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Generalized Jordan Derivations Associate with Hochschild 2-Cocycles on Triangular Matrices

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Abstract In this paper, we prove that every generalized Jordan derivation associate with a Hochschild 2-cocycle from the algebra of upper triangular matrices to its bimodule is the sum of a generalized derivation and an antiderivation.

Keywords generalized Jordan derivation; generalized derivation; Hochschild 2-cocycle.

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1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ be an algebra over $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Let $\alpha : \mathcal{A}\times\mathcal{A}\to\mathcal{M}$ be an $\mathcal{R}$-bilinear map, that is, an $\mathcal{R}$-linear map on each component. α is called a Hochschild 2-cocycle if

$$a\alpha(b,c)-\alpha(ab,c)+\alpha(a,bc)-\alpha(a,b)c=0 \tag{1}$$

for all $a,b,c\in\mathcal{A}$. Let Δ, φ and δ be $\mathcal{R}$-linear maps from $\mathcal{A}$ to $\mathcal{M}$. φ is called a generalized derivation if there exists a Hochschild 2-cocycle α such that

$$\varphi(ab)=\varphi(a)b+a\varphi(b)+\alpha(a,b) \tag{2}$$

for all $a,b\in\mathcal{A}$, and Δ is called a generalized Jordan derivation if

$$\Delta(a^2)=\Delta(a)a+a\Delta(a)+\alpha(a,a) \tag{3}$$

for all $a\in\mathcal{A}$. We denote them by (φ,α) and (Δ,α), respectively. Moreover, δ is called an antiderivation if

$$\delta(ab)=\delta(b)a+b\delta(a) \tag{4}$$

for all $a,b\in\mathcal{A}$.

Different types of generalized derivations as well as generalized Jordan derivations have been introduced by several authors. For instance, Brešar in [1] defined one kind of generalized derivations in the sense that if φ is an $\mathcal{R}$-linear map from $\mathcal{A}$ to $\mathcal{M}$ and if there exists a derivation d from $\mathcal{A}$ to $\mathcal{M}$ such that $\varphi(ab)=\varphi(a)b+ad(b)$ for all $a,b\in\mathcal{A}$, then φ is a generalized derivation;

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another kind of generalized derivations was introduced by Nakajima [2] as follows. An $\mathcal{R}$-linear map φ from $\mathcal{A}$ to $\mathcal{M}$ is called a generalized derivation if there exists an element $\omega \in \mathcal{M}$ such that $\varphi(ab) = \varphi(a)b + a\varphi(b) + a\omega b$ for all $a, b \in \mathcal{A}$. In the case that $\mathcal{A}$ has an identity element I, this is equivalent to that φ satisfies $\varphi(ab) = \varphi(a)b + a\varphi(b) - a\varphi(I)b$ for all $a, b \in \mathcal{A}$. Nakajima in [3] defined the type of generalized derivations associate with Hochschild 2-cocycles and pointed out that this type includes not only the generalized derivations mentioned above, but left multipliers and (σ, τ)-derivations as well. For more details we refer the reader to [3–6] and references therein.

Throughout this paper, by $\mathcal{M}_n(\mathcal{R})$, $n \geq 2$, we denote the algebra of all $n \times n$ matrices over $\mathcal{R}$, by $\mathcal{T}_n(\mathcal{R})$ its subalgebra of all upper triangular matrices, and by $\mathcal{D}_n(\mathcal{R})$ its subalgebra of all diagonal matrices. Benkovič in [7] showed that every Jordan derivation from $\mathcal{T}_n(\mathcal{R})$ to its bimodule is the sum of a derivation and an antiderivation. Ji and Ma [8] extended this result to generalized Jordan derivations in the usual sense, i.e., if Δ is a generalized Jordan derivation from $\mathcal{T}_n(\mathcal{R})$ to its bimodule in the sense

