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Quasi-Modular Preserving Rank One Maps on Hilbert C^*-Modules

Bin MENG
Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Jiangsu 210016*, P. R. China*

Abstract In this paper, we characterize a class of quasi-modular maps on Hilbert C^*-modules which map a "rank one" adjointable operator to another rank one operator.

Keywords Hilbert C^*-module; coordinate inverse; quasi-modular map.

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1. Introduction and preliminaries

The problem of determining linear map Φ on $\mathbf{B}(X)$ preserving certain properties has attracted attention of many mathematicians in recent decade. They have been devoted to the study of linear maps preserving spectrum, rank, nilpotency, etc.

Rank preserving problem is a basic problem in the study of linear preserving problem. Rank preserving linear maps have been studied intensively by Hou in [1]. In [2, 3] the authors used very elegant arguments to completely describe the additive mappings preserving (or decreasing) rank one.

In our study of free probability theory, we find that modular maps on Hilbert C^*-module preserving certain properties are also important [4, 5], and thus the study of the modular preserving problem becomes attractive. In [6], a complete description of modular maps preserving rank one modular operators was given. Naturally in the present paper we consider quasi-modular maps on modules preserving rank one, which are much more complicated than the modular maps case.

A Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $\mathcal{M}$ equipped with an $\mathcal{A}$-valued inner product $\langle , \rangle$ which is $\mathcal{A}$-linear in the first and conjugate $\mathcal{A}$-linear in the second variable such that $\mathcal{M}$ is a Banach space with the norm $\|v\|_M = \|\langle v, v\rangle\|^{\frac{1}{2}}$, $\forall v \in \mathcal{M}$. Hilbert C^*-modules first appeared in the work of Kaplansky [7], who used them to prove that derivations of type I AW^*-algebras are inner. Now a good text book about Hilbert C^*-module is [8]. In this paper we mainly consider Hilbert $\mathcal{A}$-module $\mathcal{A}\otimes H$ (or denoted by $H_{\mathcal{A}}$), where H is a separable infinite dimentional Hilbert space and $\mathcal{A}$ is a unital commutative C^*-algebra. $H_{\mathcal{A}}$ plays a special role

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E-mail address: bmengnuaa@gmail.com

in the theory of Hilbert C^*-modules [8]. Obviously, $H_{\mathcal{A}}$ is countably generated and possesses an orthonormal basis $\{1 \otimes e_i\} := \varepsilon$, where $\{e_i\}$ is an orthonormal basis in H.

We introduce a class of modular operators which is analogous to rank one operators on a Hilbert space. For all $x, y \in H_{\mathcal{A}}$, define $\theta_{x,y} : H_{\mathcal{A}} \to H_{\mathcal{A}}$ by $\theta_{x,y}(\xi) := \langle \xi, y \rangle x$, for every $\xi \in H_{\mathcal{A}}$. Note that $\theta_{x,y}$ is quite different from rank one linear operators on Hilbert space. For instance, we cannot infer $x = 0$ or $y = 0$ from $\theta_{x,y} = 0$. For convenience, we still call $\theta_{x,y}$ rank one. We denote the set $\{\sum_{i=1}^{n} \alpha_i \theta_{x_i,y_i}, \forall n \in \mathbb{N}, \alpha_i \in \mathcal{A}\}$ by $\mathcal{F}(H_{\mathcal{A}})$.

In the present paper, we will describe the locally quasi-modular map $\Phi : \mathcal{F}(H_{\mathcal{A}}) \to \mathcal{F}(H_{\mathcal{A}})$ which maps $\theta_{x,y}$ to some $\theta_{s,t}$, and in this case Φ is always quasi-modular. The methods are analogous to those in [1–3] but much more complicated.

2. Definitions and lemmas

In this section we mainly introduce some definitions and prove some lemmas.

Definition 2.1 ([6]) *Let ε be a orthonormal basis in $H_{\mathcal{A}}$. $x \neq 0$ in $H_{\mathcal{A}}$ will be called coordinately invertible if for all $e \in \varepsilon$, $\langle e, x \rangle$ is invertible in $\mathcal{A}$ unless $\langle e, x \rangle = 0$.*

We denote the set of all the coordinately invertible elements in $H_{\mathcal{A}}$ by $CI(H_{\mathcal{A}})$ or CI for short and the set of all the invertible elements in $\mathcal{A}$ by $\mathrm{Inv}(\mathcal{A})$. Obviously, $\varepsilon \subseteq CI$.

Coordinately invertible elements in Hilbert C^*-module are analogous to the elements in Hilbert space to some extents.

Lemma 2.2 ([6]) *Suppose $y \in CI$ and $\theta_{x,y} = 0$, then $x = 0$.*

Lemma 2.3 ([6]) *Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module, where $\mathcal{A}$ is a unital C^*-algebra, and let $\phi, \sigma : \mathcal{M} \to \mathcal{A}$ be $\mathcal{A}$-linear operators. Suppose that σ vanishes on the kernel of ϕ. Then there exists $b \in \mathcal{A}$ such that $\sigma = \phi \cdot b$.*

Corollary 2.4 ([6]) *Let $g_1, g_2 \in \mathcal{M}$. If for all $x \in \mathcal{M}$, $\langle x, g_1 \rangle = 0$ implies $\langle x, g_2 \rangle = 0$, then there is $a \in \mathcal{A}$ such that $g_2 = a g_1$.*

The following Lemma 2.5, which will be used frequently, has been obtained in [6]. Nevertheless we give its proof for the sake of completeness.

Lemma 2.5 *Let $\mathcal{A}$ be a unital C^*-algebra, $x_1, x_2 \in \mathcal{M}$, $g_1, g_2 \in CI$ satisfying $\theta_{x_1,g_1} + \theta_{x_2,g_2} = \theta_{x_3,g_3}$. Then at least one of the following is true:*

(i) There exists an invertible $\alpha_1 \in \mathcal{A}$ such that $g_1 = \alpha_1 g_2$;

(ii) There are $\beta_1, \beta_2 \in \mathcal{A}$ such that $x_1 = \beta_1 x_3$, $x_2 = \beta_2 x_3$.

Proof We will complete the proof by considering the following four cases.

Case 1 For all $\xi \in H_{\mathcal{A}}$, $\langle \xi, g_2 \rangle = 0$ implies $\langle \xi, g_1 \rangle = 0$. From Corollary 2.4, there exists $\alpha_1 \in \mathcal{A}$ such that $g_1 = \alpha_1 g_2$. Furthermore from $g_1, g_2 \in CI$, we infer that $\alpha_1 \in \mathcal{A}$ is invertible.

