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On Fuzzy Quasi-Ideals of Ordered Semigroups

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Abstract In this paper, fuzzy quasi-ideals of ordered semigroups are characterized by the properties of their level subsets. Furthermore, we introduce the notion of completely semiprime fuzzy quasi-ideals of ordered semigroups and characterize strongly regular ordered semigroups in terms of completely semiprime fuzzy quasi-ideals. Finally, we investigate the characterizations and decompositions of left and right simple ordered semigroups by means of fuzzy quasi-ideals.

Keywords fuzzy quasi-ideal; completely semiprime fuzzy quasi-ideal; strongly regular ordered semigroup; left simple ordered semigroup; right simple ordered semigroup.

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1. Introduction

Let S be a nonempty set. A fuzzy subset of S is, by definition, an arbitrary mapping $f : S \longrightarrow [0,1]$, where $[0,1]$ is the usual interval of real numbers. The important concept of a fuzzy set put forth by Zadeh in 1965 (see [1]) has opened up keen insights and applications in a wide range of scientific fields. Recently, a theory of fuzzy sets on ordered semigroups has been developed [2–8]. Following the terminology given by Zadeh, if S is an ordered semigroup, fuzzy sets in ordered semigroups S have been first considered by Kehayopulu and Tsingelis in [2]. They then defined several analogous "fuzzy" notations that have been proved to be useful in the theory of ordered semigroups. The concept of ordered fuzzy points of an ordered semigroup S was first introduced by Xie and Tang [9], and prime fuzzy ideals of an ordered semigroup were studied in [10].

As we know, quasi-ideals play an important role in the study of ring, semigroup and ordered semigroup structures. The concept of quasi-ideals in rings and semigroups were studied by Stienfeld in [11]. Furthermore, Kehayopulu and Tsingelis extended the concept of quasi-ideals in

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ordered semigroups as a non-empty subset Q of an ordered semigroup S such that: (1) $(QS] \cap (SQ] \subseteq Q$ and (2) If $a \in Q$ and $S \ni b \leq a$, then $b \in Q$ (see [5]). The fuzzy quasi-ideals in semigroups were studied in [12] and [13], where the basic properties of semigroups in terms of fuzzy quasi-ideals are given. Recently, Kehayopulu extended those similar " fuzzy" results to ordered semigroups. As a continuation of the study undertaken by Kehayopulu, we first characterize fuzzy quasi-ideals of ordered semigroups by the properties of their level subsets in the present paper. Furthermore, we introduce the notion of completely semiprime fuzzy quasi-ideals of ordered semigroups and characterize strongly regular ordered semigroups in terms of completely semiprime fuzzy quasi-ideals. Finally, we also study the characterizations and decompositions of left and right simple ordered semigroups by means of fuzzy quasi-ideals. As an application of the results of this paper, the corresponding results of semigroup (without order) are also obtained.

2. Notations and preliminaries

Throughout this paper unless stated otherwise S stands for an ordered semigroup. We denote by Z^+ the set of all positive integers. A function f from S to the real closed interval [0,1] is a fuzzy subset of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) = 1$ for all $x \in S$ (the fuzzy subset S is also denoted by 1 (see [4])). Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$, $f \cup g$ are defined by

$$(f \cap g)(x) = \min(f(x), g(x)) = f(x) \wedge g(x),$$

$$(f \cup g)(x) = \max(f(x), g(x)) = f(x) \vee g(x)$$

for all $x \in S$, respectively. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily see that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $x \in S$, we define $A_x := \{(y,z) \in S \times S | x \leq yz\}$. For any $f, g \in F(S)$, the product $f \circ g$ is defined by

$$(\forall x \in S)\ (f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} [\min\{f(y), g(z)\}], & \text{if } A_x \neq \emptyset\ , \\ 0 & , \quad \text{if } A_x = \emptyset. \end{cases}$$

It is well known ([2, Theorem]), that this operation " $\circ$" is associative.