$$\Delta(a^2) = \Delta(a)a + a\Delta(a) - a\Delta(I)a$$

for all $a \in \mathcal{T}_n(\mathcal{R})$, then Δ is the sum of a generalized derivation φ and an antiderivation δ, where φ is such that

$$\varphi(ab) = \varphi(a)b + a\varphi(b) - a\varphi(I)b$$

for all $a, b \in \mathcal{T}_n(\mathcal{R})$. In this note we generalize the result above to show that every generalized Jordan derivation (Δ, α) associate with Hochschild 2-cocycle α from $\mathcal{T}_n(\mathcal{R})$ to its bimodule is the sum of a generalized derivation (φ, α) and an antiderivation. We shall assume, without further mention, that all algebras and all modules considered in this paper is 2-torsionfree.

2. Proof of the main result

Let $\mathcal{A}$ be an algebra with identity over $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. As usual, we regard $\mathcal{M}$ as an $\mathcal{R}$-bimodule by actions $rm = mr = (rI)m = m(rI)$ for all $r \in \mathcal{R}$ and $m \in \mathcal{M}$, where I is the identity of $\mathcal{A}$. We begin with the following lemma which is a modification of Lemma 2 in [3]. For the sake of completeness, we present the proof here.

Lemma 2.1 *Let Δ be an $\mathcal{R}$-linear map from $\mathcal{A}$ to its bimodule $\mathcal{M}$ and α: $\mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a Hochschild 2-cocycle. Then the following relations are equivalent:*

1) (Δ, α) is a generalized Jordan derivation.

2) For all $a, b \in \mathcal{A}$, we have

$$\Delta(ab + ba) = \Delta(a)b + a\Delta(b) + \Delta(b)a + b\Delta(a) + \alpha(a, b) + \alpha(b, a). \tag{5}$$

3) For all $a, b \in \mathcal{A}$, we have

$$\Delta(aba) = \Delta(a)ba + a\Delta(b)a + ab\Delta(a) + a\alpha(b, a) + \alpha(a, ba). \tag{6}$$

Proof 1)$\Longrightarrow$2). Since $\Delta(a^2) = \Delta(a)a + a\Delta(a) + \alpha(a, a)$ for all $a \in \mathcal{A}$, replacing a by $a + b$ gives (5).

2)$\Longrightarrow$3). Substituting $ab+ba$ for b in (5) and using the 2-cocycle condition, we have

$$\begin{aligned}2\Delta(aba) =& \Delta(a(ab+ba)+(ab+ba)a)-\Delta(a^2b+ba^2)\\ =& 2[\Delta(a)ba+a\Delta(b)a+ab\Delta(a)]+\\ & a[\alpha(a,b)+\alpha(b,a)]+\alpha(a,ab)+\alpha(a,ba)+\\ & [\alpha(a,b)+\alpha(b,a)]a+\alpha(ab,a)+\alpha(ba,a)-\\ & [\alpha(a,a)b+\alpha(a^2,b)+b\alpha(a,a)+\alpha(b,a^2)]\\ =& 2[\Delta(a)ba+a\Delta(b)a+ab\Delta(a)]+\\ & [a\alpha(a,b)-\alpha(a^2,b)+\alpha(a,ab)-\alpha(a,a)b]-\\ & [b\alpha(a,a)-\alpha(ba,a)+\alpha(b,a^2)-\alpha(b,a)a]+\\ & a\alpha(b,a)+\alpha(a,ba)+\alpha(a,b)a+\alpha(ab,a)\\ =& 2[\Delta(a)ba+a\Delta(b)a+ab\Delta(a)]+\\ & a\alpha(b,a)+\alpha(a,ba)+\alpha(a,b)a+\alpha(ab,a).\end{aligned}$$

Since $a\alpha(b,a)+\alpha(a,ba)=\alpha(ab,a)+\alpha(a,b)a$ and $\mathcal{M}$ is 2-torsionfree, we have the relation (6).