Case 2 For all $\xi \in H_{\mathcal{A}}$, $\langle \xi, g_1 \rangle = 0$ implies $\langle \xi, g_2 \rangle = 0$. Still from Corollary 2.4, there is $\alpha_2 \in \mathcal{A}$

such that $g_2 = \alpha_2 g_1$ and α_2 is invertible.

Case 3 There exists $\xi_0 \in H_{\mathcal{A}}$ such that $\langle \xi_0, g_2\rangle = 0$ but $\langle \xi_0, g_1\rangle \neq 0$. We can find $e \in \varepsilon$ such that $\langle e, g_2\rangle = 0$ but $\langle e, g_1\rangle \neq 0$. Then from $\langle e, g_1\rangle x_1 + \langle e, g_2\rangle x_2 = \langle e, g_3\rangle x_3$, it follows $\langle e, g_1\rangle x_1 = \langle e, g_3\rangle x_3$. Since $g_1 \in CI$, we have $x_1 = \langle e, g_1\rangle^{-1}\langle e, g_3\rangle x_3$. We put $\beta_1 = \langle e, g_1\rangle^{-1}\langle e, g_3\rangle$ and get $\theta_{\beta_1 x_3, g_1} + \theta_{x_2, g_2} = \theta_{x_3, g_3}$. Thus $\theta_{x_2, g_2} = \theta_{x_3, g_3 - \beta_1^* g_1}$. Now choosing $e' \in \varepsilon$, we get $\langle e', g_2\rangle x_2 = \langle e', g_3 - \beta_1^* g_1\rangle x_3$ and thus $x_2 = \langle e', g_2\rangle^{-1}\langle e', g_3 - \beta_1^* g_1\rangle x_3$. Putting $\beta_2 = \langle e', g_2\rangle^{-1}\langle e', g_3 - \beta_1^* g_1\rangle$ then we obtain (ii).

Case 4 There exists $\xi_0 \in H_{\mathcal{A}}$ such that $\langle \xi_0, g_1\rangle = 0$ but $\langle \xi_0, g_2\rangle \neq 0$. Similarly to Case 3, we get (ii) again. □

Corollary 2.6 *With the notations in the above lemma, suppose $g_1 \neq \alpha g_2$, for all $\alpha \in \mathcal{A}$. If $g_3 \in CI$, then there exist $\beta_1, \beta_2 \in \mathcal{A}$ which are invertible such that $x_1 = \beta_1 x_3$, $x_2 = \beta_2 x_3$. Furthermore, $x_1 = \beta_0 x_2$ for some $\beta_0 \in \mathrm{Inv}(\mathcal{A})$.*

Proof Denote $\{x|\langle x, g_i\rangle = 0\}$ by $\ker g_i$, $i = 1, 2$. Since $g_1 \neq \alpha g_2$, $g_2 \neq \beta g_1$, we have $\ker g_1 \nsubseteq \ker g_2$, $kerg_2 \nsubseteq \ker g_1$. So there exists $e_1 \in \varepsilon$, such that $\langle e_1, g_1\rangle \neq 0$ but $\langle e_1, g_2\rangle = 0$. Then $\langle e_1, g_1\rangle x_1 + \langle e_1, g_2\rangle x_2 = \langle e_1, g_3\rangle x_3$, i.e. $\langle e_1, g_1\rangle x_1 = \langle e_1, g_3\rangle x_3$ and thus $x_1 = \langle e_1, g_1\rangle^{-1}\langle e_1, g_3\rangle x_3$. $\langle e_1, g_1\rangle^{-1}\langle e_1, g_3\rangle = \beta_1$ is invertible.

Similarly, if there exits $e_2 \in \varepsilon$ such that $\langle e_2, g_1\rangle = 0$ but $\langle e_2, g_2\rangle \neq 0$, then

$$x_2 = \langle e_2, g_2\rangle^{-1}\langle e_2, g_3\rangle x_3.$$

Putting $\beta_2 = \langle e_2, g_2\rangle^{-1}\langle e_2, g_3\rangle$, we get the desired result. □

Definition 2.7 $\Phi : \mathcal{F}(H_{\mathcal{A}}) \to \mathcal{F}(H_{\mathcal{A}})$ *is a map. If for any $x \in H_{\mathcal{A}}$, $y \in CI$, there are $s \in H_{\mathcal{A}}$, $t \in CI$ such that $\Phi(\theta_{x,y}) = \theta_{s,t}$, $\Phi(\theta_{y,x}) = \theta_{t,s}$, $x \neq 0$ implies $s \neq 0$, and s, t can be chosen in CI whenever $x, y \in CI$, then Φ will be called rank one preserving.*

Definition 2.8 $\Phi : \mathcal{F}(H_{\mathcal{A}}) \to \mathcal{F}(H_{\mathcal{A}})$ *is an additive map and for arbitrary $\theta_{x,y}$, $\Phi(\lambda\theta_{x,y}) = \tau_{x,y}(\lambda)\Phi(\theta_{x,y})$ where $\tau_{x,y} : \mathcal{A} \to \mathcal{A}$ is a surjective multiplicative $*$-transform. Then Φ will be called a locally quasi-modular map.*

If in addition, there exists $\tau : \mathcal{A} \to \mathcal{A}$, which is a surjective multiplicative $*$-transform such that $\Phi(\lambda T) = \tau(\lambda)\Phi(T)$, for all $T \in \mathcal{F}(H_{\mathcal{A}})$, then Φ will be called a τ-quasi-modular map.

It is well known that two locally linearly dependent linear operators are linearly dependent. The following lemma is an analogue of this result in modular operator case.

Lemma 2.9 *Let $\mathcal{A}$ be a unital commutative C^*-algebra, and A, B be injective τ-quasi-modular continuous maps on $H_{\mathcal{A}}$. Suppose for all $e, e' \in \varepsilon$, there exists $\lambda_e \in \mathcal{A}$ such that $Be = \lambda_e Ae$. Then $B = \lambda A$ for some $\lambda \in \mathcal{A}$.*

Proof Let $e_1, e_2 \in \varepsilon$. From the assumptions, we know there exist $\lambda_1, \lambda_2 \in \mathcal{A}$ such that $Be_1 = \lambda_1 Ae_1$ and $Be_2 = \lambda_2 Ae_2$. $B(e_1 + e_2) = \lambda_3 A(e_1 + e_2)$. On the other hand $B(e_1 + e_2) =$

$\lambda_1 Ae_1 + \lambda_2 Ae_2$. Thus

$$(\lambda_3 - \lambda_1)Ae_1 + (\lambda_3 - \lambda_2)Ae_2 = 0.$$

Suppose $\tau(\beta_1) = \lambda_3 - \lambda_1$, $\tau(\beta_2) = \lambda_3 - \lambda_2$. Then $A(\beta_1 e_1) + A(\beta_2 e_2) = 0$, i.e., $A(\beta_1 e_1 + \beta_2 e_2) = 0$. Since A is injective, we get $\beta_1 e_1 + \beta_2 e_2 = 0$ and $\beta_1 = \beta_2 = 0$. Therefore, $\lambda_1 = \lambda_2 = \lambda_3$.

