - \lambda_n I)x = 0$. Since $T + K - \lambda_n I$ is invertible, we have $x = 0$, thus K_n is injective. We know that in finite dimensional linear space $N(T - \lambda_n I)$, K_n is injective if and only if K_n is surjective. Hence $K_n N(T - \lambda_n I) = N(T - \lambda_n I)$. Then $\bigcup_{n=1}^{\infty} N(T - \lambda_n I) \subseteq \bigcup_{n=1}^{\infty} R(K_n^m) \subseteq R(K^m)$, thus $\sum_{n=1}^{\infty} \oplus N(T - \lambda_n I) \subseteq R(K^m)$. We have that $\sum_{n=1}^{\infty} \dim N(T - \lambda_n I) \leq \dim R(K^m)$. We conclude that $\dim R(K^m) = \infty$ because $\dim N(T - \lambda_n I) > 0$ for any $n \in N$. It is in contradiction to the fact that K^m is finite rank. This shows that if $\lambda_0 \in \operatorname{iso}\sigma(T+K)$, then $\lambda_0 \in \operatorname{iso}\sigma(T) \cup \rho(T)$.

The proof of (4) is similar to that of (3). $\square$

We recall that an "isoloid" operator is the one such that the isolated points of the spectrum are all eigenvalues, while an "a-isoloid" operator is the one such that the isolated points of its approximate point spectrum are all eigenvalues.

We turn to a variant of the essential approximate point spectrum, involving a condition introduced by Saphar [9] and the zero jump condition of Kato [10]. Let

$$\rho_1(T) = \{\lambda \in \mathbb{C} : \dim N(T-\lambda I) < \infty \text{ and there exists } \epsilon > 0 \text{ such that } T-\mu I \in SF_+^-(H)$$
$$N(T-\mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\mu I)^n] \text{ if } 0 < |\mu-\lambda| < \epsilon\}$$

and let $\sigma_1(T) = \mathbb{C}\backslash\rho_1(T)$. Then $\sigma_1(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$.

Theorem 2.2 *Suppose that $T \in B(H)$. If K is a power finite rank operator commuting with T, then $T+K$ is isoloid and satisfies property (ω) if and only if $\sigma_b(T) \cap \sigma_a(T+K) = \sigma_1(T) \cup \partial\sigma_{CFI}(T)$.*

Proof Suppose that $\sigma_b(T)\cap\sigma_a(T+K) = \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Let $\lambda_0 \in \sigma_a(T+K)\backslash\sigma_{aw}(T+K)$. Then $T-\lambda_0 I \in SF_+^-(H)$ (Theorem 1.1), hence $n(T-\lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T-\lambda I \in SF_+^-(H)$, $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ and $\mathrm{ind}(T-\lambda_0 I) = \mathrm{ind}(T-\lambda I)$ if $0 < |\lambda-\lambda_0| < \epsilon$. We assert that $\lambda_0 \notin \partial\sigma_{CFI}(T)$. Otherwise, we have $\lambda_0 \in \mathrm{int}\,\rho_{CFI}(T)$ since $\mathrm{ind}(T-\lambda_0 I) = \mathrm{ind}(T-\lambda I) = 0$ if $0 < |\lambda-\lambda_0|$ is small enough. It is in contradiction to the definition of boundary point. Then $\lambda_0 \notin \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Thus $\lambda_0 \notin \sigma_b(T)\cap\sigma_a(T+K)$. But since $\lambda_0 \in \sigma_a(T+K)$, we know $\lambda_0 \notin \sigma_b(T)$. This induces that $T+K-\lambda_0 I$ is Browder. Now we prove that $\lambda_0 \in \pi_{00}(T+K)$. For the converse, let $\lambda_0 \in \pi_{00}(T+K)$. Then $\lambda_0 \in \mathrm{iso}\,\sigma(T)\cup\rho(T)$ and $n(T-\lambda_0 I) < \infty$ (Lemma 2.1). Without loss of generality, we may let $\lambda_0 \in \mathrm{iso}\,\sigma(T)$. Now we can see that $\lambda_0 \notin \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Then $T-\lambda_0 I$ is Browder, and hence $T+K-\lambda_0 I$ is Browder, which means that $\lambda_0 \in \sigma_a(T+K)\backslash\sigma_{aw}(T+K)$. In the following, we will prove that $T+K$ is isoloid. Let $\lambda_0 \in \mathrm{iso}\,\sigma(T+K)$ but $n(T+K-\lambda_0 I) = 0$. Then $\lambda_0 \in \mathrm{iso}\,\sigma(T)\cup\rho(T)$. The fact that K is power finite rank tells that $n(T-\lambda_0 I) < \infty$. This shows that $\lambda_0 \notin \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Thus $T+K-\lambda_0 I$ is bounded below or $T-\lambda_0 I$ is Browder. In each case, we may get that $T+K-\lambda_0 I$ is invertible, which contradicts the fact that $\lambda_0 \in \mathrm{iso}\,\sigma(T+K)$. Therefore $T+K$ is isoloid.

