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Central Extension for the Triangular Derivation Lie Algebra

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Abstract In this paper, we study a class of subalgebras of the Lie algebra of vector fields on n-dimensional torus, which are called the Triangular derivation Lie algebra. We give the structure and the central extension of Triangular derivation Lie algebra.

Keywords triangular derivation Lie algebra; central extension; 2-cocycle.

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1. Introduction

In recent years a new area in Lie theory has emerged - the theory of constructs and bounded modules for infinite-dimensional Lie algebras with a dense $\mathbb{Z}^n$-grading [1–8]. The classical case $n = 1$ includes Kac-Moody algebra and the Virasoro algebra. Moreover, one of the most natural Lie algebras with a dense $\mathbb{Z}^2$-grading is the Lie algebra of the derivations on a 2-dimensional torus:

$$\mathcal{D} = \mathrm{Der}\mathbb{C}\left[t_1^{\pm 1}, t_2^{\pm 1}\right],$$

which is also called the Lie algebra of vector fields on 2-dimensional torus. There is an interesting subalgebra of $\mathcal{D}$, which is called the Triangular derivation Lie algebra. We will introduce this algebra in the following text. In this paper, we want to study the structure and the central extension of triangular derivation Lie algebra.

Denote by $\mathbb{C}$ the field of complex numbers. Suppose that $A = \mathbb{C}\left[t_1^{\pm 1}, \ldots, t_d^{\pm 1}\right]$ is a ring of Laurent polynomials with d commutative indeterminate elements, that is, commutative torus. Denote by $\mathrm{Der}A$ the Lie algebra which is constructed by all derivations of torus A, called full derivation Lie algebra of torus A. Let $e_1, e_2, \ldots, e_d$ denote the column vectors of the identity matrix I_d, and let $(\cdot,\cdot)$ be the normal inner product on $\mathbb{C}^d$, i.e., $(e_i, e_j) = \delta_{ij}$, $\forall i, j = 1, \ldots, d$. Let $\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_d$ be the lattice over $\mathbb{C}^d$. For $n = n_1e_1 + \cdots + n_de_d \in \Gamma$, denote $t^n = t_1^{n_1} \cdots t_d^{n_d}$. Let $D_i = t_i \frac{\mathrm{d}}{\mathrm{d}t_i}$, $i = 1, \ldots, d$. Then $D_i \in \mathrm{Der}A$ is a degree derivation, that is, $D_i(t^n) = n_i t^n$. For $u = u_1e_1 + \cdots + u_de_d \in \mathbb{C}^d$ and $r = r_1e_1 + \cdots + r_de_d \in \Gamma$, denote $D(u, r) = t^r \Sigma_{i=1}^d u_i D_i$. Clearly,

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$D(u,r)\in \mathrm{Der}A$, and we have

Proposition 1.1 ([2]) *$\mathrm{Der}A$ is a Γ-graded Lie algebra, and $\mathrm{Der}A=\oplus_{n\in\Gamma}(\mathrm{Der}A)_n$, where $(\mathrm{Der}A)_n=\{D(u,n):u\in\mathbb{C}^d, n\in\Gamma\}$ with the Lie structure over $\mathrm{Der}A$ as follows:*

$$[D(u,r),D(v,s)]=D(w,r+s),\quad u,v\in\mathbb{C}^d,\ r,s\in\Gamma, \tag{1}$$

where $w=(u,s)v-(v,r)u$.

By Proposition 1.1, we know that $\mathrm{Der}A=\mathrm{span}\{D(u,r):u\in\mathbb{C}^d, r\in\Gamma\}$.

If $d=1$, $\mathrm{Der}A$ is called Witt algebra. The universal central extension of Witt algebra is called Virasoro algebra. Virasoro algebra plays important roles in affine Lie algebras, vertex operator algebras and many other fields. Further research has been carried on Virasoro algebra, including the promotion of Virasoro algebra in a variety of ways. The most natural opinion is to promote $d=1$ into $d\geq 1$. But if $d\geq 2$, $\mathrm{Der}A$ has no non-trivial central extension. In other words, this kind of idea which is the most natural one cannot be achieved. We try to seek out the subalgebra of $\mathrm{Der}A$ which has non-trivial central extension and make it be a new form of promotion of Virasoro algebra. In view of the importance of $\mathrm{Der}A$-module and the significance of promotion of Virasoro algebra, we recall the Triangular derivation Lie algebra for $d\geq 2$ as follows.

Definition 1.2 *The subset*

$$\mathfrak{g}=\mathrm{span}\{D(u,r):u\in\mathbb{C}^d, r\in\Gamma \text{ satisfying if } i<j, u_ir_j=0\}$$

of $\mathrm{Der}A$ is called the d-dimensional triangular derivation Lie algebra.