For arbitrary $e \in \varepsilon$ and $e \neq e_1, e_2$. $Be = \lambda_e Ae$. Repeating the above process, we have $\lambda_e = \lambda_1 = \lambda_2 = \lambda_3$.

For arbitrary $f \in H_{\mathcal{A}}$, $f = \sum_i \alpha_i e_i$. Then we have

$$\begin{aligned} B(f) &= B(\sum_i \alpha_i e_i) = \sum_i \tau(\alpha_i) Be_i = \sum_i \tau(\alpha_i)\lambda_3 Ae_i \\ &= \lambda_3 A(\sum_i \alpha_i e_i) = \lambda_3 A(f). \end{aligned}$$

Denote λ_3 by λ and we get $B = \lambda A$.

Now we introduce some notations. $L_x := \{\theta_{x,g} \mid g \in H_{\mathcal{A}}\}$, $R_f := \{\theta_{x,f} \mid x \in H_{\mathcal{A}}\}$, $L_x^{CI} := \{\theta_{x,g} \mid g \in CI\}$, $R_f^{CI} := \{\theta_{y,f} \mid y \in CI\}$, $L_x^{\varepsilon} := \{\theta_{x,e} \mid e \in \varepsilon\}$, $R_f^{\varepsilon} := \{\theta_{e,f} \mid e \in \varepsilon\}$. $\square$

In the following, $\mathcal{A}$ is a unital commutative C^*-algebra and Φ is always a locally quasi-modular preserving rank one map on $H_{\mathcal{A}}$. The following lemma plays important roles in this paper.

Lemma 2.10 *For every $x \in CI$ there exists either $y \in CI$ such that $\Phi(L_x^{\varepsilon}) \subseteq L_y^{CI}$ or $f \in CI$ such that $\Phi(L_x^{\varepsilon}) \subseteq R_f^{CI}$.*

Proof For $e_1, e_2 \in \varepsilon$, $\Phi(\theta_{x,e_1}) = \theta_{y_1,g_1}$, $\Phi(\theta_{x,e_2}) = \theta_{y_2,g_2}$, where $y_1, y_2, g_1, g_2 \in CI$. Since $e_1 + e_2 \in CI$, there are $y_{12}, g_{12} \in CI$ such that $\Phi(\theta_{x,e_1+e_2}) = \theta_{y_1,g_1} + \theta_{y_2,g_2} = \theta_{y_{12},g_{12}}$. From Corollary 2.6, there exist $\alpha_{12}, \beta_{12} \in \mathrm{Inv}(\mathcal{A})$ such that $y_1 = \alpha_{12} y_2$ or $g_1 = \beta_{12} g_2$.

Case 1 We suppose $g_1 = \beta_{12} g_2$ and $y_1 \neq \alpha y_2$, for all $\alpha \in \mathrm{Inv}(\mathcal{A})$. For arbitrary $e_i \in \varepsilon$, $i \neq 1, 2$, $\Phi(\theta_{x,e_i}) = \theta_{y_i,g_i}$ where $g_i, y_i \in CI$. Assume that $g_1 \neq \beta g_i$, for all $\beta \in \mathrm{Inv}(\mathcal{A})$. Then there exists $\alpha_i \in \mathrm{Inv}(\mathcal{A})$ such that $y_1 = \alpha_i y_i$.

$$\begin{aligned} \Phi(\theta_{x,e_1+e_2+e_i}) &= \theta_{y_1,g_1} + \theta_{y_2,g_2} + \theta_{y_i,g_i} = \theta_{\alpha_i y_i,\beta_{12} g_2} + \theta_{y_2,g_2} + \theta_{y_i,g_i} \\ &= \theta_{\beta_{12}^* \alpha_i y_i + y_2, g_2} + \theta_{y_i,g_i}. \end{aligned}$$

Since $g_1 = \beta_{12} g_2, g_1 \neq \beta g_i$ for all $\beta \in \mathrm{Inv}(\mathcal{A})$ and Φ preserving rank one, we know there exists $\alpha_0 \in \mathrm{Inv}(\mathcal{A})$ such that $y_i = \alpha_0(\beta_{12}^* \alpha_i y_i + y_2)$ and $(1 - \alpha_0 \beta_{12}^* \alpha_i) y_i = \alpha_0 y_2$. Consequently

$$y_2 = \alpha_0^{-1}(1 - \beta_{12}^* \alpha_0 \alpha_i)\alpha_i^{-1} y_1.$$

On the other hand, $y_1, y_2 \in CI$, so $\alpha_0^{-1}(1 - \beta_{12}^* \alpha_0 \alpha_i)\alpha_i^{-1} \in \mathrm{Inv}(\mathcal{A})$, which contradicts our assumption. Thus we have proved $g_1 = \beta_{1i} g_i$ for some $\beta_{1i} \in \mathrm{Inv}(\mathcal{A})$.

Then $\Phi(\theta_{x,e_i}) = \theta_{y_i,g_i} = \theta_{\beta_{12}^{-1*} y_i, g_1}$. So we have $\Phi(L_x^{\varepsilon}) \subseteq R_{g_1}^{CI}$.

Case 2 We suppose $y_1 = \alpha_{12} y_2$ for some $\alpha_{12} \in \mathrm{Inv}(\mathcal{A})$ and $g_1 \neq \beta g_2$, for all $\beta \in \mathrm{Inv}(\mathcal{A})$.

For arbitrary $e_i \in \varepsilon$, $i \neq 1, 2$, $\Phi(\theta_{x,e_i}) = \theta_{y_i,g_i}$, for some $y_i, g_i \in CI$.