Conversely, suppose $T+K$ is isoloid and satisfies property (ω). The inclusion "$\sigma_b(T)\cap\sigma_a(T+K) \supseteq \sigma_1(T)\cup\partial\sigma_{CFI}(T)$" is easy to prove. Let $\lambda_0 \notin \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Then $n(T-\lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T-\lambda I \in SF_+^-(H)$ and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda-\lambda_0| < \epsilon$. Hence $\lambda \notin \sigma_a(T+K)$ or $\lambda \in \sigma_a(T+K)\backslash\sigma_{aw}(T+K)$. Since property (ω) holds for $T+K$, it follows that $T-\lambda I$ is bounded below if $0 < |\lambda-\lambda_0| < \epsilon$, which shows that $\lambda_0 \in \mathrm{iso}\,\sigma_a(T)\cup\rho_a(T)$. There are two cases to consider.

Case 1 Suppose $\lambda_0 \in \mathrm{iso}\,\sigma_a(T)$. If $R(T-\lambda_0 I)$ is closed, then $T-\lambda_0 I \in SF_+^-(H)$, thus $\lambda_0 \notin \sigma_a(T+K)$ or $\lambda_0 \in \sigma_a(T+K)\backslash\sigma_{aw}(T+K)$. The fact that property (ω) holds for $T+K$ tells that $\lambda_0 \notin \sigma_a(T+K)\cap\sigma_b(T+K)$, then $\lambda_0 \notin \sigma_a(T+K)\cap\sigma_b(T)$. In the case that $R(T-\lambda_0 I)$ is not closed. Then $\lambda_0 \notin \sigma_{CFI}(T)$. From the fact that $\lambda_0 \notin \partial\sigma_{CFI}(T)$, we know that $\lambda_0 \notin \overline{\sigma_{CFI}(T)}$. Then $T-\lambda I$ is invertible if $0 < |\lambda-\lambda_0|$ is small enough. This means that $\lambda_0 \in \mathrm{iso}\,\sigma(T)\cup\rho(T)$. Thus $\lambda_0 \in \mathrm{iso}\,\sigma(T+K)\cup\rho(T+K)$ (Lemma 2.1). Since property (ω) holds for $T+K$ and $T+K$ is isoloid, $T+K-\lambda_0 I$ is Browder. Then $T-\lambda_0 I$ is Browder, which means that $\lambda_0 \notin \sigma_b(T)\cap\sigma_a(T+K)$.

Case 2 If $\lambda_0 \notin \sigma_a(T)$, then $\lambda_0 \notin \sigma_a(T+K)$ or $\lambda_0 \in \sigma_a(T+K)\backslash\sigma_{aw}(T+K)$. We may suppose that $\lambda_0 \in \sigma_a(T+K)\backslash\sigma_{aw}(T+K)$. Since property (ω) holds for $T+K$, $T+K-\lambda_0 I$ is Browder. Then $T-\lambda_0 I$ is Browder. Again we prove that $\lambda_0 \notin \sigma_b(T)\cap\sigma_a(T+K)$. $\Box$

Remark 2.3 (1) "$T+K$ is isoloid" is essential. For example, let $K=0$ and $T \in B(\ell^2)$ be defined by:

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots).$$

Then K is power finite rank and $TK = KT$. Also $T+K$ satisfies property (ω) and is not isoloid. But $\sigma_b(T)\cap\sigma_a(T+K) = \{0\}$, $\sigma_1(T)\cup\partial\sigma_{CFI}(T) = \emptyset$.

