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Topologically Mixing and Hypercyclicity of Tuples

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Abstract In this paper, we characterize conditions under which a tuple of bounded linear operators is topologically mixing. Also, we give conditions for a tuple to be hereditarily hypercyclic with respect to a tuple of syndetic sequences.

Keywords tuple; hypercyclic vector; topologically mixing; thick set; hypercylicity criterion; hereditarily hypercyclic; syndetic sequence.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on an infinite dimensional Banach space X.

Definition 1.1 *Let $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be an n-tuple of operators acting on the Banach space X. We will let*

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}\cdots{T_n}^{k_n} : k_i \geq 0,\ i = 1, \ldots, n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $\mathrm{Orb}(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $\mathrm{Orb}(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e.,

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus \cdots \oplus S_k : S_1 = \cdots = S_k \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, \ldots, x_k \in X$ such that $\{W(x_1 \oplus \cdots \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the k copies of X, $X \oplus \cdots \oplus X$.

For simplicity we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Also, remember that if T_1, T_2 are commutative bounded linear operators on a Banach space X, and $\{m_j\}$, $\{n_j\}$ are two sequences of natural numbers, then we say $\{{T_1}^{m_j}{T_2}^{n_j} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j}{T_2}^{n_j}x : j \geq 0\}$ is dense in X.

Definition 1.2 *We say that the pair $\mathcal{T} = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets U and V of X there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$. Similarly, we say that*

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$\mathcal{T}_d^{(2)}$ is topologically transitive provided for any given nonempty open sets U_1, V_1, U_2, V_2 in X, there exist two positive integers m and n such that $T_1^m T_2^n(U_1) \cap V_1 \neq \emptyset$ and $T_1^m T_2^n(U_2) \cap V_2 \neq \emptyset$.

Definition 1.3 *A pair (T_1, T_2) is called topologically mixing if for any given open sets U and V, there exist two positive integers M and N such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ for all $m \geq M$ and $n \geq N$.*

Definition 1.4 *We say that a pair $\mathcal{T} = (T_1, T_2)$ is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_k\})$ of integers provided for all pair of subsequences $(\{m_{k_j}\}, \{n_{k_j}\})$ of $(\{m_k\}, \{n_k\})$, the sequence $\{{T_1}^{m_{k_j}} {T_2}^{n_{k_j}} : j \geq 1\}$ is hypercyclic. We say that a pair $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences.*

Definition 1.5 *A strictly increasing sequence of positive integers $\{n_k\}$ is said to be syndetic if* $\sup_k\{n_{k+1} - n_k\} < \infty$.

Definition 1.6 *A set $S \subset \mathbb{Z}^2{}_+$ is called thick if for every $m, n \in \mathbb{N}$, there exist some $i_m, j_n \in \mathbb{N}$ such that $\{(p, q) : p = i_m, i_m + 1, \ldots, i_m + m;\ q = j_n, j_n + 1, \ldots, j_n + n\} \subset S$. Also, we say that S is co-finite if the complement of S in $\mathbb{Z}^2{}_+$ is finite.*

Definition 1.7 *Let $\mathcal{T} = (T_1, T_2)$ be a pair of continuous operators acting on a separable infinite dimensional Banach space X. For any nonempty open sets U, V in X, we define $N(U, V) \doteq \{(m, n) : m, n \in \mathbb{N},\ T_1^m T_2^n(U) \cap V \neq \emptyset\} = \{(m, n) : m, n \in \mathbb{N},\ T_1^{-m} T_2^{-n}(V) \cap U \neq \emptyset\}$.*

Here, we want to extend some properties of topologically mixing operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some topics we refer to [1–19].

2. Main results

A nice criterion, namely the Hypercyclicity Criterion for tuples, was given by N. S. Feldman [5].

Theorem 2.1 (The Hypercyclicity Criterion for a tuple) *Suppose X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear mappings on X. If there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_j\}$ and $\{n_j\}$ such that:*

(1) $T_1^{m_j} T_2^{n_j} \to 0$ on Y as $j \to \infty$;

(2) There exists a sequence of function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_j} T_2^{n_j} S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.

In this section we characterize the equivalent conditions for a pair of operators, satisfying the Hypercyclicity Criterion. Also, we give conditions under which a tuple is hereditarily hypercyclic with respect to a tuple of syndetic sequences. We will use $HC(T)$ for the collection of hypercyclic vectors for the tuple $\mathcal{T}$ of operators.