Assume $y_1 \neq \alpha y_i$, for all $\alpha \in \mathrm{Inv}(\mathcal{A})$, then $g_1 = \beta_{1i} g_i$, for some $\beta_{1i} \in \mathrm{Inv}(\mathcal{A})$.

$$\begin{aligned}\Phi(\theta_{x,e_1+e_2+e_i}) &= \theta_{y_1,g_1} + \theta_{y_2,g_2} + \theta_{y_i,g_i} = \theta_{y_1,\beta_{1i}g_i} + \theta_{\alpha_{12}^{-1}y_1,g_2} + \theta_{y_i,\beta_{1i}^{-1}g_1}\\ &= \theta_{y_1,\beta_{1i}g_i+\alpha_{12}^{-1*}g_2} + \theta_{y_i,\beta_{1i}^{-1}g_1}.\end{aligned}$$

Since $x, e_1+e_2+e_i \in CI$, we know there exists $\beta_0 \in \mathrm{Inv}(\mathcal{A})$ such that $\beta_{1i}^{-1}g_1 = \beta_0(g_1 + \alpha_{12}^{-1*}g_2)$ and $g_2 = \beta_0^{-1}\alpha_{12}^*(\beta_{1i}^{-1} - \beta_0)g_1$ which contradicts $g_1 \neq \beta g_2$, for all $\beta \in \mathrm{Inv}(\mathcal{A})$. Thus we have proved $y_1 = \alpha_{1i} y_i$, for some $\alpha_{1i} \in \mathrm{Inv}(\mathcal{A})$. Then $\Phi(\theta_{x,e_i}) = \theta_{y_i,g_i} = \theta_{\alpha_{1i}^{-1}y_1,g_i} = \theta_{y_1,\alpha_{1i}^{-1*}g_i}$ and $\Phi(L_x^\varepsilon) \subseteq L_{y_1}^{CI}$.

Case 3 If for all $e_j \in \varepsilon$, $j \neq 1$, $\Phi(\theta_{x,e_j}) = \theta_{y_j,g_j}$ such that $y_1 = \alpha_{1j} y_j$, $g_1 = \beta_{1j} g_j$, then both $\Phi(L_x^\varepsilon) \subseteq L_{y_1}^{CI}$ and $\Phi(L_x^\varepsilon) \subseteq R_{g_1}^{CI}$ hold. □

Lemma 2.11 *At least one of the following is true*

(1) For all $x \in CI$, there exists $y \in CI$ such that $\Phi(L_x^\varepsilon) \subseteq L_y^{CI}$;

(2) For all $x \in CI$, there exists $f \in CI$ such that $\Phi(L_x^\varepsilon) \subseteq R_f^{CI}$.

Proof If for some $x \in CI$, $\Phi(L_x^\varepsilon) \subseteq \mathrm{Inv}(\mathcal{A})\theta_{y,g}$ where $y, g \in CI$, then both $\Phi(L_x^\varepsilon) \subseteq L_y^{CI}$ and $\Phi(L_x^\varepsilon) \subseteq R_g^{CI}$ hold.

Now we assume that there exist $x_0, x_1 \in CI$ such that $\Phi(L_{x_0}^\varepsilon) \subseteq L_{y_0}^{CI}$, $\Phi(L_{x_1}^\varepsilon) \subseteq R_{g_1}^{CI}$ and $\Phi(L_{x_0}^\varepsilon) \nsubseteq \mathrm{Inv}(\mathcal{A})\theta_{y,g}$, $\Phi(L_{x_1}^\varepsilon) \nsubseteq \mathrm{Inv}(\mathcal{A})\theta_{y,g}$ for any $\theta_{y,g}$. We say there exists an $e_0 \in \varepsilon$ such that $\Phi(\theta_{x_0,e_0}) = \theta_{y_0,g}$ where $g \neq \alpha g_1$ (If such an e_0 does not exist, then both $\Phi(L_{x_0}^\varepsilon) \subseteq L_{y_0}^{CI}$ and $\Phi(L_{x_0}^\varepsilon) \subseteq R_{g_1}^{CI}$). We can find an $e_1 \in \varepsilon$ such that $\Phi(\theta_{x_1,e_1}) = \theta_{z,g_1}$ where $z \neq \alpha y_0$ for all $\alpha \in \mathrm{Inv}(\mathcal{A})$. We put $\Phi(\theta_{x_0,e_1}) = \theta_{y_0,m}$ for some $m \in CI$. It follows that $m = \lambda g_1$ for some $\lambda \in \mathrm{Inv}(\mathcal{A})$ from $\Phi(\theta_{x_0+x_1,e_1}) = \theta_{z,g_1} + \theta_{y_0,m}$ and $z \neq \alpha y_0$ for any $\alpha \in \mathcal{A}$. And then $\Phi(\theta_{x_0,e_1}) = \theta_{y_0,\lambda g_1}$.

On the other hand $\Phi(\theta_{x_1,e_0}) = \theta_{y_1,g_1}$ for some $y_1 \in CI$. Then $\Phi(\theta_{x_0+x_1,e_0}) = \theta_{y_0,g} + \theta_{y_1,g_1}$. From $g \neq \alpha g_1$, we infer that $y_1 = \mu y_0$ for some $\mu \in \mathrm{Inv}(\mathcal{A})$ and $\Phi(\theta_{x_1,e_0}) = \theta_{\mu y_0,g_1}$.

Now we have

$$\Phi(\theta_{x_0+x_1,e_0+e_1}) = \theta_{y_0,g} + \theta_{y_0,\lambda g_1} + \theta_{\mu y_0,g_1} + \theta_{z,g_1} = \theta_{y_0,g} + \theta_{\lambda^* y_0+\mu y_0+z,g_1}$$

We say that $\theta_{y_0,g} + \theta_{\lambda^* y_0+\mu y_0+z,g_1}$ cannot be a rank one operator since $g_1 \neq \alpha g, z \neq \beta y_0$ for all $\alpha, \beta \in \mathrm{Inv}(\mathcal{A})$ and $e_1 + e_2 \in CI$ which contradicts Φ preserving rank one. Thus we get the desired results. □

Corollary 2.12 *At least one of the following is true*

(i) For all $f \in CI$ there exists $g \in CI$ such that $\Phi(R_f^\varepsilon) \subseteq R_g^{CI}$;

(ii) For all $f \in CI$ there exists $y \in CI$ with $\Phi(R_f^\varepsilon) \subseteq L_y^{CI}$.

Lemma 2.13 *(i) If for all $x \in CI$, there exists $y \in CI$ such that $\Phi(L_x^\varepsilon) \subseteq L_y^{CI}$, then for all $f \in CI$, there exists $g \in CI$ such that $\Phi(R_f^\varepsilon) \subseteq R_g^{CI}$;*

(ii) If for all $x \in CI$, there exists $g \in CI$ such that $\Phi(L_x^\varepsilon) \subseteq R_g^{CI}$, then for all $f \in CI$ there exists $z \in CI$ such that $\Phi(R_f^\varepsilon) \subseteq L_z^{CI}$.

Proof We only prove (i). If $\Phi(R_f^\varepsilon)\subseteq \mathrm{Inv}(\mathcal{A})\theta_{y,g}$ for some $y,g\in CI$, then both $\Phi(R_f^\varepsilon)\subseteq R_g^{CI}$ and $\Phi(R_f^\varepsilon)\subseteq L_y^{CI}$ hold.