(2) In Theorem 2.2, "K commutes with T" is essential. For example, let $T, K \in B(\ell^2)$ be defined by:

$$T(x_1, x_2, x_3, \ldots) = (0, 0, x_2, x_3, \ldots),$$

$$K(x_1, x_2, x_3, \ldots) = (0, x_1, 0, 0, \ldots).$$

Clearly, K is a power finite rank operator, $TK \neq KT$. We can see that $T+K$ is isoloid and satisfies property (ω). On the other hand, it is easily seen that $\sigma_b(T)\cap\sigma_a(T+K) = \{\lambda\in\mathbb{C} : |\lambda| = 1\}$, while $\sigma_1(T)\cup\partial\sigma_{CFI}(T) = \{\lambda\in\mathbb{C} : |\lambda| = 1\}\cup\{0\}$.

(3) "K is power finite rank" is essential. For example, let $T = 0, K \in B(\ell^2)$ be defined by:

$$K(x_1, x_2, x_3, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots).$$

Then K is not power finite rank and $TK = KT$; Also $\sigma_b(T)\cap\sigma_a(T+K) = \sigma_2(T)\cup\partial\sigma_{CFI}(T)$. But property (ω) fails for $T+K$.

As an immediate consequence we have:

Corollary 2.4 *Suppose that $T \in B(H)$. If $K \in B(H)$ is a finite rank operator commuting with T, then $T+K$ is isoloid and satisfies property (ω) if and only if $\sigma_b(T)\cap\sigma_a(T+K) = \sigma_1(T)\cup\partial\sigma_{CFI}(T)$.*

In [11], Aiena and Biondi studied the stability of property (ω). We will give another proof of their main theorem:

Corollary 2.5 *Suppose that $T \in B(H)$ is a-isoloid and K is a finite rank operator commuting with T such that $\sigma_a(T) = \sigma_a(T+K)$. If T satisfies property (ω), then $T+K$ satisfies property (ω).*

Proof Using Corollary 2.4, we need to prove that $\sigma_b(T)\cap\sigma_a(T+K) = \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Since T is a-isoloid and property (ω) holds for T, we can get that $\sigma_b(T)\cap\sigma_a(T) = \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Then $\sigma_b(T)\cap\sigma_a(T+K) = \sigma_b(T)\cap\sigma_a(T) = \sigma_1(T)\cup\partial\sigma_{CFI}(T)$. $\Box$

Recall that T is finite-isoloid if $\mathrm{iso}\,\sigma(T) \subseteq \{\lambda\in\mathbb{C} : 0 < n(T-\lambda I) < \infty\}$. Similarly to the proof of Theorem 2.2, we get:

Corollary 2.6 *Suppose that T is finite-isoloid. If K is a compact operator commuting with T, then $T+K$ is isoloid and satisfies property (ω) if and only if $\sigma_b(T)\cap\sigma_a(T+K) = \sigma_1(T)\cup\partial\sigma_{CFI}(T)\cup\{\lambda\in\mathbb{C} : n(T+K-\lambda I) = \infty\}$.*

"T is finite-isoloid" is essential in Corollary 2.6. For example, let $T = 0, K \in B(\ell^2)$ be defined by:

$$K(x_1, x_2, x_3, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots).$$

Clearly, K is compact and $TK = KT$. Since $n(T) = \infty$, T is not finite-isoloid. It is easy to check that $\sigma_b(T) \cap \sigma_a(T+K) = \{0\} = \sigma_2(T) \cup \partial\sigma_{CFI}(T) \cup \{\lambda \in \mathbb{C} : n(T+K-\lambda I) = \infty\}$. On the other hand, property (ω) fails for $T+K$.

Corollary 2.7 *Suppose that $T \in B(H)$. If $K \in B(H)$ is a power finite rank operator commuting with T, then $T+K$ is finite-isoloid and satisfies property (ω) if and only if $\sigma_b(T) \cap \sigma_a(T+K) = [\sigma_1(T) \cap \operatorname{acc}\sigma(T)] \cup \partial\sigma_{CFI}(T)$.*