Theorem 2.2 *Let $\mathcal{T}=(T_1,T_2)$ be a pair of bounded linear operators acting on the Banach space X such that $(T_1^m T_2^n)^*$ has no eigenvalue for all $m,n>0$. Let W be a nonempty open set in X such that for any nonempty open subsets U,V of W, the set $N(U,V)$ is a thick set. Then $\mathcal{T}_d^{(2)}$ is hypercyclic.*

Proof Let $\mathcal{B}=\{B_n : n\in\mathbb{N}\}$ be a countable open base of X and define

$$G_i=\bigcup_{m,n} T_1^{-m}T_2^{-n}(B_i).$$

If $G_i\cap\overline{W}\neq\emptyset$, then clearly $G_i\cap\overline{W}$ is dense in $\overline{W}$ under the relative topology, since else there exists a nonempty set $U\subset W$ such that $U\cap\overline{G_i\cap\overline{W}}=\emptyset$. On the otherhand, since $G_i\cap\overline{W}\neq\emptyset$, we have $W\cap G_i\neq\emptyset$, and so

$$T_1^{-m_0}T_2^{-n_0}(B_i)\cap W\neq\emptyset$$

for some pair of nonnegative integers (m_0,n_0). If we define

$$V=T_1^{-m_0}T_2^{-n_0}(B_i)\cap W,$$

then $V\subset W$. Now by using the hypothesis, there exist a pair (m_0',n_0') of integers such that

$$T_1^{m_0'}T_2^{n_0'}(U)\cap V\neq\emptyset.$$

Thus

$$\emptyset\neq U\cap T_1^{-m_0'}T_2^{-n_0'}T_1^{-m_0}T_2^{-n_0}(B_i)\cap W\subset U\cap\overline{G_i\cap\overline{W}},$$

that is a contradiction. Hence $G_i\cap\overline{W}$ is dense in $\overline{W}$ for all i, and so $\cap_i G_i$ is a dense G_δ subset of $\overline{W}$. Thus, indeed

$$HC(\mathcal{T})\cap W=\cap_i G_i.$$

So, if $x\in\cap_i G_i$, then $\mathrm{Orb}(\mathcal{T},x)\cap W$ is dense in W. Now, since $(T_1^m T_2^n)^*$ has no eigenvalue for all $m,n>0$, thus by Corollary 5.6 in [5], $\mathcal{T}$ is hypercyclic. Hence, $\mathcal{T}$ is topologically transitive and so, for any nonempty open sets U',V' in X, there exist two pairs of nonnegative integers (m_1,n_1) and (m_2,n_2) with $m_1<m_2$ and $n_1<n_2$ such that

$$U''\doteq T_1^{-m_1}T_2^{-n_1}(U')\cap W\neq\emptyset,$$

and

$$V''\doteq T_1^{-m_2}T_2^{-n_2}(V')\cap W\neq\emptyset.$$

By our assumption $N(U'',V'')$ is a thick set, thus $N(U',V')$ is also a thick set, since

$$N(U'',V'')+(m_2-m_1,n_2-n_1)\subseteq N(U',V').$$

This implies that $\mathcal{T}_d^{(2)}$ is hypercyclic and so the proof is completed. □

Theorem 2.3 *Let $\mathcal{T}=(T_1,T_2)$ be a pair of bounded linear operators acting on the Banach space X such that $(T_1^m T_2^n)^*$ has no eigenvalue for all $m,n>0$. Let W be a nonempty open set in X such that for any nonempty open subsets U,V of W, the set $N(U,V)$ is co-finite. Then $\mathcal{T}$ is topologically mixing.*

Proof By Theorem 2.2, $\mathcal{T}_d^{(2)}$ and so $\mathcal{T}$ is hypercyclic. Thus, for any U, V of open subsets of X, there exist the pairs (m_1, n_1) and (m_2, n_2) of nonnegative integers such that

$$T_1^{-m_1} T_2^{-n_1}(U) \cap W \neq \emptyset$$

and

$$T_1^{-m_2} T_2^{-n_2}(V) \cap W \neq \emptyset.$$

Define the sets U_1 and V_1 by

$$U_1 = T_1^{-m_1} T_2^{-n_1}(U) \cap W$$

and

$$V_1 = T_1^{-m_2} T_2^{-n_2}(V) \cap W.$$

Then by the hypothesis of the theorem, $N(U_1, V_1)$ is a co-finite set and consequently $N(U, V)$ is also a co-finite set. Thus $\mathcal{T}$ is topologically mixing. $\square$