3)$\Longrightarrow$1). Taking $a=b=I$ in (6) yields $\Delta(I)=-\alpha(I,I)$. Then putting $b=I$ in (6) gives (1), since $a\Delta(I)a=-a\alpha(I,I)a=-a\alpha(I,a)$. This completes the proof. $\Box$

For any idempotent $e\in\mathcal{A}$, (3) gives

$$\Delta(e)=\Delta(e)e+e\Delta(e)+\alpha(e,e). \tag{7}$$

For any $a\in\mathcal{A}$ satisfying $ae=ea=0$, Lemma 2.1 implies $0=\Delta(ae+ea)=\Delta(a)e+a\Delta(e)+\Delta(e)a+e\Delta(a)+\alpha(a,e)+\alpha(e,a)$. Multiplying e from the right yields

$$\Delta(a)e+a\Delta(e)e+e\Delta(a)e+\alpha(a,e)e+\alpha(e,a)e=0. \tag{8}$$

By the fact $0=\Delta(eae)=e\Delta(a)e+e\alpha(a,e)=e\Delta(a)e+\alpha(e,a)e$, (8) becomes $\Delta(a)e+a\Delta(e)e+\alpha(a,e)e=0$. Notice that $a\Delta(e)=a[\Delta(e)e+e\Delta(e)+\alpha(e,e)]=a\Delta(e)e+a\alpha(e,e)=a\Delta(e)e-\alpha(a,e)+\alpha(a,e)e$, and hence we obtain

$$\Delta(a)e+a\Delta(e)+\alpha(a,e)=0=\Delta(e)a+e\Delta(a)+\alpha(e,a) \tag{9}$$

for any idempotent $e,a\in\mathcal{A}$ such that $ae=ea=0$.

Now we assume that (Δ,α) is a generalized Jordan derivation from $\mathcal{T}_n(\mathcal{R})$ to its bimodule $\mathcal{M}$. Let e_{ij} be the element in $\mathcal{M}_n(\mathcal{R})$ with entries I at the position i,j and 0 otherwise for any $1\le i,j\le n$. By (7) we have

$$\Delta(e_{ii})=\Delta(e_{ii})e_{ii}+e_{ii}\Delta(e_{ii})+\alpha(e_{ii},e_{ii}) \tag{10}$$

and

$$e_{ki}\Delta(e_{ii})e_{ij}=-e_{ki}\alpha(e_{ii},e_{ii})e_{ij} \tag{11}$$

for all i and $k\le i\le j$. From (5) we obtain that

$$\begin{aligned}\Delta(e_{ij}) &= \Delta(e_{ii}e_{ij}+e_{ij}e_{ii})\\ &= \Delta(e_{ii})e_{ij}+e_{ii}\Delta(e_{ij})+\Delta(e_{ij})e_{ii}+e_{ij}\Delta(e_{ii})+\alpha(e_{ii},e_{ij})+\alpha(e_{ij},e_{ii})\end{aligned} \tag{12}$$

whenever $1 \le i < j \le n$. Furthermore, (9) gives that

$$\Delta(e_{kj})e_{ii} + e_{kj}\Delta(e_{ii}) + \alpha(e_{kj}, e_{ii}) = 0 = \Delta(e_{ii})e_{kj} + e_{ii}\Delta(e_{kj}) + \alpha(e_{ii}, e_{kj}) \tag{13}$$

for all $k, j \neq i$.

Now define an $\mathcal{R}$-linear map φ from $\mathcal{T}_n(\mathcal{R})$ to $\mathcal{M}$ by

$$\varphi(e_{ij}) = \Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \alpha(e_{ii}, e_{ij}), \quad 1 \le i \le j \le n. \tag{14}$$

According to (10), we have $\varphi(e_{ii}) = \Delta(e_{ii})$ for all $1 \le i \le n$.