In the next section we will prove that $\mathfrak{g}$ is a subalgebra of $\mathrm{Der}A$ indeed. Then we study its structure and the central extension.

2. The structure of triangular derivation Lie algebra

Set

$$\mathbb{C}^k=\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_k, \Gamma_k=\mathbb{C}^k\cap\Gamma,\quad k=1,\ldots,d.$$

For $1\leq i,k\leq d$, $\mathbb{G}\leq\mathbb{Z}$ is an addition subgroup of $\mathbb{Z}$. We define the subspace of $\mathrm{Der}A$:

$$S_k^i(\mathbb{G}):=D(\mathbb{C}e_k,\mathbb{G}e_i):=\mathrm{span}\{D(e_k,ge_i):g\in\mathbb{G}\}.$$

$$\mathfrak{S}_k:=\mathrm{span}\{D(e_k,r):r\in\Gamma_k\}.$$

Lemma 2.1 *Let $1\leq i,k\leq d, \mathbb{G}\leq\mathbb{Z}$ be an addition subgroup of $\mathbb{Z}$. Then $S_k^i(\mathbb{G})$ is the subalgebra of $\mathrm{Der}A$. In addition we have that if $i\neq k$, $S_k^i(\mathbb{G})$ is Abelian; and if $\mathbb{G}\neq 0$, $S_k^k(\mathbb{G})$ is perfect.*

Proof Take any $D(e_k,ae_i),D(e_k,be_i)\in S_k^i(\mathbb{G})$, we have

$$\begin{aligned}[D(e_k,ae_i),D(e_k,be_i)]&=D(((e_k,be_i)e_k-(e_k,ae_i)e_k),(a+b)e_i)\\&=(e_k,(b-a)e_i)D(e_k,(a+b)e_i)\\&=(b-a)\delta_{k,i}D(e_k,(a+b)e_i)\in S_k^i(\mathbb{G}).\end{aligned}$$

So $S_k^i(\mathbb{G})$ is a subalgebra since operation is closed in $\mathrm{Der}A$. When $i \neq k$, $\delta_{k,i} = 0$, thus $S_k^i(\mathbb{G})$ is Abelian. If $\mathbb{G} \neq 0$, then $\mathbb{G}$ has infinitely many elements. For any but fixed $D(e_k, ge_k) \in S_k^k(\mathbb{G})$, make $a \in \mathbb{G}\setminus \left\{2^{-1}g\right\}$, $b = g - a$, then $b - a = g - 2a \neq 0$. Thus

$$\begin{aligned} D(e_k, ge_k) &= D(e_k, (a+b)e_k) \\ &= (b-a)^{-1} D(e_k, (a+b)e_k) \\ &= (g-2a)^{-1} \left[D(e_k, ae_k), D(e_k, be_k)\right] \in \left[S_k^k(\mathbb{G}), S_k^k(\mathbb{G})\right], \end{aligned}$$

which gives that $S_k^k(\mathbb{G}) = \left[S_k^k(\mathbb{G}), S_k^k(\mathbb{G})\right]$. $\square$

Lemma 2.2 *Let $1 \leq k \leq d$. Then $\mathfrak{S}_k$ is a perfect subalgebra of* $\mathrm{Der}A$ *and*

$$\mathfrak{S}_k = \left\langle \oplus_{i=1}^k S_k^i(\mathbb{Z}) \right\rangle.$$

Proof For any $r, s \in \Gamma_k \oplus 0_{d-k}$, since

$$\begin{aligned} [D(e_k, r), D(e_k, s)] &= D(((e_k, s)e_k - (e_k, r)e_k), r + s) \\ &= (e_k, s - r) D(e_k, r + s) \in \mathfrak{S}_k, \end{aligned}$$

$\mathfrak{S}_k$ is perfect subalgebra of $\mathrm{Der}A$.

For any $D(e_k, \rho) \in \mathfrak{S}_k$, make $r = \rho_1 e_1 + \cdots + \rho_{k-1} e_{k-1}, + r_k e_k$, where $r_k \in \mathbb{Z}\setminus \left\{2^{-1}\rho_k\right\}$, $s = \rho - r$, then $(s_k - r_k) = \rho_k - 2r_k \neq 0$ and

$$D(e_k, \rho) = D(e_k, r + s) = (r_k - s_k)^{-1} \left[D(e_k, r), D(e_k, s)\right] \in \left[\mathfrak{S}_k, \mathfrak{S}_k\right].$$

Thus we have $\mathfrak{S}_k = [\mathfrak{S}_k, \mathfrak{S}_k]$, so that, $\mathfrak{S}_k$ is perfect.