Proof Suppose that $\sigma_b(T) \cap \sigma_a(T+K) = [\sigma_1(T) \cap \operatorname{acc}\sigma(T)] \cup \partial\sigma_{CFI}(T)$, then $\sigma_b(T) \cap \sigma_a(T+K) \subseteq \sigma_1(T) \cup \partial\sigma_{CFI}(T)$. And $\sigma_b(T) \cap \sigma_a(T+K) \supseteq \sigma_1(T) \cup \partial\sigma_{CFI}(T)$ is easy to prove. Now we can get that $\sigma_b(T) \cap \sigma_a(T+K) = \sigma_1(T) \cup \partial\sigma_{CFI}(T)$. From Theorem 2.2, $T+K$ is isoloid and satisfies property (ω). In the following we can prove $T+K$ is finite-isoloid. Let $\lambda_0 \in \operatorname{iso}\sigma(T+K)$. Then $\lambda_0 \in \operatorname{iso}\sigma(T) \cup \rho(T)$. This shows that $\lambda_0 \notin \operatorname{acc}\sigma(T) \cup \partial\sigma_{CFI}(T)$, thus $\lambda_0 \notin \sigma_b(T) \cap \sigma_a(T+K)$, which means that $n(T+K-\lambda_0 I) < \infty$.

Conversely, using Theorem 2.2, we need to prove that $\sigma_b(T) \cap \sigma_a(T+K) \subseteq \operatorname{acc}\sigma(T) \cup \partial\sigma_{CFI}(T) = \operatorname{acc}\sigma(T)$. Let $\lambda_0 \notin \operatorname{acc}\sigma(T)$. Then $\lambda_0 \in \operatorname{iso}\sigma(T) \cup \rho(T)$. Without loss of generality, let $\lambda_0 \in \operatorname{iso}\sigma(T)$. From Lemma 2.1, we get that $\lambda_0 \in \operatorname{iso}\sigma(T+K) \cup \rho(T+K)$. The fact that property (ω) holds for $T+K$ and $T+K$ is finite-isoloid tells us that $T+K-\lambda_0 I$ is Browder. Then $T-\lambda_0 I$ is Browder, which shows that $\lambda_0 \notin \sigma_b(T) \cap \sigma_a(T+K)$. $\square$

Let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$.

Corollary 2.8 *Suppose that $T \in B(H)$. If $K \in B(H)$ is a power finite rank operator commuting with T, then the following statements are equivalent:*

(1) For any $f \in H(T)$, $f(T)+K$ satisfies property (ω) and is isoloid, $\sigma_a(f(T)+K) = \sigma(f(T)+K)$;

(2) For any $f \in H(T)$, $f(T)$ satisfies property (ω) and is isoloid, $\sigma_a(f(T)) = \sigma(f(T))$;

(3) $\sigma_b(T) = \sigma_1(T)$.

Proof (1)$\Rightarrow$(2). Clearly.

(2)$\Rightarrow$(3). We only need to prove that $\sigma_b(T) \subseteq \sigma_1(T)$. Let $\lambda_0 \notin \sigma_1(T)$. Then $n(T-\lambda_0 I) < \infty$ and $T-\lambda I \in SF_+^-(H)$ and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda - \lambda_0|$ is sufficiently small. Since property (ω) holds for T, $T-\lambda I$ is bounded below. From the fact that $\sigma(T) = \sigma_a(T)$, we know that $T-\lambda I$ is invertible. This shows that $\lambda_0 \in \operatorname{iso}\sigma(T) \cup \rho(T)$. Then $T-\lambda_0 I$ is Browder since T is isoloid and satisfies property (ω), which means that $\lambda_0 \notin \sigma_b(T)$, hence $\sigma_b(T) = \sigma_1(T)$.

(3)$\Rightarrow$(1). Suppose that $\sigma_b(T) = \sigma_1(T)$, we can get that $f(\sigma_1(T)) = \sigma_1(f(T))$. In fact, we have that $f(\sigma_1(T)) = f(\sigma_b(T)) = \sigma_b(f(T)) \supseteq \sigma_1(f(T))$. For the converse inclusion, let $\mu_0 \notin \sigma_1(f(T))$. Then $n(f(T)-\mu_0 I) < \infty$ and there exists $\epsilon > 0$ such that $f(T)-\mu I \in SF_+^-(H)$