Lemma 2.2 *φ is a generalized derivation associate with Hochschild 2-cocycle α.*

Proof It is enough to check that

$$\varphi(e_{ij}e_{kl}) = \varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \alpha(e_{ij}, e_{kl}) \tag{15}$$

for all $i \le j$ and $k \le l$. We consider two cases.

Case 1 $j \neq k$. Our goal is to show that $\varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \alpha(e_{ij}, e_{kl}) = 0$, for $\varphi(e_{ij}e_{kl}) = 0$. By (14) we have

$$\begin{aligned}
&\varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \alpha(e_{ij}, e_{kl}) \\
&\quad = [\Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \alpha(e_{ii}, e_{ij})]e_{kl} + \\
&\qquad e_{ij}[\Delta(e_{kk})e_{kl} + e_{kk}\Delta(e_{kl}) + \alpha(e_{kk}, e_{kl})] + \alpha(e_{ij}, e_{kl}) \\
&\quad = e_{ii}\Delta(e_{ij})e_{kl} + \alpha(e_{ii}, e_{ij})e_{kl} + e_{ij}\Delta(e_{kk})e_{kl} + e_{ij}\alpha(e_{kk}, e_{kl}) + \alpha(e_{ij}, e_{kl}).
\end{aligned} \tag{16}$$

Since $e_{ij}\alpha(e_{kk}, e_{kl}) = -\alpha(e_{ij}, e_{kl}) + \alpha(e_{ij}, e_{kk})e_{kl} = -\alpha(e_{ij}, e_{kl}) + [e_{ii}\alpha(e_{ij}, e_{kk}) - \alpha(e_{ii}, e_{ij})e_{kk}]e_{kl}$, (16) becomes

$$e_{ii}\Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kk})e_{kl} + e_{ii}\alpha(e_{ij}, e_{kk})e_{kl}. \tag{17}$$

If $i \neq k$, then (13) implies

$$e_{ii}[\Delta(e_{ij})e_{kk} + e_{ij}\Delta(e_{kk}) + \alpha(e_{ij}, e_{kk})]e_{kl} = 0.$$

If $i = k$, then (12) gives

$$\begin{aligned}
&e_{ii}\Delta(e_{ij})e_{il} + e_{ij}\Delta(e_{ii})e_{il} + e_{ii}\alpha(e_{ij}, e_{ii})e_{il} \\
&\quad = e_{ii}\Delta(e_{ij})e_{il} + e_{ij}\Delta(e_{ii})e_{il} + [\alpha(e_{ij}, e_{ii}) + \alpha(e_{ii}, e_{ij})e_{ii}]e_{il} \\
&\quad = [e_{ii}\Delta(e_{ij}) + e_{ij}\Delta(e_{ii}) + \alpha(e_{ii}, e_{ij})]e_{il} + \alpha(e_{ij}, e_{ii})e_{il} \\
&\quad = [\Delta(e_{ij}) - \Delta(e_{ii})e_{ij} - \Delta(e_{ij})e_{ii} - \alpha(e_{ij}, e_{ii})]e_{il} + \alpha(e_{ij}, e_{ii})e_{il} \\
&\quad = -\alpha(e_{ij}, e_{ii})e_{il} + \alpha(e_{ij}, e_{ii})e_{il} = 0.
\end{aligned}$$

Hence (15) holds true in the first case.

Case 2 $j = k$. Now we have to show that

$$\varphi(e_{il}) = \varphi(e_{ij})e_{jl} + e_{ij}\varphi(e_{jl}) + \alpha(e_{ij}, e_{jl}).$$