Let $r = r_1 e_1 + \cdots + r_k e_k \in \Gamma_k$. We have the following

Assertion: If $r_k \neq 0$, then $D(e_k, r) \in \left\langle \oplus_{i=1}^k S_k^i(\mathbb{Z}) \right\rangle$.

Note that $D(e_k, r_k e_k) \in S_k^k(\mathbb{Z})$. Suppose that $D(e_k, r_{j+1} e_{j+1} + \cdots r_k e_k) \in \left\langle \oplus_{i=j+1}^k S_k^i(\mathbb{Z}) \right\rangle$, $\forall 1 \leq j < k$. Then we have

$$\begin{aligned} D(e_k, r_j e_j + \cdots + r_k e_k) &= r_k^{-1} \left[D(e_k, r_j e_j), D(e_k, r_{j+1} e_{j+1} + \cdots r_k e_k)\right] \\ &\in \left\langle \oplus_{i=j}^k S_k^i(\mathbb{Z}) \right\rangle. \end{aligned}$$

So $D(e_k, r_1 e_1 + \cdots + r_k e_k) \in \left\langle \oplus_{i=1}^k S_k^i(\mathbb{Z}) \right\rangle$, which proves the above assertion.

Also since

$$D(e_k, r_1 e_1 + \cdots + r_{k-1} e_{k-1}) = 2^{-1} r_k^{-1} \left[D(e_k, -r_k e_k), D(e_k, r)\right] \in \left\langle \oplus_{i=1}^k S_k^i(\mathbb{Z}) \right\rangle,$$

we have $\mathfrak{S}_k \subset \left\langle \oplus_{i=1}^k S_i^i(\mathbb{Z}) \right\rangle$, while $\left\langle \oplus_{i=1}^k S_i^i(\mathbb{Z}) \right\rangle \subset \mathfrak{S}_k$ is obvious. The proof is completed. $\square$

2.1. The characterization of triangular derivation Lie algebra

For $1 \leq k \leq d$, we denote

$$\mathfrak{T}_k := \mathrm{span}\left\{D(u, r) : u \in \mathbb{C}^k, r \in \Gamma_k \text{ satisfying if } i < j, u_i r_j = 0\right\}.$$

Lemma 2.3 *For $1 \leq i < j \leq d$, we have $[\mathfrak{S}_i, \mathfrak{S}_j] \subset \mathfrak{S}_j$.*

Proof Let $r \in \Gamma_i$, $s \in \Gamma_j$. Note that

$$[D(e_i, r), D(e_j, s)] = (e_i, s)D(e_j, r + s) \in \mathfrak{S}_j.$$

Thus the lemma is obtained. □

Lemma 2.4 *Let $1 \leq k \leq d$. Then $\mathfrak{T}_k$ is a prefect subalgebra of* Der*A. We have also $\mathfrak{S}_k$ is an ideal of $\mathfrak{T}_k$, and*

$$\mathfrak{T}_k = \oplus_{i=1}^{k} \mathfrak{S}_i.$$

Proof By the definition, it is obvious that $\oplus_{i=1}^{k} \mathfrak{S}_i \subset \mathfrak{T}_k$. We only need to prove that $\mathfrak{T}_k \subset \oplus_{i=1}^{k} \mathfrak{S}_i$.

It is clear that $\mathfrak{T}_1 = \mathfrak{S}_1$. Suppose $\mathfrak{T}_{k-1} = \oplus_{i=1}^{k-1} \mathfrak{S}_i$.

Let $D(u, r) \in \mathfrak{T}_k$, where $u = u_1e_1 + \cdots + u_ke_k$, $r = r_1e_1 + \cdots + r_ke_k$. Without loss of generality, we can assume that u_k, r_k are not all zero.

If $r_k \neq 0$, we have by $u_ir_k = 0$, $i < k$, that $u = u_ke_k$. Thus $D(u, r) \in \mathfrak{S}_k$.

Next, we only need to consider the case of $u_k \neq 0$,$r_k = 0$. Since $r_k = 0$, it follows

$$D(u_1e_1 + \cdots + u_{k-1}e_{k-1}, r) \in \mathfrak{T}_{k-1} = \oplus_{i=1}^{k-1} \mathfrak{S}_i.$$

Since

$$D(u_ke_k, r) \in \mathfrak{S}_k,$$

we see that

$$D(u, r) = D(u_1e_1 + \cdots + u_{k-1}e_{k-1}, r) + D(u_ke_k, r) \in \oplus_{i=1}^{k} \mathfrak{S}_i.$$

By induction, we prove that $\mathfrak{T}_k \subset \oplus_{i=1}^{k} \mathfrak{S}_i$, and so $\mathfrak{T}_k = \oplus_{i=1}^{k} \mathfrak{S}_i$.