Lemma 2.1 ([2, Proposition 1]) *Let $(S, \cdot, \leq)$ be an ordered semigroup and f_1, f_2, g_1, g_2 fuzzy subsets of S such that $f_1 \subseteq g_1$, $f_2 \subseteq g_2$. Then $f_1 \circ f_2 \subseteq g_1 \circ g_2$.*

For any subset A of S, we denote by f_A the characteristic function of A, that is the mapping of S into $[0,1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \ \text{ for some } \ h \in H\}.$$

For two subsets A, B of S, we have: (1)$A \subseteq (A]$; (2) If $A \subseteq B$, then $(A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $((A]] = (A]$; (5) $((A](B]] = (AB]$ (see [14]). A nonempty subset A of S is called a left (resp., right) ideal of S if (1) $SA \subseteq A$ (resp., $AS \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and right ideal of S, then it is called an (two-sided) ideal of S (see [14]). Clearly, every left (right) ideal of S is a quasi-ideal of S. We denote by $Q(a)$ the quasi-ideal of S generated by a ($a \in S$). Then $Q(a) = (a \cup (aS \cap Sa)]$ (see [5]).

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a fuzzy left (resp., right) ideal of S if (1) $f(xy) \geq f(y)$ (resp., $f(xy) \geq f(x)$) for all $x, y \in S$ and (2) $x \leq y$ implies $f(x) \geq f(y)$. Equivalent definition: (1) $S \circ f \subseteq f$ (resp., $f \circ S \subseteq f$) and (2) $x \leq y$ implies $f(x) \geq f(y)$ (see [15]). A fuzzy ideal of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subset f of S is called a fuzzy quasi-ideal of S if (1) $(f \circ S) \cap (S \circ f) \subseteq f$ and (2) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in S$ (see [5]). Clearly, every fuzzy left (right) ideal of S is a fuzzy quasi-ideal of S.

Lemma 2.2 ([5]) *Let S be an ordered semigroup and $\emptyset \neq Q \subseteq S$. Then Q is a quasi-ideal of S if and only if the characteristic function f_Q of Q is a fuzzy quasi-ideal of S.*

The reader is referred to [9, 18] for notation and terminology not defined in this paper.

3. Fuzzy quasi-ideals of ordered semigroups

In this section, we study and characterize fuzzy quasi-ideals of ordered semigroups by the properties of their level subsets.

Definition 3.1 ([9]) *Let f be any fuzzy subset of an ordered semigroup S. The set*

$$f_t := \{x \in S |\ f(x) \geq t\}, \text{ where } t \in [0,1]$$

is called a level subset of f.

Theorem 3.2 *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy quasi-ideal of S if and only if all nonempty level subsets f_t ($t \in (0, 1]$) are quasi-ideals of S.*

Proof $\Rightarrow$. Let f be a fuzzy quasi-ideal of S and $t \in (0, 1]$ such that $f_t \neq \emptyset$. Suppose that $x \in S$ is such that $x \in (f_t S] \cap (S f_t]$. Then $x \in (f_t S]$ and $x \in (S f_t]$, and we have $x \leq yz$ and $x \leq y_1 z_1$ for some $y, z_1 \in f_t$ and $z, y_1 \in S$. Then $(y, z) \in A_x$ and $(y_1, z_1) \in A_x$. By hypothesis, we have

$$\begin{aligned} f(x) &\geq ((f \circ S) \cap (S \circ f))(x) = \min\{(f \circ S)(x), (S \circ f)(x)\} \\ &= \min\{\bigvee_{(p,q) \in A_x} \min\{f(p), S(q)\}, \bigvee_{(u,v) \in A_x} \min\{S(u), f(v)\}\} \\ &\geq \min\{\min\{f(y), S(z)\}, \min\{S(y_1), f(z_1)\}\} \\ &= \min\{\min\{f(y), 1\}, \min\{1, f(z_1)\}\} \\ &= \min\{f(y), f(z_1)\}. \end{aligned}$$

Since $y, z_1 \in f_t$, we have $f(y) \geq t$ and $f(z_1) \geq t$. Then

$$f(x) \geq \min\{f(y), f(z_1)\} \geq t,$$

and so $x \in f_t$. Thus $(f_tS] \cap (Sf_t] \subseteq f_t$. Furthermore, let $x \in f_t, S \ni y \leq x$. Then $y \in f_t$. Indeed: Since $x \in f_t, f(x) \geq t$, and f is a fuzzy quasi-ideal of S, we have $f(y) \geq f(x) \geq t$, so $y \in f_t$. Therefore, f_t is a quasi-ideal of S.