Assume $i < j < l$. Then (14) gives us

$$\varphi(e_{ij})e_{jl} + e_{ij}\varphi(e_{jl}) + \alpha(e_{ij}, e_{jl})$$

$$
\begin{aligned}
&= [\Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \alpha(e_{ii}, e_{ij})]e_{jl} + \\
&\quad e_{ij}[\Delta(e_{jj})e_{jl} + e_{jj}\Delta(e_{jl}) + \alpha(e_{jj}, e_{jl})] + \alpha(e_{ij}, e_{jl}) \\
&= \Delta(e_{ii})e_{il} + e_{ii}\Delta(e_{ij})e_{jl} + \alpha(e_{ii}, e_{ij})e_{jl} + \\
&\quad e_{ij}\Delta(e_{jj})e_{jl} + e_{ij}\Delta(e_{jl}) + e_{ij}\alpha(e_{jj}, e_{jl}) + \alpha(e_{ij}, e_{jl}). \qquad (18)
\end{aligned}
$$

From (11), we have $e_{ij}\Delta(e_{jj})e_{jl} = -e_{ij}\alpha(e_{jj}, e_{jj})e_{jl} = -e_{ij}\alpha(e_{jj}, e_{jl})$, and since $\alpha(e_{ii}, e_{ij})e_{jl} = e_{ii}\alpha(e_{ij}, e_{jl}) - \alpha(e_{ij}, e_{jl}) + \alpha(e_{ii}, e_{il})$, (18) becomes

$$
\begin{aligned}
&\Delta(e_{ii})e_{il} + \alpha(e_{ii}, e_{il}) + e_{ii}\Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}) + e_{ii}\alpha(e_{ij}, e_{jl}) \\
&\quad = \varphi(e_{il}) - e_{ii}\Delta(e_{il}) + e_{ii}\Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}) + e_{ii}\alpha(e_{ij}, e_{jl}) \\
&\quad = \varphi(e_{il}) - e_{ii}[\Delta(e_{il}) - \Delta(e_{ij})e_{jl} - e_{ij}\Delta(e_{jl}) - \alpha(e_{ij}, e_{jl})] \\
&\quad = \varphi(e_{il}) - e_{ii}[\Delta(e_{jl})e_{ij} + e_{jl}\Delta(e_{ij}) + \alpha(e_{jl}, e_{ij})] \\
&\quad = \varphi(e_{il}) - e_{ii}\Delta(e_{jl})e_{ij} - e_{ii}\alpha(e_{jl}, e_{ij}), \qquad (19)
\end{aligned}
$$

where the third equality results from the fact that $\Delta(e_{il}) = \Delta(e_{ij}e_{jl} + e_{jl}e_{ij}) = \Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}) + \Delta(e_{jl})e_{ij} + e_{jl}\Delta(e_{ij}) + \alpha(e_{ij}, e_{jl}) + \alpha(e_{jl}, e_{ij})$. By (13) we have $\Delta(e_{ii})e_{jl} + e_{ii}\Delta(e_{jl}) + \alpha(e_{ii}, e_{jl}) = 0$. Multiplying e_{ij} from the right yields $e_{ii}\Delta(e_{jl})e_{ij} = -\alpha(e_{ii}, e_{jl})e_{ij} = -e_{ii}\alpha(e_{jl}, e_{ij})$, whence (19) equals $\varphi(e_{il})$.
Next we assume that $i = j < l$, then $\varphi(e_{ii}e_{il}) = \varphi(e_{il})$ and

$$
\begin{aligned}
&\varphi(e_{ii})e_{il} + e_{ii}\varphi(e_{il}) + \alpha(e_{ii}, e_{il}) \\
&\quad = \Delta(e_{ii})e_{il} + e_{ii}[\Delta(e_{ii})e_{il} + e_{ii}\Delta(e_{il}) + \alpha(e_{ii}, e_{il})] + \alpha(e_{ii}, e_{il}) \\
&\quad = \Delta(e_{ii})e_{il} + e_{ii}\Delta(e_{il}) + \alpha(e_{ii}, e_{il}) + e_{ii}\Delta(e_{ii})e_{il} + e_{ii}\alpha(e_{ii}, e_{il}) \\
&\quad = \varphi(e_{il}) + e_{ii}\Delta(e_{ii})e_{il} + e_{ii}\alpha(e_{ii}, e_{il}) = \varphi(e_{il}),
\end{aligned}
$$