By Lemma 2.3 $[\mathfrak{S}_i, \mathfrak{S}_j] \subset \mathfrak{S}_j \subset \mathfrak{T}_k, \forall 1 \leq i \leq j \leq k$, so $\mathfrak{T}_k$ is a subalgebra. And $\mathfrak{S}_k$ is an ideal. Note that $\mathfrak{S}_i$, $1 \leq i \leq k$ is perfect. We show that $\mathfrak{T}_k$ is perfect. □

Theorem 2.5 *The Triangular derivation Lie algebra $\mathfrak{g}$ of d-dimensional commutative torus is a $\mathbb{Z}^d$-graded perfect algebra. Moreover, g has $\mathbb{Z}^k$-graded perfect subalgebras $\mathfrak{T}_k$ such that*

$$0 < \mathfrak{T}_1 < \cdots < \mathfrak{T}_{d-1} < \mathfrak{g},$$

where $\mathfrak{T}_{k-1}$ is an ideal of $\mathfrak{T}_k$, and $\mathfrak{T}_k/\mathfrak{T}_{k-1} \simeq \mathfrak{S}_k$. Specially we have,

$$\mathfrak{g} = \oplus_{i=1}^{d} \mathfrak{S}_k = \oplus_{k=1}^{d} \left\langle \oplus_{i=1}^{k} S_k^i(\mathbb{Z}) \right\rangle.$$

3. The central extension of triangular derivation Lie algebra

Let $\mathfrak{L}$ be a perfect Lie algebra. The central extension of $\mathfrak{L}$ is a Lie algebra $\bar{\mathfrak{L}}$ and a surjective homomorphism $\pi : \mathfrak{L} \to \bar{\mathfrak{L}}$ satisfying $\ker \pi$ is included in the center of $\bar{\mathfrak{L}}$. If $\bar{\mathfrak{L}}$ is still perfect, then we call $\bar{\mathfrak{L}}$ a covering central extension of $\mathfrak{L}$. Let $(\bar{\mathfrak{L}}, \pi)$ be a covering central extension of $\mathfrak{L}$, if for the central extension of any $\mathfrak{L}$, there exists unique Lie algebra homomorphism $\varphi : \bar{\mathfrak{L}} \to \hat{\mathfrak{L}}$ such that $\omega\varphi = \pi$, then we call $(\bar{\mathfrak{L}}, \pi)$ the universal central extension of $\mathfrak{L}$. Every perfect Lie algebra has a universal central extension, moreover, it is unique in the sense of isomorphism.

A bilinear function ψ on Lie algebra $\mathfrak{L}$, satisfying the following conditions:

$$\psi(x,x)=0,\quad \forall x\in\mathfrak{L},$$

$$\psi(x,[y,z])=\psi([x,y],z)-\psi([x,z],y),\quad \forall x,y,z\in\mathfrak{L}$$

is called a 2-cocycle on $\mathfrak{L}$. A 2-cocycle can uniquely determine a one-dimensional central extension: given a 2-cocycle ψ on $\mathfrak{L}$, we can define a central extension $\mathfrak{L}\oplus\mathbb{C}c$ of $\mathfrak{L}$ as follows:

$$[x+\lambda c,y+\mu c]_0=[x,y]+\psi(x,y)c,\quad \forall x,y\in\mathfrak{L},\lambda,\mu\in\mathbb{C},$$

where $[\cdot,\cdot]$ is the Lie operation on $\mathfrak{L}$, $[\cdot,\cdot]_0$ is the Lie operation on $\mathfrak{L}\oplus\mathbb{C}c$. Every one-dimensional central extension of $\mathfrak{L}$ can be obtained in this way.

If a 2-cocycle ψ is induced by a linear function f on $\mathfrak{L}$, that is, $\psi=\alpha_f$, where

$$\alpha_f(x,y)=f([x,y]),\quad \forall x,y\in\mathfrak{L},$$

then ψ is called trivial, while the corresponding central extension is called trivial central extension. Two 2-cocycle ϕ and ψ are called equivalent if $\phi-\psi$ is trivial.