Theorem 2.4 *Let $\mathcal{T} = (T_1, T_2)$ be a pair of bounded linear operators acting on the Banach space X such that $(T_1^m T_2^n)^*$ has no eigenvalue for all $m, n > 0$. Let W be a nonempty open set in X containing two dense sets Y, Z, and suppose that there exist a pair of nonnegative integers (m_k, n_k), and a sequence of mappings $\{S_k : Z \to X\}$ such that: for every $z \in Z$, $S_k z \to 0$, $T_1^{m_k} T_2^{n_k} S_k z \to z$, and $T_1^{m_k} T_2^{n_k} \to 0$ pointwise on Y. Then $\mathcal{T}_d^{(2)}$ is hypercyclic.*

Proof Consider nonempty open sets U_1, U_2, V_1, V_2 in W and let $y \in Y$ and $z \in Z$. Also, let $\epsilon > 0$ be such that $B(y, \epsilon) \subset U_1$, and $B(z, 2\epsilon) \subset V_1$. Since all three sequences

$$\{S_k z\}_k, \ \{z - T_1^{m_k} T_2^{n_k} S_k z\}_k, \ \{T_1^{m_k} T_2^{n_k} y\}_k$$

tend to 0, there exists $M > 0$ such that

$$S_k z, \ T_1^{m_k} T_2^{n_k} y, \ z - T_1^{m_k} T_2^{n_k} S_k z \in B(0, \epsilon)$$

for all $k > M$. So,

$$y + S_k z \in U_1$$

and

$$T_1^{m_k} T_2^{n_k}(y + S_k z) \in V_1$$

for all $k > M$. Hence, we get

$$T_1^{m_k} T_2^{n_k}(U_1) \cap V_1 \neq \emptyset$$

for all $k > M$. Note that we can let M be large enough such that simultaneously

$$T_1^{m_k} T_2^{n_k}(U_i) \cap V_i \neq \emptyset$$

for $i = 1, 2$, and all $k > M$. This implies that

$$[T_1^{m_k} T_2^{n_k} \oplus T_1^{m_k} T_2^{n_k}(U_1 \oplus U_2)] \cap (V_1 \oplus V_2) \neq \emptyset$$

for all $k > M$. Now, by the same method used earlier, there exists a hypercyclic vector $x \oplus y \in X \oplus X$ for $\mathcal{T}_d^{(2)}$ such that

$$\{S(x \oplus y) : S \in \mathcal{T}_d^{(2)}\} \cap W \oplus W$$

is dense in $W \oplus W$. This implies that $\mathcal{T}_d^{(2)}$ is hypercyclic and so the proof is completed. □

Corollary 2.5 *Under the conditions of Theorem 2.4, if $m_k = n_k = k$, then $\mathcal{T}$ is topologically mixing.*

Theorem 2.6 *Let $\mathcal{T} = (T_1, T_2)$ be a pair of bounded linear operators acting on the Banach space X. If $\mathcal{T}$ satisfies the Hypercyclicity Criterion and $(\{m_k\}, \{n_k\})$ is a pair of syndetic sequences, then $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{m_k\}, \{n_k\})$.*

Proof Since $(\{m_k\}, \{n_k\})$ is syndetic, there exists a pair (m, n) of integers such that $m_{k+1} - m_k < m$ and $n_{k+1} - n_k < n$ for all k. This implies that for any k there exists $(m'_k, n'_k) \in (\{m_k\}, \{n_k\})$ such that $km \leq m'_k < (k+1)m$ and $kn \leq n'_k < (k+1)n$. Let B be an open neighborhood of 0 and U be any nonempty open subset of X. Define

$$W = \bigcap_{i=1}^{2m-1} \bigcap_{j=1}^{2n-1} T_1^{-i} T_2^{-j}(B)$$

and

$$V = T_1^{-2m} T_2^{-2n}(U).$$

Since $\mathcal{T}$ satisfies the Hypercyclicity Criterion, $\mathcal{T}_d^{(2)}$ is topologically transitive. Indeed, let $\mathcal{T}$ satisfy the Hypercyclicity Criterion and let $(\{m_j\}, \{n_j\})$, X_0, Y_0, and $S_j : Y_0 \longrightarrow X$ be as given in the Hypercyclicity Criterion. Note that Hypercyclicity Criterion will also be satisfied by any pair of subsequence $(\{m_{j_k}\}, \{n_{j_k}\})$ of $(\{m_j\}, \{n_j\})$. Now, let U_1, V_1, U_2, and V_2 be any nonempty open subsets of X. Pick $x \in X_0$, $y \in Y_0$ and $\epsilon > 0$ so that $B(x, \epsilon) \subset U_1$ and $B(y, 2\epsilon) \subset V_1$. By the conditions of Hypercyclicity Criterion, there exists an integer r large enough, satisfying:

$$T_1^{m_r} T_2^{n_r} x, \ S_r y, \ T_1^{m_r} T_2^{n_r} S_r y - y \in B(0, \epsilon).$$