and $N(f(T)-\mu I)\subseteq\bigcap_{n=1}^{\infty}R[(f(T)-\mu I)^n]$ if $0<|\mu-\mu_0|<\epsilon$. Let $f(T)-\mu_0 I=(T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_k I)^{n_k}g(T)$, where $\lambda_i\neq\lambda_j$ and $g(T)$ is invertible. By continuity of $f(\lambda)$, there exists $\delta>0$ such that $0<|f(\lambda)-f(\lambda_i)|=|f(\lambda)-\mu_0|<\epsilon$ if $0<|\lambda-\lambda_0|<\delta$. Then $f(T)-f(\lambda)I\in SF_+^-(H)$ and $N(f(T)-f(\lambda)I)\subseteq\bigcap_{n=1}^{\infty}R[(f(T)-f(\lambda)I)^n]$. Thus $f(\lambda)\notin\sigma_k(f(T))=f(\sigma_k(T))$ (see [12, Satz 6]), so $\lambda\notin\sigma_k(T)$. Since $f(T)-f(\lambda)I$ is upper semi-Fredholm, $T-\lambda I$ is upper semi-Fredholm and hence $\lambda\notin\sigma_1(T)$, it follows that $T-\lambda I$ is Browder. Then $N(T-\lambda I)=N(T-\lambda I)\cap\bigcap_{n=1}^{\infty}R[(T-\lambda I)^n]=\{0\}$, and we know that $T-\lambda I$ is invertible. Now we have that $\lambda_i\in\operatorname{iso}\sigma(T)\cup\rho(T)$ and $n(T-\lambda_i I)<\infty$, then $\lambda_i\notin\sigma_1(T)$. Hence $f(\sigma_1(T))\subseteq\sigma_1(f(T))$. This shows that $\sigma_b(f(T))=\sigma_1(f(T))$, and $\sigma_a(f(T)+K)\supseteq\sigma_b(f(T))$, then $\sigma_b(f(T))\cap\sigma_a(f(T)+K)=\sigma_b(f(T))=\sigma_1(f(T))\cup\partial\sigma_{CFI}(f(T))$. By Theorem 2.2, $f(T)+K$ is isoloid and property (ω) holds for $f(T)+K$. Since $\sigma_a(f(T)+K)\supseteq\sigma_b(f(T))(=\sigma_b(f(T)+K))$, we know that $\sigma_a(f(T)+K)=\sigma(f(T)+K)$. $\Box$

In the sequel we shall consider nilpotent perturbations of operators. It is easy to check that if N is a nilpotent operator commuting with T, then $\sigma(T)=\sigma(T+N)$ and $\sigma_a(T)=\sigma_a(T+N)$. It is easily proved that:

Theorem 2.9 *Suppose $N\in B(H)$ is a nilpotent operator that commutes with $T\in B(H)$. Then $T+N$ is isoloid and satisfies property (ω) if and only if $\sigma_b(T)\cap\sigma_a(T)=\sigma_1(T)\cup\partial\sigma_{CFI}(T)$.*

"N is a nilpotent operator" is essential. Let $T=0, N\in B(\ell^2)$ be defined by:

$$N(x_1,x_2,x_3,\ldots)=(\frac{x_2}{2},\frac{x_3}{3},\frac{x_4}{4},\ldots).$$

Clearly, property (ω) fails for $T+N$, while $\sigma_b(T)\cap\sigma_a(T)=\{0\}=\sigma_1(T)\cup\partial\sigma_{CFI}(T)$.

For the stability of property (ω) for quasi-nilpotent operators or Riesz operators, we can get:

Theorem 2.10 *Suppose that $T\in B(H)$. If $K\in B(H)$ is a Riesz operator commuting with T, then $T+K$ is isoloid and satisfies property (ω) if and only if $\sigma_b(T)\cap\sigma_a(T+K)=[\sigma_1(T)\cap acc\sigma(T+K)]\cup\partial\sigma_{CFI}(T)\cup\{\lambda\in\mathbb{C}:n(T+K-\lambda I)=\infty\}$.*

A bounded operator $T\in B(H)$ is said to have property $H(P)$ if for every complex number λ there exists a positive integer d_λ for which $H_0(T-\lambda I)=N((T-\lambda I)^{d_\lambda})$, where $H_0(T)=\{x\in H:\lim_{n\to\infty}\|T^n x\|^{\frac{1}{n}}=0\}$. This class has been studied in [13] and for the constant function $d_\lambda=1$ has been also studied in [14]. Also property $H(P)$ is satisfied by p-hyponormal operators and log-hyponormal operators, M-hyponormal operators, ω-hyponormal operators, totally paranormal operators and totally $*$-paranormal operators. As an application, we shall consider the stability of property (ω) for $H(P)$ operators.