3.1. The 2-cocycle of triangular derivation Lie algebra

For any chosen $1\le k\le d$, we have by $D(e_k,r)\in\mathfrak{S}_k$ and from (1) that

$$D(e_k,r)=r_k^{-1}\left[D(e_k,0),D(e_k,r)\right],\quad r_k\neq 0, \tag{2}$$

and

$$D(e_k,r)=2^{-1}\left[D(e_k,-e_k),D(e_k,r+e_k)\right],\quad r_k=0. \tag{3}$$

Let ψ be a 2-cocycle of $\mathfrak{T}_d$. We define a linear function $f_\psi:\mathfrak{T}_d\to\mathbb{C}$ as

$$f_\psi(D(e_k,r))=\begin{cases} r_k^{-1}\psi(D(e_k,0),D(e_k,r)) & r_k\neq 0\\ 2^{-1}\psi(D(e_k,-e_k),D(e_k,r+e_k)) & r_k=0\end{cases},D(e_k,r)\in\mathfrak{S}_k,1\le k\le d.$$

Then $\phi=\psi-\alpha_{f_\psi}$ is a 2-cocycle on $\mathfrak{T}_d$, which is equivalent to ψ.

Lemma 3.1 *Let $1\le i,j\le d$. Then $\phi(D(e_i,0),D(e_j,0))=0$.*

Proof If $i=j$, then it is clear that $\phi(D(e_i,0),D(e_i,0))=0$.

If $j\neq i$, then

$$\begin{aligned}\phi(D(e_i,0),D(e_j,0))=&\phi(D(e_i,0),2^{-1}\left[D(e_j,-e_j),D(e_j,e_j)\right])\\ =&2^{-1}\phi(\left[D(e_i,0),D(e_j,-e_j)\right],D(e_j,e_j))-\\ &2^{-1}\phi(\left[D(e_i,0),D(e_j,e_j)\right],D(e_j,-e_j))\\ =&2^{-1}\phi(0,D(e_j,e_j))-2^{-1}\phi(0,D(e_j,-e_j))=0.\end{aligned}$$

The proof is completed. □

Lemma 3.2 *Let $1\le k\le d$, $D(e_k,r)\in\mathfrak{S}_k$. Then*

$$\phi(D(e_k,0),D(e_k,r))=0, \tag{4}$$

and if $r_k = 0$, *we still have*

$$\phi(D(e_k, -e_k), D(e_k, r + e_k)) = 0, \tag{5}$$

and

$$\phi(D(e_k, e_k), D(e_k, r - e_k)) = 0. \tag{6}$$

Proof If $r_k \neq 0$, by the definition of ϕ, we can obtain (4).

If $r_k = 0$, by Lemma 3.1, we can assume without loss of generality that $r \neq 0$. Then for $1 \leq j < k$ such that $r_j \neq 0$, we have

$$\begin{aligned}\phi(D(e_k, 0), D(e_k, r)) =& \phi(D(e_k, 0), r_j^{-1} [D(e_j, 0), D(e_k, r)]) \\ =& r_j^{-1} \phi([D(e_k, 0), D(e_j, 0)], D(e_k, r)) - \\ & r_j^{-1} \phi([D(e_k, 0), D(e_k, r)], D(e_j, 0)) \\ =& r_j^{-1} \phi(0, D(e_k, r)) - r_j^{-1} \phi(0, D(e_j, 0)) = 0.\end{aligned}$$

That is, (4) is obtained. By the definition of ϕ, we can obtain (5). Since

$$\begin{aligned}&\phi(D(e_k, e_k), D(e_k, r - e_k)) \\ &= \phi(D(e_k, e_k), [D(e_k, -e_k), D(e_k, r)]) \\ &= \phi([D(e_k, e_k), D(e_k, -e_k)], D(e_k, r)) - \\ &\quad \phi([D(e_k, e_k), D(e_k, r)], D(e_k, -e_k)) \\ &= -2\phi(D(e_k, 0), D(e_k, r)) + \phi(D(e_k, r + e_k), D(e_k, -e_k)),\end{aligned}$$

finally, we can obtain (6) by (4) and (5). □

Lemma 3.3 *Let* $1 \leq j, k \leq d$, $D(e_k, r) \in \mathfrak{S}_k$. *Then*

$$\phi(D(e_j, 0), D(e_k, r)) = 0.$$

Proof If $r_k \neq 0$, it follows from (4) that

$$\begin{aligned}\phi(D(e_j, 0), D(e_k, r)) =& \phi(D(e_j, 0), r_k^{-1} [D(e_k, 0), D(e_k, r)]) \\ =& r_k^{-1} \phi([D(e_j, 0), D(e_k, 0)], D(e_k, r)) - \\ & r_k^{-1} \phi([D(e_j, 0), D(e_k, r)], D(e_k, 0)) \\ =& 0 - r_k^{-1} r_j \phi(D(e_k, r), D(e_k, 0)) = 0.\end{aligned}$$