So we get $x + S_r y \in B(x, \epsilon) \subset U_1$ and

$$\|T_1^{m_r} T_2^{n_r} S_r y - y\| + \|T_1^{m_r} T_2^{n_r} x\| < 2\epsilon.$$

Thus, we have

$$T_1^{m_r} T_2^{n_r} S_r y - y \in B(y, 2\epsilon) \subset V_1.$$

Hence, $T_1^{m_r} T_2^{n_r}(U_1) \cap V_1$ is nonempty for r large enough. This implies that there exists a pair of subsequence $(\{m_{j_k}\}, \{n_{j_k}\})$ of $(\{m_j\}, \{n_j\})$ such that for all k we have

$$T_1^{m_{j_k}} T_2^{n_{j_k}}(U_1) \cap V_1 \neq \emptyset.$$

Now, since Hypercyclicity Criterion will also be satisfied for $(\{m_{j_k}\}, \{n_{j_k}\})$, by using the same method we can see that there exists k_0 large enough such that

$$T_1^{m_{j_{k_0}}} T_2^{n_{j_{k_0}}}(U_2) \cap V_2 \neq \emptyset.$$

Since $(m_{j_{k_0}}, n_{j_{k_0}})$ is an element of the sequence $(\{m_{j_k}\}, \{n_{j_k}\})$, we have

$$T_1^{m_{j_{k_0}}} T_2^{n_{j_{k_0}}}(U_1) \cap V_1 \neq \emptyset$$

and so $\mathcal{T}_d^{(2)}$ is topologically transitive. Thus, there exists a pair (p, q) of integers such that

$$T_1^p T_2^q(U) \cap W \neq \emptyset$$

and

$$T_1^p T_2^q(W) \cap V \neq \emptyset.$$

Note that $mr_0 \leq p \leq m(r_0+1)$ and $ns_0 \leq p \leq n(s_0+1)$ for some pair of integers (r_0, s_0). In the intervals $[mr_0, m(r_0+1)]$, $[ns_0, n(s_0+1)]$, consider elements m'_{k_0} and n'_{k_0} of the sequences $\{m_k\}$ and $\{n_k\}$, respectively. Now, let the case $p < m'_{k_0}$ and $q < n'_{k_0}$ and set

$$t(p) = m'_{k_0+1} - p, \quad t'(p) = 2m - (m'_{k_0+1} - p),$$

$$t(q) = n'_{k_0+1} - q, \quad t'(q) = 2n - (n'_{k_0+1} - q).$$

Clearly,

$$1 \leq t(p), \quad t'(p) \leq 2m - 1,$$

$$1 \leq t(q), \quad t'(q) \leq 2n - 1.$$

This implies that

$$W \subset T_1^{-t(p)} T_2^{-t(q)}(B) \cap T_1^{-t'(p)} T_2^{-t'(q)}(B).$$

So, clearly we get

$$T_1^p T_2^q (T_1^{-t'(p)} T_2^{-t'(q)}(B)) \cap T_1^{-2m} T_2^{-2n}(V) \neq \emptyset,$$

from which we can conclude that

$$T_1^{p+t(p)} T_2^{q+t(q)}(U) \cap B = T_1^{m'_{k_0+1}} T_2^{n'_{k_0+1}}(U) \cap B \neq \emptyset$$

and

$$T_1^{p-t'(p)+2m} T_2^{q-t'(q)+2n}(B) \cap V = T_1^{m'_{k_0+1}} T_2^{n'_{k_0+1}}(B) \cap V \neq \emptyset.$$

So, in the case of $p < m'_{k_0}$ and $q < n'_{k_0}$, $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{m_k\}, \{n_k\})$. In other cases, for example, if $q \geq n'_{k_0}$, then by substituting $t(q)$ by $q - n'_{k_0}$, and $t'(q)$ by $2n - (q - n'_{k_0})$, we can get a similar result. So the proof is completed. $\square$

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