Theorem 2.11 *Suppose that $T^*\in H(P)$. Then for any $f\in H(T)$ and any power finite rank K commuting with T, property (ω) holds for $f(T)+K$.*

Proof First we assert that $\sigma_1(T)=\sigma_b(T)$. In fact, we only need to prove $\sigma_b(T)\subseteq\sigma_1(T)$. Suppose that $\lambda_0\notin\sigma_1(T)$, then $n(T-\lambda_0 I)<\infty$ and there exists $\epsilon>0$ such that $T-\lambda I\in SF_+^-(H)$ and $N(T-\lambda I)\subseteq\bigcap_{n=1}^{\infty}R[(T-\lambda I)^n]$ if $0<|\lambda-\lambda_0|<\epsilon$. By $p(T^*-\lambda I)<\infty$, we know that

$\text{ind}(T-\lambda I)=0$, then $T-\lambda I$ is Weyl if $0<|\lambda-\lambda_0|<\epsilon$. Since $T^*\in H(P)$ and $q(T-\lambda I)<\infty$, we know that $T-\lambda I$ is Browder, then $N(T-\lambda I)=N(T-\lambda I)\cap\bigcap_{n=1}^{\infty}R[(T-\lambda I)^n]=\{0\}$, this means that $T-\lambda I$ is invertible. Now we have that $\lambda_0\in\text{iso}\,\sigma(T)$, then $\lambda_0\in\text{iso}\,\sigma(T^*)$. Since $T^*\in H(P)$, λ_0 is a polar point of resolvent set of T^*. We get that λ_0 is a polar point of resolvent set of T. Using the fact that $n(T-\lambda_0 I)<\infty$, we know $T-\lambda_0 I$ is Browder, that is $\lambda_0\notin\sigma_b(T)$. Thus $\sigma_b(T)\subseteq\sigma_1(T)$.

By Corollary 2.8, we know that property (ω) holds for $f(T)+K$. □

If $T\in H(P)$, $K\in B(H)$ is a power finite rank operator commuting with T, then property (ω) may fail for $T+K$.

Theorem 2.12 *Suppose that $T\in H(P)$, if K is a power finite rank operator and $TK=KT$, then $T+K$ is isoloid and satisfies property (ω) $\Leftrightarrow$ $\sigma_{CFI}(T)=\sigma_b(T)\cap\rho_a(T+K)$.*

Proof Suppose that $T+K$ satisfies property (ω) and is isoloid. The inclusion $\sigma_{CFI}(T)\supseteq\sigma_b(T)\cap\rho_a(T+K)$ is easy to prove. Let $\lambda_0\notin\sigma_b(T)\cap\rho_a(T+K)$. Without loss of generality, we suppose that $\lambda_0\in\sigma_a(T+K)$.

Case 1 Suppose $\lambda_0\notin\sigma_{aw}(T+K)$, $T+K-\lambda_0 I$ is Browder since property (ω) holds for $T+K$, this induces that $T-\lambda_0 I$ is Browder, which means $\lambda_0\notin\sigma_{CFI}(T)$.

Case 2 Suppose $\lambda_0\in\sigma_{aw}(T+K)$, then $\lambda_0\in\sigma_{aw}(T)$. If $R(T-\lambda_0 I)$ is not closed, we can get $\lambda_0\notin\sigma_{CFI}(T)$. In the case that $R(T-\lambda_0 I)$ is closed, then $n(T-\lambda_0 I)=\infty$, we claim that $d(T-\lambda_0 I)=\infty$. Otherwise if $d(T-\lambda_0 I)<\infty$, then $n(T-\lambda_0 I)\le d(T-\lambda_0 I)<\infty$ since $p(T-\lambda_0 I)<\infty$. It is a contradiction. Then $n(T-\lambda_0 I)=d(T-\lambda_0 I)=\infty$. Again we have $\lambda_0\notin\sigma_{CFI}(T)$.

Conversely, suppose $\sigma_{CFI}(T)=\sigma_b(T)\cap\rho_a(T+K)$. By Theorem 2.2, we need to prove that "$\sigma_b(T)\cap\sigma_a(T+K)\subseteq\sigma_1(T)\cup\partial\sigma_{CFI}(T)$." Let $\lambda_0\notin\sigma_1(T)\cup\partial\sigma_{CFI}(T)$. Then $n(T-\lambda_0 I)<\infty$ and there exists $\epsilon>0$ such that $T-\lambda I\in SF_+^-(H)$ and $N(T-\lambda I)\subseteq\bigcap_{n=1}^{\infty}R[(T-\lambda I)^n]$ if $0<|\lambda-\lambda_0|<\epsilon$. By $p(T-\lambda I)<\infty$, we know that $T-\lambda I$ is bounded below. This induces that $\lambda_0\in\text{iso}\,\sigma_a(T)\cup\rho_a(T)$.