If $r_k = 0$, we have by (5) that

$$\begin{aligned}\phi(D(e_j, 0), D(e_k, r)) =& \phi(D(e_j, 0), 2^{-1} [D(e_k, -e_k), D(e_k, r + e_k)]) \\ =& 2^{-1} \phi([D(e_j, 0), D(e_k, -e_k)], D(e_k, r + e_k)) - \\ & 2^{-1} \phi([D(e_j, 0), D(e_k, r + e_k)], D(e_k, -e_k)) \\ =& 0 - 2^{-1} r_j \phi(D(e_k, r + e_k), D(e_k, -e_k)) = 0.\end{aligned}$$

The proof is completed. □

Lemma 3.4 *Let $D(e_k, r), D(e_k, s) \in \mathfrak{S}_k$ satisfying $r + s \neq 0$. Then*

$$\phi(D(e_k, r), D(e_k, s)) = 0.$$

Proof Since $r + s \neq 0$, there exists $1 \leq j \leq k$ such that $r_j + s_j \neq 0$. This tells us that r_j, s_j are not all zero. Without loss of generality, we assume $s_j \neq 0$. Noting that

$$\begin{aligned}\phi(D(e_k, r), D(e_k, s)) =& \phi(D(e_k, r), s_j^{-1} [D(e_j, 0), D(e_k, s)]) \\ =& s_j^{-1}\phi([D(e_k, r), D(e_j, 0)], D(e_k, s)) - \\ & s_j^{-1}\phi([D(e_k, r), D(e_k, s)], D(e_j, 0)) \\ =& - r_j s_j^{-1}\phi(D(e_k, r), D(e_k, s)) - \\ & s_j^{-1}(s_k - r_k)\phi(D(e_k, r + s), D(e_j, 0)) \\ =& - r_j s_j^{-1}\phi(D(e_k, r), D(e_k, s)) - 0,\end{aligned}$$

one has

$$(1 + r_j s_j^{-1})\phi(D(e_k, r), D(e_k, s)) = 0.$$

If $r_j + s_j \neq 0$, we have $1 + r_j s_j^{-1} \neq 0$. Thus

$$\phi(D(e_k, r), D(e_k, s)) = 0.$$

This completes the proof. □

Lemma 3.5 *Let $1 \leq i < j \leq d$, $D(e_i, r) \in \mathfrak{S}_i, D(e_j, s) \in \mathfrak{S}_j$. Then*

$$\phi(D(e_i, r), D(e_j, s)) = 0.$$

Proof Case 1. $r + s \neq 0$.

If there exists $s_k \neq 0$ such that $r_k + s_k \neq 0$, then

$$\begin{aligned}\phi(D(e_i, r), D(e_j, s)) =& \phi(D(e_i, r), s_k^{-1} [D(e_k, 0), D(e_j, s)]) \\ =& s_k^{-1}\phi([D(e_i, r), D(e_k, 0)], D(e_j, s)) - \\ & s_k^{-1}\phi([D(e_i, r), D(e_j, s)], D(e_k, 0)) \\ =& - r_k s_k^{-1}\phi(D(e_i, r), D(e_j, s)) - \\ & s_k^{-1} s_i \phi(D(e_j, r + s), D(e_k, 0)) \\ =& - r_k s_k^{-1}\phi(D(e_i, r), D(e_j, s)) - 0.\end{aligned}$$

Thus $\phi(D(e_i, r), D(e_j, s)) = 0$.

If there exists $s_k = 0$ such that $r_k + s_k \neq 0$, then $r_k \neq 0$. So we have

$$\begin{aligned}\phi(D(e_i, r), D(e_j, s)) =& - \phi(D(e_j, s), D(e_i, r)) \\ =& - \phi(D(e_j, s), r_k^{-1} [D(e_k, 0), D(e_i, r)]) \\ =& r_k^{-1} s_k \phi(D(e_i, r), D(e_j, s)) = 0.\end{aligned}$$

Case 2. $r + s = 0$.

Since $i<j$, $r_j=0$. Applying (6) gives

$$\begin{aligned}\phi(D(e_i,-r),D(e_j,r)) =&\phi(D(e_i,-r),2^{-1}\left[D(e_j,-e_j),D(e_j,r+e_j)\right])\\ =&2^{-1}\phi(\left[D(e_i,-r),D(e_j,-e_j)\right],D(e_j,r+e_j))-\\ &2^{-1}\phi(\left[D(e_i,-r),D(e_j,r+e_j)\right],D(e_j,-e_j))\\ =&2^{-1}\phi(0,D(e_j,r+e_j))-2^{-1}r_i\phi(D(e_j,e_j),D(e_j,-e_j))\\ =&0.\end{aligned}$$