$\Leftarrow$. Suppose that for every $t \in (0,1]$ such that $f_t \neq \emptyset$ the set f_t is a quasi-ideal of S. Let $x \in S$. Then $((f \circ S) \cap (S \circ f))(x) \leq f(x)$. In fact, if $f(x) < ((f \circ S) \cap (S \circ f))(x)$, then there exists $t \in (0,1)$ such that

$$f(x) < t < ((f \circ S) \cap (S \circ f))(x) = \min\{(f \circ S)(x), (S \circ f)(x)\},$$

and so $(f \circ S)(x) > t$ and $(S \circ f)(x) > t$. Then

$$\bigvee_{(p,q) \in A_x} \min\{f(p), S(q)\} > t \text{ and } \bigvee_{(p,q) \in A_x} \min\{S(p), f(q)\} > t,$$

which implies that there exist $a, b, c, d \in S$ with $(a,b) \in A_x$ and $(c,d) \in A_x$ such that $f(a) > t$ and $f(d) > t$. Then $a, d \in f_t$ and so $ab \in f_tS$, $cd \in Sf_t$. Hence $x \in (f_tS]$ and $x \in (Sf_t]$. By hypothesis, $x \in (f_tS] \cap (Sf_t] \subseteq f_t$, and so $x \in f_t$. Then $f(x) \geq t$. This is a contradiction. Thus, $(f \circ S) \cap (S \circ f) \subseteq f$. Moreover, let $x, y \in S$. If $x \leq y$, then $f(x) \geq f(y)$. Indeed: Let $\lambda = f(y)$. Then $y \in f_\lambda$. Since f_λ is a quasi-ideal of S, we have $x \in f_\lambda$. Then $f(x) \geq \lambda = f(y)$. Therefore, f is a fuzzy quasi-ideal of S.

Example 3.3 We consider the ordered semigroup $S = \{a, b, c, d, e\}$ defined by the following multiplication "$\cdot$" and the order "$\leq$":

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

$$\leq := \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (c,c), (d,d), (e,e)\}.$$

Then all quasi-ideals of S are:

$$\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,d\}, \{a,c,d\}, \{a,b,e\}, \{a,c,f\} \text{ and } S \text{ (see [16])}.$$

Let f be a fuzzy subset of S such that $f(a) = 0.9, f(b) = 0.8, f(d) = 0.7, f(c) = f(e) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.6], \\ \{a,b,d\}, & \text{if } t \in (0.6, 0.7], \\ \{a,b\}, & \text{if } t \in (0.7, 0.8], \\ \{a\}, & \text{if } t \in (0.8, 0.9], \\ \emptyset, & \text{if } t \in (0.9, 1]. \end{cases}$$

Thus all nonempty level subsets f_t ($t \in (0,1]$) of f are quasi-ideals of S and by Theorem 3.2, f is a fuzzy quasi-ideal of S.

4. Characterizations of strongly regular ordered semigroups

An ordered semigroup $(S,\cdot,\leq)$ is called left (resp., right) regular if for each $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp., $a \leq a^2x$), i.e., $a \in (Sa^2]$ (resp., $a \in (a^2S]$) (see [4]). An ordered semigroup $(S,\cdot,\leq)$ is called strongly regular if it is both left regular and right regular. Clearly, an ordered semigroup S is strongly regular if and only if $a \in (Sa^2] \cap (a^2S]$ for every $a \in S$.

Definition 4.1 *Let S be an ordered semigroup and f a fuzzy quasi-ideal of S. Then f is called completely semiprime fuzzy quasi-ideal of S if $f(a) \geq f(a^2)$ for any $a \in S$.*

Theorem 4.2 *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) S is strongly regular;

(2) For each fuzzy quasi-ideal f of S, $f(a) = f(a^2)$ for any $a \in S$;

(3) For each fuzzy quasi-ideal f of S, $f(a^n) = f(a^{n+1})$ for any $a \in S, n \in Z^+$;

(4) Every fuzzy quasi-ideal f of S is completely semiprime.