In the following we suppose $\Phi(R_f^\varepsilon)\nsubseteq \mathrm{Inv}(A)\theta_{y,g}$ for any $y,g\in CI$. We assume, to reach a contradiction, that we have simultaneously $\Phi(L_x^\varepsilon)\subseteq L_y^{CI}$ and $\Phi(R_f^\varepsilon)\subseteq L_z^{CI}$ where $x,y,f,z\in CI$. We can find $e_1,e_2\in\varepsilon$ such that $\Phi(\theta_{x,e_1})=\theta_{y,g_1}$ and $\Phi(\theta_{x,e_2})=\theta_{y,g_2}$ where $g_1,g_2\in CI, g_1\neq\alpha g_2$ for all $\alpha\in\mathrm{Inv}(\mathcal{A})$. From Φ preserving rank one we can find $x_1\in CI$ with $\Phi(L_{x_1}^\varepsilon)\subseteq L_{y_1}^{CI}$ such that $y_1\neq\alpha y$ for all $\alpha\in\mathrm{Inv}(\mathcal{A})$. We put $\Phi(\theta_{x_1,e_1})=\theta_{y_1,u}$ for some $u\in CI$. Then we have $\Phi(\theta_{x+x_1,e_1})=\theta_{y,g_1}+\theta_{y_1,u}=\theta_{y_0,g_0}$ where $g_0\in CI$. From Corollary 2.6 and $y_1\neq\alpha_0 y$ we have $g_1=\lambda u$ for some $\lambda\in\mathrm{Inv}(\mathcal{A})$ and $\Phi(\theta_{x_1,e_1})=\theta_{y_1,\lambda^{-1}g_1}$.

For arbitrary $x'\in CI$, with $\Phi(\theta_{x',e_1})=\theta_{y',g'}$ such that $y',g'\in CI$. If $y'=\alpha' y$ for some $\alpha'\in\mathrm{Inv}(\mathcal{A})$, then $y'\neq\alpha y_1$ for all $\alpha\in\mathrm{Inv}(\mathcal{A})$. By considering $\Phi(\theta_{x'+x_1,e_1})$ we get $g'=\beta_1 g_1$ for some $\beta_1\in\mathrm{Inv}(\mathcal{A})$. If $y'\neq\alpha y$ for all $\alpha\in\mathrm{Inv}(\mathcal{A})$, then we consider $\Phi(\theta_{x'+x,e_1})$ and can get the same result. Anyway we have proved $\Phi(R_{e_1}^{CI})\subseteq R_{g_1}^{CI}$.

We assume $\Phi(R_{e_1}^\varepsilon)\subseteq L_z^{CI}$ for some $z\in CI$. For all $e\in\varepsilon$ with $\Phi(\theta_{e,e_1})=\theta_{z,g_0}$ such that $z,g_0\in CI$. As the argument in the above paragraph, we consider $\Phi(\theta_{e+x,e_1})$ or $\Phi(\theta_{e+x',e_1})$. Then we get $g_0=\alpha_0 g_1$ for some $\alpha_0\in\mathrm{Inv}(\mathcal{A})$. Therefore, $\Phi(R_{e_1}^\varepsilon)\subseteq\mathrm{Inv}(\mathcal{A})\theta_{z,g_1}$ which contradicts our assumption. Thus $\Phi(R_{e_1}^\varepsilon)\nsubseteq L_z^{CI}$ for any $z\in CI$ and $\Phi(R_{e_1}^\varepsilon)\subseteq R_g^{CI}$ for some $g\in CI$. From Lemma 2.11 we get that for all $y\in CI$, $\Phi(R_y^\varepsilon)\subseteq R_g^{CI}$ which contradicts $\Phi(R_f^\varepsilon)\subseteq L_z^{CI}$. $\Box$

Lemma 2.14 *If for $x\in CI$, $\Phi(L_x^\varepsilon)\subseteq L_y^{CI}$, then $\Phi(L_x)\subseteq L_y$.*

Proof For all $f\in H_{\mathcal{A}}$, $f=\sum_i\alpha_i e_i$ where $\alpha_i\in\mathcal{A}$, $e_i\in\varepsilon$, then

$$\begin{aligned}\Phi(\theta_{x,f})&=\Phi(\theta_{x,\sum\limits_i\alpha_i e_i})=\Phi(\sum_i\alpha_i^*\theta_{x,e_i})=\sum_i\tau_{x,e_i}(\alpha_i^*)\Phi(\theta_{x,e_i})\\&=\theta_{y,\sum\limits_i\tau_{x,e_i}(\alpha_i)g_i}.\end{aligned}$$

Therefore, $\Phi(L_x)\subseteq L_y$. $\Box$

Similarly we can prove

Lemma 2.15 *If $\Phi(L_x^\varepsilon)\subseteq R_f^{CI}$, for some $f\in CI$, then $\Phi(L_x)\subseteq R_f$.*

3. Main results

In this section, we characterize the rank one preserving quasi-modular maps on Hilbert C^*-modules. We find that their forms are very similar to those of rank one preserving maps on linear spaces. We also get that a rank one preserving locally quasi-modular map is always quasi-modular.

Theorem 3.1 *$\Phi:\mathcal{F}(H_{\mathcal{A}})\to\mathcal{F}(H_{\mathcal{A}})$ is a surjective preserving rank one locally quasi-modular map. Then one of the following is true: (i) For all $x,f\in H_{\mathcal{A}}$, $\Phi(\theta_{x,f})=\theta_{Ax,Cf}$ where A,C are injective quasi-modular maps on $H_{\mathcal{A}}$;*

(ii) For all $x,f\in H_{\mathcal{A}}$, $\Phi(\theta_{x,f})=\theta_{Cf,Ax}$ where A,C are injective conjugate quasi-modular maps on $H_{\mathcal{A}}$.

Proof We consider the case of $x \in CI$ first. In this case we have $\Phi(\theta_{x,f}) = \theta_{y,C_xf}$. For all $f \in H_{\mathcal{A}}$, from Lemma 2.14 we have the following claims.

Claim 1 C_x is a map. In fact, putting $f_1 = f_2$, we have $\Phi(\theta_{x,f_1}) = \theta_{y,C_xf_1}$ and $\Phi(\theta_{x,f_2}) = \theta_{y,C_xf_2}$. We infer $C_xf_1 = C_xf_2$ from $y \in CI$.

Claim 2 C_x is injective. Otherwise, there exists $f_0 \neq 0$ but $C_xf_0 = 0$. Then $\Phi(\theta_{x,f_0}) = \theta_{y,C_xf_0} = 0$ which contradicts Φ preserving rank one.