The proof is completed. □

Lemma 3.6 *Let $D(e_k,r)\in\mathfrak{S}_k$. If there exists $1\le j<k$ such that $r_j=0$, then*

$$\phi(D(e_k,-r),D(e_k,r))=0.$$

Proof By Lemma 3.5, we know

$$\begin{aligned}\phi(D(e_k,-r),D(e_k,r)) =&\phi(D(e_k,-r),[D(e_j,-e_j),D(e_k,r+e_j)])\\ =&\phi(\left[D(e_k,-r),D(e_j,-e_j)\right],D(e_k,r+e_j))-\\ &\phi(\left[D(e_k,-r),D(e_k,r+e_j)\right],D(e_j,-e_j))\\ =&\phi(0,D(e_k,r+e_j))-2r_k\phi(D(e_k,e_j),D(e_j,-e_j))\\ =&0-0=0.\ \square\end{aligned}$$

Lemma 3.7 *Let $3\le k\le d$, $D(e_k,r)\in\mathfrak{S}_k$. If $r_k=0$, then*

$$\phi(D(e_k,-r),D(e_k,r))=0.$$

Proof If $r\neq0$, we can take $1\le i<k$ such that $r_i\neq0$. Since $k\ge3$, it follows by Lemma 3.6 that

$$\phi(D(e_k,-r_ie_i),D(e_k,r_ie_i))=0.$$

Thus

$$\begin{aligned}\phi(D(e_k,-r),D(e_k,r)) =&\phi(D(e_k,-r),r_i^{-1}\left[D(e_i,r-r_ie_i),D(e_k,r_ie_i)\right])\\ =&r_i^{-1}\phi(\left[D(e_k,-r),D(e_i,r-r_ie_i)\right],D(e_k,r_ie_i))-\\ &r_i^{-1}\phi(\left[D(e_k,-r),D(e_k,r_ie_i)\right],D(e_i,r-r_ie_i))\\ =&r_i^{-1}\phi(r_iD(e_k,-r_ie_i),D(e_k,r_ie_i))-\\ &r_i^{-1}\phi(0,D(e_i,r-r_ie_i))\\ =&\phi(D(e_k,-r_ie_i),D(e_k,r_ie_i))=0.\ \square\end{aligned}$$

Lemma 3.8 *Let $3\le k\le d,D(e_k,r)\in\mathfrak{S}_k$. Then*

$$\phi(D(e_k,-r),D(e_k,r))=0.$$

Proof By Lemma 3.7, without loss of generality, we assume $r_k\neq0$. We have

$$\phi(D(e_k,-r_je_j),D(e_k,r_je_j))=0.$$

By Lemma 3.6 we know

$$\phi(D(e_k,-r+r_je_j),D(e_k,r-r_je_j))=0.$$

Thus,

$$\begin{aligned}\phi(D(e_k,-r),D(e_k,r)) =&\phi(D(e_k,-r),r_k^{-1}\left[D(e_k,r_je_j),D(e_k,r-r_je_j)\right])\\ =&r_k^{-1}\phi(\left[D(e_k,-r),D(e_k,r_je_j)\right],D(e_k,r-r_je_j))-\\ &r_k^{-1}\phi(\left[D(e_k,-r),D(e_k,r-r_je_j)\right],D(e_k,r_je_j))\\ =&\phi(D(e_k,-r+r_je_j),D(e_k,r-r_je_j))-\\ &2\phi(D(e_k,-r_je_j),D(e_k,r_je_j))\\ =&0. \quad \square\end{aligned}$$

Lemma 3.9 We have the following equation:

$$\phi(D(e_2,-r_1e_1),D(e_2,r_1e_1))=r_1\phi(D(e_2,-e_1),D(e_2,e_1)).$$

Proof It follows from the following equation.

$$\begin{aligned}\phi(D(e_2,-r_1e_1),D(e_2,r_1e_1)) =&\phi(D(e_2,-r_1e_1),[D(e_1,(r_1-1)e_1),D(e_2,e_1)])\\ =&\phi(\left[D(e_2,-r_1e_1),D(e_1,(r_1-1)e_1)\right],D(e_2,e_1))-\\ &\phi(\left[D(e_2,-r_1e_1),D(e_2,e_1)\right],D(e_1,(r_1-1)e_1))\\ =&r_1\phi(D(e_2,-e_1),D(e_2,e_1))-\phi(0,D(e_1,(r_1-1)e_1))\\ =&r_1\phi(D(e_2,-e_1),D(e_2,e_1)). \quad \square\end{aligned}$$