Case 1 Suppose $\lambda_0\in\rho_a(T)$, that is $\lambda_0\notin\sigma_a(T)$, then $\lambda_0\notin\sigma_a(T+K)$ or $\lambda_0\in\sigma_a(T+K)$. We may suppose that $\lambda_0\in\sigma_a(T+K)$. From the fact that $\sigma_{CFI}(T)=\sigma_b(T)\cap\rho_a(T+K)$, $\lambda_0\notin\sigma_{CFI}(T)$, hence $\lambda_0\notin\sigma(T)$, which means that $\lambda_0\notin\sigma_b(T)$.

Case 2 Suppose $\lambda_0\in\text{iso}\,\sigma_a(T)$, then $\lambda_0\in\text{iso}\,\sigma_a(T+K)\cup\rho_a(T+K)$ (Lemma 2.1). Without loss of generality, let $\lambda_0\in\sigma_a(T+K)$. Since $\sigma_{CFI}(T)=\sigma_b(T)\cap\rho_a(T+K)$, we know that $\lambda_0\notin\sigma_{CFI}(T)$. Then $T-\lambda I$ is invertible if $0<|\lambda-\lambda_0|$ is small enough. Thus $\lambda_0\in\text{iso}\,\sigma(T)$. Since $T\in H(P)$, it follows that λ_0 is a polar point of resolvent set of T. Using the fact that $n(T-\lambda_0 I)<\infty$, we know that $T-\lambda_0 I$ is Browder, that is $\lambda_0\notin\sigma_b(T)$. □

Corollary 2.13 *Suppose that $T\in H(P)$. If $\sigma_{CFI}(T)=\emptyset$, then $T+K$ is isoloid and satisfies property (ω) for any power finite rank operator K commuting with T.*

Corollary 2.14 *Suppose that $T \in H(P)$ is finite-isoloid. If K is a Riesz operator commuting with T, then $T+K$ is isoloid and satisfies property $(\omega) \Leftrightarrow \sigma_{CFI}(T)=\sigma_b(T)\cap\rho_a(T+K)$.*

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References

[1] H. WEYL. *Überbeschränkte quadratische Formen, deren Differenz vollstetig ist.* Rend. Circ. Mat. Palermo, 1909, **27**: 373–392. (in German)
[2] R. HARTE, W. Y. LEE. *Another note on Weyl's theorem.* Trans. Amer. Math. Soc., 1997, **349**(5): 2115–2124.
[3] V. RAKOČEVIČ. *Operators obeying a-Weyl's theorem.* Rev. Roumaine Math. Pures Appl., 1989, **34**(10): 915–919.
[4] V. RAKOČEVIČ. *On a class of operators.* Mat. Vesnik, 1985, **37**(4): 423–426.
[5] D. S. DJORDJEVIĆ. *Operators obeying a-Weyl's theorem.* Publ. Math. Debrecen, 1999, **55**(3-4): 283–298.
[6] V. RAKOČEVIČ. *Semi-Browder operators and perturbations.* Studia Math., 1997, **122**(2): 131–137.
[7] Xiaohong CAO. *Weyl spcetrum of the products of operators.* J. Korean Math. Soc., 2008, **45**(3): 771–780.
[8] P. AIENA, P. PEÑA. *Variantion on Weyl's theorem.* J. Math. Anal. Appl., 2006, **324**(1): 566–579.
[9] P. SAPHAR. *Contribution à l'étude des applications linéaires dans un espace de Banach.* Bull. Soc. Math. France, 1964, **92**: 363–384. (in French)
[10] D. KATO. *Perturbation Theory for Linear Operator.* Springer-Verlag New York, Inc., New York, 1966.
[11] P. AIENA, M. T. BIONDI. *Property (ω) and perturbations.* J. Math. Anal. Appl., 2007, **336**: 683–692.
[12] C. SCHMOEGER. *Ein spektralabbidungssatz.* Arch. Math. (Basel), 1990, **55**(5): 484–489.
[13] M. OUDGHIRI. *Weyl's and Browder's theorems for operators satisfying the SVEP.* Studia Math., 2004, **163**(1): 85–101.
[14] P. AIENA, F. VILLAFÃNE. *Weyl's theorems for some classes of operators.* Integral Equations Operator Theory, 2005, **53**(4): 453–466.