since $e_{ii}\Delta(e_{ii})e_{il} = -e_{ii}\alpha(e_{ii}, e_{ii})e_{il} = -e_{ii}\alpha(e_{ii}, e_{il})$. Now, let $i < j = l$. We have $\varphi(e_{ij}e_{jj}) = \varphi(e_{ij})$ and

$$
\begin{aligned}
&\varphi(e_{ij})e_{jj} + e_{ij}\varphi(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
&\quad = [\Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \alpha(e_{ii}, e_{ij})]e_{jj} + e_{ij}\Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
&\quad = \Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij})e_{jj} + \alpha(e_{ii}, e_{ij})e_{jj} + e_{ij}\Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
&\quad = \Delta(e_{ii})e_{ij} + \alpha(e_{ii}, e_{ij}) + e_{ii}\Delta(e_{ij})e_{jj} + e_{ij}\Delta(e_{jj}) + e_{ii}\alpha(e_{ij}, e_{jj}), \qquad (20)
\end{aligned}
$$

where the last equality is due to $\alpha(e_{ii}, e_{ij})e_{jj} = e_{ii}\alpha(e_{ij}, e_{jj}) - \alpha(e_{ij}, e_{jj}) + \alpha(e_{ii}, e_{ij})$. According to (14), (20) equals

$$
\begin{aligned}
&\varphi(e_{ij}) - e_{ii}\Delta(e_{ij}) + e_{ii}\Delta(e_{ij})e_{jj} + e_{ij}\Delta(e_{jj}) + e_{ii}\alpha(e_{ij}, e_{jj}) \\
&\quad = \varphi(e_{ij}) - e_{ii}[\Delta(e_{ij}) - \Delta(e_{ij})e_{jj} - e_{ij}\Delta(e_{jj}) - \alpha(e_{ij}, e_{jj})] \\
&\quad = \varphi(e_{ij}) - e_{ii}[\Delta(e_{jj})e_{ij} + e_{jj}\Delta(e_{ij}) + \alpha(e_{jj}, e_{ij})] \\
&\quad = \varphi(e_{ij}) - e_{ii}\Delta(e_{jj})e_{ij} - e_{ii}\alpha(e_{jj}, e_{ij}). \qquad (21)
\end{aligned}
$$

From (13), we have $\Delta(e_{ii})e_{jj} + e_{ii}\Delta(e_{jj}) + \alpha(e_{ii}, e_{jj}) = 0$. Multiplying e_{ij} from the right gives $e_{ii}\Delta(e_{jj})e_{ij} = -\alpha(e_{ii}, e_{jj})e_{ij} = -e_{ii}\alpha(e_{jj}, e_{ij})$, whence (21) equals $\varphi(e_{ij})$. Finally, if $i = j = l$, then (15) follows from (10) and $\varphi(e_{ii}) = \Delta(e_{ii})$. Therefore, (15) holds true in every case and the proof is completed. $\square$

Now set $\delta = \Delta - \varphi$. Then $\delta(e_{ij}) = \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii}) + \alpha(e_{ij}, e_{ii})$ for all $1 \le i < j \le n$ and $\delta(\mathcal{D}_n(\mathcal{R})) = 0$. Since

$$\begin{aligned}
&\delta(e_{ij}e_{kl} + e_{kl}e_{ij})\\
&\quad = \Delta(e_{ij}e_{kl} + e_{kl}e_{ij}) - \varphi(e_{ij}e_{kl} + e_{kl}e_{ij})\\
&\quad = \Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kl}) + \Delta(e_{kl})e_{ij} + e_{kl}\Delta(e_{ij}) + \alpha(e_{ij}, e_{kl}) + \alpha(e_{kl}, e_{ij}) -\\
&\quad\ \ [\varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \varphi(e_{kl})e_{ij} + e_{kl}\varphi(e_{ij}) + \alpha(e_{ij}, e_{kl}) + \alpha(e_{kl}, e_{ij})]\\
&\quad = \delta(e_{ij})e_{kl} + e_{ij}\delta(e_{kl}) + \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}),
\end{aligned}$$