Lemma 3.10 *Let $D(e_2,r)\in\mathfrak{S}_2$ satisfy $r_2\neq 0$. Then*

$$\phi(D(e_2,-r),D(e_2,r))=-2r_1\phi(D(e_2,-e_1),D(e_2,e_1)).$$

Proof It follows from the equation:

$$\begin{aligned}\phi(D(e_2,-r),D(e_2,r)) =&\phi(D(e_2,-r),r_2^{-1}\left[D(e_2,r_1e_1),D(e_2,r_2e_2)\right])\\ =&r_2^{-1}\phi(\left[D(e_2,-r),D(e_2,r_1e_1)\right],D(e_2,r_2e_2))-\\ &r_2^{-1}\phi(\left[D(e_2,-r),D(e_2,r_2e_2)\right],D(e_2,r_1e_1))\\ =&\phi(D(e_2,-r_2e_2),D(e_2,r_2e_2))-2\phi(D(e_2,-r_1e_1),D(e_2,r_1e_1))\\ =&-2\phi(D(e_2,-r_1e_1),D(e_2,r_1e_1))\\ =&-2r_1\phi(D(e_2,-e_1),D(e_2,e_1)). \quad \square\end{aligned}$$

Lemma 3.11 *Let $D(e_1,r_1e_1)\in\mathfrak{S}_1$. Then*

$$\phi(D(e_1,-r_1e_1),D(e_1,r_1e_1))=\frac{r_1^3-r_1}{6}\phi(D(e_1,-2e_1),D(e_1,2e_1)).$$

Proof By Lemmas 3.1 and 3.2, we know the lemma holds if $r_1=0,1$. Obviously, if $r_1=2$, the lemma is identity. Let $r_1\geq 3$. Then

$$\phi(D(e_1,-r_1e_1),D(e_1,r_1e_1)) =\phi(D(e_1,-r_1e_1),(r_1-2)^{-1}\left[D(e_1,e_1),D(e_1,(r_1-1)e_1)\right])$$

$$
\begin{aligned}
&=(r_1-2)^{-1}\phi([D(e_1,-r_1e_1),D(e_1,e_1)],D(e_1,(r_1-1)e_1))-\\
&\quad(r_1-2)^{-1}\phi([D(e_1,-r_1e_1),D(e_1,(r_1-1)e_1)],D(e_1,e_1))\\
&=(r_1-2)^{-1}(r_1+1)\phi(D(e_1,-(r_1-1)e_1),D(e_1,(r_1-1)e_1))\\
&\quad\cdots\\
&=\frac{r_1+1}{r_1-2}\cdot\frac{(r_1-1)+1}{(r_1-2)-1}\cdots\frac{4}{1}\phi(D(e_1,-2e_1),D(e_1,2e_1))\\
&=\frac{(r_1+1)!/3!}{(r_1-2)!}\phi(D(e_1,-2e_1),D(e_1,2e_1))\\
&=\frac{r_1^3-r_1}{6}\phi(D(e_1,-2e_1),D(e_1,2e_1)).
\end{aligned}
$$

The proof is completed. □

By the above several lemmas, we can easily prove the following main result.

Theorem 3.12 *The one-dimensional central extension of the triangular derivation Lie algebra* $\mathfrak{g}$ *is* $\mathfrak{g}\oplus\mathbb{C}c$*:*

$$
\begin{aligned}
&\Big[\sum_{k=1}^{d}D(a_ke_k,r^{(k)})+\lambda c,\sum_{k=1}^{d}D(b_ke_k,s^{(k)})+\mu c\Big]\\
&\quad=\Big[\sum_{k=1}^{d}D(a_ke_k,r^{(k)}),\sum_{k=1}^{d}D(b_ke_k,s^{(k)})\Big]+\\
&\quad\quad(\phi(D(a_1e_1,r^{(1)}),D(b_1e_1,s^{(1)}))+\phi(D(a_2e_2,r^{(2)}),D(b_2e_2,s^{(2)})))c\\
&\quad=\Big[\sum_{k=1}^{d}D(a_ke_k,r^{(k)}),\sum_{k=1}^{d}D(b_ke_k,s^{(k)})\Big]+\\
&\quad\quad\Big(a_1b_1\delta_{r^{(1)}+s^{(1)},0}\frac{(s^{(1)})^3-s^{(1)}}{6}c_1-2a_2b_2\delta_{r^{(2)}+s^{(2)},0}(1-\delta_{r_2^{(2)},0})s_1^{(2)}c_2+\\
&\quad\quad a_2b_2\delta_{r^{(2)}+s^{(2)},0}\delta_{r_2^{(2)},0}s_1^{(2)}c_2\Big)c,
\end{aligned}
$$

where $D(a_ke_k,r^{(k)})$, $D(b_ke_k,s^{(k)})\in\mathfrak{S}_k$, $1\le k\le d$; c_1, c_2 *are given constants.*

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