Claim 3 C_x is additive. For $f_1, f_2 \in H_{\mathcal{A}}$, $\Phi(\theta_{x,f_1+f_2}) = \theta_{y,C_x(f_1+f_2)}$. On the other hand $\Phi(\theta_{x,f_1+f_2}) = \Phi(\theta_{x,f_1}) + \Phi(\theta_{x,f_2}) = \theta_{y,C_xf_1} + \theta_{y,C_xf_2}$. It follows $C_x(f_1+f_2) = C_xf_1 + C_xf_2$ from $y \in CI$.

Claim 4 C_x is a locally quasi-modular map. In fact, for all $\lambda \in \mathcal{A}$, $\Phi(\lambda\theta_{x,f}) = \tau_{x,f}(\lambda)\Phi(\theta_{x,f}) = \tau_{x,f}(\lambda)\theta_{y,C_xf}$. At the same time $\Phi(\lambda\theta_{x,f}) = \Phi(\theta_{x,\lambda^*f}) = \theta_{y,C_x(\lambda^*f)}$. So we have $C_x(\lambda^*f) = \tau_{x,f}(\lambda^*)C_xf$.

Claim 5 $\tau_{x,f}$ is independent of f. Since $\tau_{x,f}$ is surjective, $C_x(H_{\mathcal{A}})$ is a submodule of $H_{\mathcal{A}}$. Without loss of generality, we suppose $\{C_x(h_i)\}$ is the orthonormal basis in $C_x(H_{\mathcal{A}})$. For a $\lambda \in \mathcal{A}$, we have $C_x(\lambda h_1 + \lambda h_2) = \tau_{x,h_1+h_2}(\lambda)(C_xh_1 + C_xh_2)$ and $C_x(\lambda h_1 + \lambda h_2) = \tau_{x,h_1}(\lambda)C_xh_1 + \tau_{x,h_2}(\lambda)C_x(h_2)$. Thus $[\tau_{x,h_1+h_2}(\lambda) - \tau_{x,h_1}(\lambda)]C_xh_1 + [\tau_{x,h_1+h_2}(\lambda) - \tau_{x,h_2}(\lambda)]C_xh_2 = 0$. It follows from $[\tau_{x,h_1+h_2}(\lambda) - \tau_{x,h_1}(\lambda)]C_xh_1 = [\tau_{x,h_1+h_2}(\lambda) - \tau_{x,h_2}(\lambda)]C_xh_2 = 0$ that $\tau_{x,h_1+h_2}(\lambda) = \tau_{x,h_1}(\lambda) = \tau_{x,h_2}(\lambda)$. Thus for every h_i, $\tau_{x,h_i} = \tau_{x,h_1+h_2}$.

For all $h \in H_{\mathcal{A}}$, $C_x(h) = \sum_i \alpha_i C_x(h_i)$. Since τ_{x,h_i} is surjective, there exists β_i such that $\alpha_i = \tau_{x,h_i}(\beta_i)$. Then $C_x(h) = \sum_i \alpha_i C_x(h_i) = \sum_i \tau_{x,h_i}(\beta_i)C_x(h_i) = \sum_i C_x(\beta_i h_i)$. We infer $h = \sum_i \beta_i h_i$ from the fact that C_x is injective.

Now for arbitrary $\lambda \in \mathcal{A}$, we have

$$\begin{aligned}\Phi(\lambda\theta_{x,h}) &= \theta_{y,C_x(\lambda^*h)} = \theta_{y,C_x(\lambda^*\sum\limits_i \beta_i h_i)} \\ &= \sum_i \theta_{y,C_x(\lambda^*\beta_i h_i)} = \sum_i \theta_{y,\tau_{x,h_1+h_2}(\lambda^*\beta_i)C_xh_i} \\ &= \sum_i \theta_{y,\tau_{x,h_1+h_2}(\lambda^*)\tau_{x,h_1+h_2}(\beta_i)C_xh_i} \\ &= \theta_{y,\tau_{x,h_1+h_2}(\lambda^*)C_x(h)} = \tau_{x,h_1+h_2}(\lambda)\Phi(\theta_{x,h}).\end{aligned}$$

Therefore, $\tau_{x,h}$ is independent of h and we denote it by τ_x.

Claim 6 τ_x is a injective homomorphism from $\mathcal{A}$ onto $\mathcal{A}$. We show τ_x is additive first. In fact, for all $\lambda_1, \lambda_2 \in \mathcal{A}$, $\Phi((\lambda_1+\lambda_2)\theta_{x,f}) = \tau_x(\lambda_1+\lambda_2)\Phi(\theta_{x,f}) = \tau_x(\lambda_1+\lambda_2)\theta_{y,C_xf}$. On the other hand, $\Phi((\lambda_1+\lambda_2)\theta_{x,f}) = \Phi(\lambda_1\theta_{x,f}) + \Phi(\lambda_2\theta_{x,f}) = [\tau_x(\lambda_1) + \tau_x(\lambda_2)]\theta_{y,C_xf}$. From $y \in CI$, we infer $\tau_x(\lambda_1+\lambda_2)C_xf = [\tau_x(\lambda_1) + \tau_x(\lambda_2)]C_xf$. When we choose $f \in \varepsilon$, from Lemma 2.10, we get $C_xf \in CI$. Then $\tau_x(\lambda_1) + \tau_x(\lambda_2) = \tau_x(\lambda_1+\lambda_2)$ and τ_x is additive.

Next we show τ_x is injective. Otherwise, there exists $0 \neq \lambda_0 \in \mathcal{A}$ but $\tau_x(\lambda_0) = 0$. So $\Phi(\lambda_0\theta_{x,f}) = \tau_x(\lambda_0)\theta_{y,C_xf} = 0$ which contradicts Φ preserving rank one.

Claim 7 τ_x is independent of x. We choose $x_1, x_2 \in CI$ with $\Phi(L^{\varepsilon}_{x_1}) \subseteq L^{CI}_{y_1}$, $\Phi(L^{\varepsilon}_{x_2}) \subseteq L^{CI}_{y_2}$ such that $\langle y_1, y_2\rangle = 0$. We will complete the proof by 5 steps.

Step 1. We consider $\Phi(\theta_{x_1+x_2,e}) = \theta_{y_1,C_{x_1}e} + \theta_{y_2,C_{x_2}e}$. From $\langle y_1, y_2\rangle = 0$, there exists $\sigma \in \mathrm{Inv}(\mathcal{A})$ such that $C_{x_1}e = \sigma C_{x_2}e$.