we have that δ is a Jordan derivation. Moreover, we have

Lemma 2.3 *δ is an antiderivation.*

Proof Since $\delta = \Delta - \varphi$, φ is a generalized derivation and $\delta(\mathcal{D}_n(\mathcal{R})) = 0$, it follows that

$$\begin{aligned}
\delta(e_{ij}) &= \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii}) + \alpha(e_{ij}, e_{ii})\\
&= [\delta(e_{ij}) + \varphi(e_{ij})]e_{ii} + e_{ij}[\delta(e_{ii}) + \varphi(e_{ii})] + \alpha(e_{ij}, e_{ii})\\
&= \delta(e_{ij})e_{ii} + \varphi(e_{ij})e_{ii} + e_{ij}\varphi(e_{ii}) + \alpha(e_{ij}, e_{ii})\\
&= \delta(e_{ij})e_{ii} + \varphi(e_{ij}e_{ii}) = \delta(e_{ij})e_{ii}
\end{aligned}$$

if $i < j$. Note that δ is a Jordan derivation, we then have

$$\begin{aligned}
\delta(e_{ij}) &= \delta(e_{ij}e_{jj} + e_{jj}e_{ij}) = \delta(e_{ij})e_{jj} + e_{ij}\delta(e_{jj}) + \delta(e_{jj})e_{ij} + e_{jj}\delta(e_{ij})\\
&= e_{jj}\delta(e_{ij})
\end{aligned}$$

when $i < j$. We proved that

$$\delta(e_{ij}) = \delta(e_{ij})e_{ii} \quad \text{and} \quad \delta(e_{ij}) = e_{jj}\delta(e_{ij}) \tag{22}$$

whenever $i < j$. Our goal is to prove that

$$\delta(e_{ij}e_{kl}) = \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) \tag{23}$$

for all $i \le j$ and $k \le l$. Again we consider two cases.

Case 1 $j \ne k$. We have to show that $\delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) = 0$.

If $i = k$ and $j = l$, this holds true since δ is a Jordan derivation.
If $i \ne k$ and $j \ne l$, it follows from (22) that $\delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) = \delta(e_{kl})e_{kk}e_{ij} + e_{kl}e_{jj}\delta(e_{ij}) = 0$. Next assume that $i = k$ and $j \ne l$. If $i = l$, then from (22) we have $\delta(e_{ii})e_{ij} + e_{ii}\delta(e_{ij}) = 0$, since δ vanishes on diagonal elements. If $i \ne l$, then from the fact that δ is a Jordan derivation we infer that $0 = \delta(e_{il}e_{ij} + e_{ij}e_{il})e_{jj} = \delta(e_{il})e_{ij} + e_{il}\delta(e_{ij})e_{jj} + \delta(e_{ij})e_{il}e_{jj} + e_{ij}\delta(e_{il})e_{jj} = \delta(e_{il})e_{ij} = \delta(e_{il})e_{ij} + e_{il}\delta(e_{ij})$.
In the case $i \ne k$ and $j = l$ we proceed similarly as above. Now we have

$$\delta(e_{kj})e_{ij} + e_{kj}\delta(e_{ij}) = e_{kj}\delta(e_{ij}) = e_{kk}\delta(e_{kj}e_{ij} + e_{ij}e_{kj}) = e_{kk}\delta(e_{kj}e_{ij}) = 0.$$

Case 2 $j = k$. The case when $i = j = l$ is trivial, since δ vanishes on diagonal elements.