We choose $e' \in \varepsilon$ such that $C_{x_2}e' \neq \alpha C_{x_2}e$ for all $\alpha \in \mathrm{Inv}(\mathcal{A})$. Repeating the above process, we have $C_{x_1}e' = \nu C_{x_2}e'$ for some $\nu \in \mathrm{Inv}(\mathcal{A})$. Then

$$\begin{aligned}\Phi(\theta_{x_1+x_2,e+e'}) &= \theta_{y_1,C_{x_1}e} + \theta_{y_1,C_{x_1}e'} + \theta_{y_2,C_{x_2}e} + \theta_{y_2,C_{x_2}e'}\\ &= \theta_{y_1,\sigma C_{x_2}e} + \theta_{y_1,\nu C_{x_2}e'} + \theta_{y_2,C_{x_2}e} + \theta_{y_2,C_{x_2}e'}\\ &= \theta_{\sigma^* y_1+y_2,C_{x_2}e} + \theta_{\nu^* y_1+y_2,C_{x_2}e'}.\end{aligned}$$

Since $e + e' \in CI$, $C_{x_2}e \neq \alpha C_{x_2}e'$ for all $\alpha \in \mathrm{Inv}(\mathcal{A})$, there exists $\alpha_0 \in \mathrm{Inv}(\mathcal{A})$ such that $\sigma^* y_1 + y_2 = \alpha_0(\nu^* y_1 + y_2)$, i.e., $(\sigma^* - \alpha_0\nu^*)y_1 + (1-\alpha_0)y_2 = 0$. It follows $\sigma = \nu$ from $\langle y_1, y_2\rangle = 0$ and $y_1, y_2 \in CI$.

Step 2. We put $\lambda \in \mathrm{Inv}(\mathcal{A})$. Then $\Phi(\theta_{x_1+x_2,\lambda e'}) = \theta_{y_1,C_{x_1}(\lambda e')} + \theta_{y_2,C_{x_2}(\lambda e')}$. Since $\lambda e' \in CI$ and $y_1 \neq \alpha y_2$, there exists $a(\lambda) \in \mathrm{Inv}(\mathcal{A})$ such that $C_{x_1}(\lambda e') = a(\lambda) C_{x_2}(\lambda e')$.

$$\begin{aligned}\Phi(\theta_{x_1+x_2,e+\lambda e'}) &= \theta_{y_1,C_{x_1}e} + \theta_{y_1,C_{x_1}(\lambda e')} + \theta_{y_2,C_{x_2}e} + \theta_{y_2,C_{x_2}(\lambda e')}\\ &= \theta_{y_1,\sigma C_{x_2}e} + \theta_{y_1,a(\lambda)C_{x_2}(\lambda e')} + \theta_{y_2,C_{x_2}e} + \theta_{y_2,C_{x_2}(\lambda e')}\\ &= \theta_{\sigma^* y_1+y_2,C_{x_2}e} + \theta_{a(\lambda)^* y_1+y_2,C_{x_2}(\lambda e')}\end{aligned}$$

Since $C_{x_2}(e') \neq \alpha C_{x_2}e$, for all $\alpha \in \mathrm{Inv}(\mathcal{A})$ and $e + \lambda e' \in CI$ we know there exists $c(\lambda) \in \mathrm{Inv}(\mathcal{A})$ such that $\sigma^* y_1 + y_2 = c(\lambda)(a(\lambda)^* y_1 + y_2)$, i.e., $[\sigma^* - c(\lambda)a(\lambda)^*]y_1 + [1 - c(\lambda)]y_2 = 0$. Then it follows $\sigma^* = c(\lambda)a(\lambda)^*$, $c(\lambda) = 1$ and $a(\lambda) = \sigma$. Thus we have $C_{x_1}(\lambda e') = \sigma C_{x_2}(\lambda e')$.

Step 3. When λ is not in $\mathrm{Inv}(\mathcal{A})$, we may suppose $\|\lambda\| < 1$ (If not, we can consider $\frac{\lambda}{2\|\lambda\|}$).

We have

$$\begin{aligned}\Phi(\theta_{x_1+x_2,\lambda e'}) &= \theta_{y_1,C_{x_1}(\lambda e')} + \theta_{y_2,C_{x_2}(\lambda e')} = \theta_{\tau_{x_1}(\lambda^*)y_1,C_{x_1}e'} + \theta_{\tau_{x_2}(\lambda^*)y_2,C_{x_2}e'}\\ &= \theta_{\sigma^*\tau_{x_1}(\lambda^*)y_1+\tau_{x_2}(\lambda^*)y_2,C_{x_2}e'} := \theta_{y_3,g_3}\end{aligned}$$

Note that $y_1, y_2 \in CI$ but $C_{x_1}(\lambda e'), C_{x_2}(\lambda e')$ are not in CI. We get for some $a(\lambda), b(\lambda), a(1-\lambda), b(1-\lambda) \in \mathcal{A}$,

$$\begin{cases} C_{x_1}(\lambda e') = a(\lambda)g_3,\\ C_{x_2}(\lambda e') = b(\lambda)g_3,\end{cases}$$

and

$$\begin{cases} C_{x_1}[(1-\lambda)e'] = a(1-\lambda)g_3,\\ C_{x_2}[(1-\lambda)e'] = b(1-\lambda)g_3.\end{cases}$$

On the one hand, $C_{x_1}[(1-\lambda)e'] = C_{x_1}(e' - \lambda e') = \sigma C_{x_2}e' - a(\lambda)g_3 = [\sigma - a(\lambda)]g_3$. On the other hand, since $(1-\lambda) \in \mathrm{Inv}(\mathcal{A})$, $C_{x_1}[(1-\lambda)e'] = \sigma C_{x_2}[(1-\lambda)e'] = \sigma[1 - b(\lambda)]g_3$. From $g_3 \in CI$ we have $\sigma - a(\lambda) = \sigma(1 - b(\lambda))$ i.e., $a(\lambda) = \sigma b(\lambda)$. Consequently, $C_{x_1}(\lambda e') = a(\lambda)g_3 = \sigma b(\lambda)g_3 = \sigma C_{x_2}(\lambda e')$.

Step 4. For all $\lambda \in \mathcal{A}$, $C_{x_1}(\lambda e') = \tau_{x_1}(\lambda)C_{x_1}e' = \tau_{x_1}(\lambda)\sigma C_{x_2}e'$. At the same time $C_{x_1}(\lambda e') = \sigma C_{x_2}(\lambda e') = \sigma\tau_{x_2}(\lambda)C_{x_2}e'$. We can get $\tau_{x_1}(\lambda) = \tau_{x_2}(\lambda)$ since $\sigma \in \mathrm{Inv}(\mathcal{A}), C_{x_2}e' \in CI$.