In cases $i < j = l$ or $i = j < l$, it follows from (22) that

$$\delta(e_{ij}) = \delta(e_{jj})e_{ij} + e_{jj}\delta(e_{ij}), \quad \delta(e_{il}) = \delta(e_{il})e_{ii} + e_{il}\delta(e_{ii}).$$

Finally, let $i < j < l$. Then we have $\delta(e_{jl})e_{ij} + e_{jl}\delta(e_{ij}) = \delta(e_{jl})e_{jj}e_{ij} + e_{jl}e_{jj}\delta(e_{ij}) = 0$, while

$$\begin{aligned}\delta(e_{il}) &= \delta(e_{ij}e_{jl} + e_{jl}e_{ij})\\ &= \delta(e_{ij})e_{jl} + e_{ij}\delta(e_{jl}) + \delta(e_{jl})e_{ij} + e_{jl}\delta(e_{ij})\\ &= \delta(e_{ij})e_{ii}e_{jl} + e_{ij}e_{ll}\delta(e_{jl}) + \delta(e_{jl})e_{jj}e_{ij} + e_{jl}e_{jj}\delta(e_{ij}) = 0,\end{aligned}$$

whence (23) holds. This completes the proof. □

Theorem 2.4 *Let (Δ, α) be a generalized Jordan derivation associate with Hochschild 2-cocycle α from $\mathcal{T}_n(\mathcal{R})$ to a $\mathcal{T}_n(\mathcal{R})$-bimodule $\mathcal{M}$. Then there exists a generalized derivation (φ, α), associate with the same Hochschild 2-cocycle α, and an antiderivation δ from $\mathcal{T}_n(\mathcal{R})$ to $\mathcal{M}$ with $\delta(\mathcal{D}_n(\mathcal{R})) = 0$ such that $\Delta = \delta + \varphi$. Moreover, (φ, α) and δ are uniquely determined.*

Proof It suffices to prove the uniqueness. Suppose $\Delta = \delta_1 + \varphi_1 = \delta_2 + \varphi_2$, where φ_1 and φ_2 are generalized derivations associate with α, while δ_1 and δ_2 are antiderivations vanishing on diagonals. Then $d = \delta_1 - \delta_2 = \varphi_2 - \varphi_1$ is a derivation and an antiderivation as well. Therefore, d vanishes on commutators, which implies $d(e_{ij}) = d(e_{ij}e_{jj} - e_{jj}e_{ij}) = 0$ for all $i < j$. On the other hand, $d(\mathcal{D}_n(\mathcal{R})) = \delta_1(\mathcal{D}_n(\mathcal{R})) - \delta_2(\mathcal{D}_n(\mathcal{R})) = 0$. It follows that $d(\mathcal{T}_n(\mathcal{R})) = 0$ and this completes the proof. □

Let $m \geq n \geq 2$. We may regard $\mathcal{M}_m(\mathcal{R})$ as a $\mathcal{T}_n(\mathcal{R})$-bimodule by the actions $AX = (A \oplus I_{m-n})X$, $XA = X(A \oplus I_{m-n})$, for all $A \in \mathcal{T}_n(\mathcal{R})$ and $X \in \mathcal{M}_m(\mathcal{R})$, where I_{m-n} is the identity of $\mathcal{M}_{m-n}(\mathcal{R})$. As a corollary to Theorem 2.4, we shall easily derive

Corollary 2.5 *Let $m \geq n \geq 2$. Then a generalized Jordan derivation from $\mathcal{T}_n(\mathcal{R})$ to $\mathcal{M}_m(\mathcal{R})$ is a generalized derivation.*

In the case $\alpha = 0$, we have

Corollary 2.6 ([7]) *Let $m \geq n \geq 2$. There are no proper Jordan derivations from $\mathcal{T}_n(\mathcal{R})$ to $\mathcal{M}_m(\mathcal{R})$. In particular, there are no proper Jordan derivations from $\mathcal{T}_n(\mathcal{R})$ to itself.*

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