Step 5. For every $x \in CI$, $\Phi(L_x^\varepsilon) \subseteq L_y^{CI}$ for some $y \in CI$. Then $y = \sum_i \beta_i y_i$, where $\{y_i\}$ is the orthonormal basis. Since Φ is surjective, there exists $\{x_i\} \subseteq CI$ such that $\Phi(L_{x_i}^\varepsilon) \subseteq L_{y_i}^{CI}$. Now we investigate

$$\begin{aligned}\Phi(\theta_{x_1,e} + \theta_{x,e}) &= \theta_{y_1,C_{x_1}e} + \theta_{y,C_xe} = \theta_{y_1,C_{x_1}e} + \theta_{\sum\limits_{i=1}^{\infty}\beta_i y_i,C_xe}\\ &= \theta_{y_1,C_{x_1}e} + \theta_{\beta_1 y_1,C_xe} + \theta_{\sum\limits_{i=2}^{\infty}\beta_i y_i,C_xe}\\ &= \theta_{y_1,C_{x_1}e+\beta_1^* C_xe} + \theta_{\sum\limits_{i=2}^{\infty}\beta_i y_i,C_xe}\end{aligned}$$

and we have $C_{x_1}e + \beta_1^* C_xe = \beta C_xe$ for some $\beta \in \mathcal{A}$ i.e., $C_{x_1}e = (\beta - \beta_1^*)C_xe$. Since $C_{x_1}e, C_xe \in CI$, we get $\beta - \beta_1^* \in \mathrm{Inv}(\mathcal{A})$ and denote $\beta - \beta_1^*$ by γ_1.

Similarly, we have $C_{x_i}e = \gamma_i C_xe$ for some $\gamma_i \in \mathrm{Inv}(\mathcal{A})$. Suppose $\tau_{x_i}(\alpha_i) = \beta_i$, $\tau_{x_i}(\delta_i) = (\gamma_i^{-1})^*$ for some $\alpha_i, \delta_i \in \mathcal{A}$ and

$$\begin{aligned}\Phi(\theta_{\sum\limits_i \delta_i\alpha_i x_i,e}) &= \sum_i \Phi(\theta_{\delta_i\alpha_i x_i,e}) = \sum_i \tau_{x_i}(\delta_i\alpha_i)\Phi(\theta_{x_i,e}) = \sum_i (\gamma_i^{-1})^*\beta_i\theta_{y_i,C_{x_i}e}\\ &= \sum_i (\gamma_i^{-1})^*\beta_i\theta_{y_i,\gamma_i C_xe} = \sum_i \theta_{\beta_i y_i,C_xe} = \theta_{y,C_xe} = \Phi(\theta_{x,e}).\end{aligned}$$

Then $\Phi(\theta_{\sum_i \delta_i\alpha_i x_i - x,e}) = 0$. We infer that $x = \sum_i \delta_i\alpha_i x_i$.

For all $\lambda \in \mathcal{A}$, $e \in \varepsilon$ and from the above step, $\tau_{x_1} = \tau_{x_2} = \cdots := \tau$, we have

$$\begin{aligned}\Phi(\lambda\theta_{x,e}) &= \Phi(\lambda\theta_{\sum\limits_i \delta_i\alpha_i x_i,e}) = \sum_i \Phi(\lambda\delta_i\alpha_i\theta_{x_i,e})\\ &= \sum_i \tau_{x_i}(\lambda\delta_i\alpha_i)\Phi(\theta_{x_i,e}) = \sum_i \tau(\lambda)\tau(\delta_i\alpha_i)\Phi(\theta_{x_i,e})\\ &= \sum_i \tau(\lambda)\Phi(\theta_{\delta_i\alpha_i x_i,e}) = \tau(\lambda)\Phi(\theta_{x,e}).\end{aligned}$$

Thus $\tau_x = \tau$ is independent of x.

Claim 8 C_x is independent of $x \in CI$. For $x_1, x_2 \in CI$ with $\Phi(L_{x_1}^\varepsilon) \subseteq L_{y_1}^{CI}$, $\Phi(L_{x_2}^\varepsilon) \subseteq L_{y_2}^{CI}$ such that $y_1 \neq \alpha y_2$ for all $\alpha \in \mathrm{Inv}(\mathcal{A})$.

$$\Phi(\theta_{x_1+x_2,e}) = \Phi(\theta_{x_1,e}) + \Phi(\theta_{x_2,e}) = \theta_{y_1,C_{x_1}e} + \theta_{y_2,C_{x_2}e}$$

which yields $C_{x_1}e = \alpha_e C_{x_2}e$ for all $e \in \varepsilon$. From Lemma 2.9, we can see α_e is independent of the choice of $e \in \varepsilon$. Then there exists $\alpha_0 \in \mathrm{Inv}(\mathcal{A})$ such that $C_{x_1} = \alpha_0 C_{x_2} := C$.

For arbitrary $x \in CI$ with $\Phi(L_x^\varepsilon) \subseteq L_y^{CI}$ for some $y \in CI$, then $y \neq \alpha y_1$ or $y \neq \beta y_2$, for all $\alpha, \beta \in \mathrm{Inv}(\mathcal{A})$. So $C_x = \alpha_x C_{x_1}$ (or $C_x = \beta_x C_{x_2}$) for some $\alpha_x \in \mathrm{Inv}(\mathcal{A})$.

Then we have

$$\Phi(\theta_{x,e}) = \theta_{y,C_xe} = \theta_{y,\alpha_x C_{x_1}e}$$

and

$$\Phi(\theta_{x,f}) = \theta_{y,C_xf} = \theta_{y,\alpha_x C_{x_1}f} = \theta_{\alpha_x^* y,Cf}.$$

Denote $\alpha_x^* y$ by $A'x$ and get

$$\Phi(\theta_{x,f}) = \theta_{A'x,Cf}$$

for all $x \in CI$, $f \in H_{\mathcal{A}}$.

Now for arbitrary $x \in H_{\mathcal{A}}$ (may not in CI), $x = \sum_i \alpha_i x_i$ where $x_i \in CI$ with $\Phi(\theta_{x_i,f}) = \theta_{A'x_i,Cf}$. Then

$$\Phi(\theta_{x,f}) = \Phi(\theta_{\sum\limits_i \alpha_i x_i,f}) = \sum_i \tau(\alpha_i)\theta_{A'x_i,Cf} = \theta_{\sum\limits_i \tau(\alpha_i)A'x_i,Cf}.$$

We denote $\sum_i \tau(\alpha_i)A'x_i$ by Ax. Then for all $x, f \in H_{\mathcal{A}}$, we always have $\Phi(\theta_{x,f}) = \theta_{Ax,Cf}$. Especially, we choose $f \in \varepsilon$, and then one can see A is an injective quasi-modular map.

The statement (ii) can be shown by the similar methods and its proof is omitted here. $\square$

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