Proof (1) $\Rightarrow$ (2). Let S be a strongly regular ordered semigroup and f a fuzzy quasi-ideal of S and let $a \in S$. Since S is left and right regular, we have $a \in (Sa^2]$ and $a \in (a^2S]$. Then there exist $x, y \in S$ such that $a \leq xa^2$ and $a \leq a^2y$. Then $(x,a^2),(a^2,y) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} f(a) &\geq ((f \circ S) \cap (S \circ f))(a) = \min\{(f \circ S)(a), (S \circ f)(a)\} \\ &= \min\{\bigvee_{(p,q)\in A_a} \min\{f(p), S(q)\}, \bigvee_{(u,v)\in A_a} \min\{S(u), f(v)\}\} \\ &\geq \min\{\min\{f(a^2), S(y)\}, \min\{S(x), f(a^2)\}\} \\ &= \min\{\min\{f(a^2), 1\}, \min\{1, f(a^2)\}\} \\ &= f(a^2) = f(aa) \geq \min\{f(a), f(a)\} = f(a), \end{aligned}$$

which implies that $f(a) = f(a^2)$.

(2) $\Rightarrow$ (3). Let $a \in S$ and $n \in Z^+$. Then

$$\begin{aligned} (f \circ S)(a^{4n}) &= \bigvee_{(x,y)\in A_{a^{4n}}} \min\{f(x), S(y)\} \\ &\geq \min\{f(a^{n+1}), S(a^{3n-1})\} \\ &= \min\{f(a^{n+1}), 1\} = f(a^{n+1}). \end{aligned}$$

Similarly, it can be shown that $(S \circ f)(a^{4n}) \geq f(a^{n+1})$. Therefore, by (2), we have

$$\begin{aligned} f(a^n) = f(a^{2n}) = f(a^{4n}) &\geq ((f \circ S) \cap (S \circ f))(a^{4n}) \\ &= \min\{(f \circ S)(a^{4n}), (S \circ f)(a^{4n})\} \\ &\geq \min\{f(a^{n+1}), f(a^{n+1})\} = f(a^{n+1}). \end{aligned}$$

On the other hand, let $a \in S$. Since f is a fuzzy quasi-ideal of S, we have

$$(f \circ S)(a^{n+1}) = \bigvee_{(x,y)\in A_{a^{n+1}}} \min\{f(x), S(y)\} \geq \min\{f(a^n), S(a)\} = f(a^n)$$

for any $n \in Z^+$. Similarly, $(S \circ f)(a^{n+1}) \geq f(a^n)$. Therefore, $f(a^{n+1}) \geq ((f \circ S) \cap (S \circ f))(a^{n+1}) \geq f(a^n)$.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1). Let $a \in S$. We consider the quasi-ideal $Q(a^2)$ of S generated by a^2. By Lemma 2.2, $f_{Q(a^2)}$ is a fuzzy quasi-ideal of S. By hypothesis, we have $f_{Q(a^2)}(a) = f_{Q(a^2)}(a^2)$. Since $a^2 \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$, we have $f_{Q(a^2)}(a^2) = 1$, then $f_{Q(a^2)}(a) = 1$, and we have

$$a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)].$$

Then $a \leq a^2$ or $a \in (a^2S \cap Sa^2] \subseteq (a^2S] \cap (Sa^2]$. If $a \leq a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2 \subseteq (a^2S] \cap (Sa^2]$. Thus, S is strongly regular.

5. Characterizations of left and right simple ordered semigroups

An ordered semigroup S is called left (resp., right) simple if for every left (resp., right) ideal A of S, we have $A = S$ (see [3]).

Lemma 5.1 ([3]) *Let S be an ordered semigroup. Then S is left (resp., right) simple if and only if $(Sa] = S$ (resp., $(aS] = S$) for every $a \in S$.*

An ordered semigroup $(S, \cdot, \leq)$ is called *regular* if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalently, $a \in (aSa]$, for any $a \in S$ (see [5]).

Lemma 5.2 *Let S be an ordered semigroup. If S is left and right simple, then S is regular.*

Proof Let $a \in S$. By hypothesis, $S = (aS] = (Sa]$. Then we have

$$a \in (aS] = (a(Sa]] = (aSa].$$

It thus follows that S is regular.

Theorem 5.3 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is left and right simple if and only if every fuzzy quasi-ideal of S is a constant function.*

Proof Suppose that S is a left and right simple ordered semigroup. Let f be a fuzzy quasi-ideal of S and $a \in S$. We consider the set

$$E_S := \{e \in S | e^2 \geq e\}.$$

Then E_S is nonempty. Indeed: By Lemma 5.2, there exists $x \in S$ such that $a \leq axa$. Then we have $(ax)^2 = (axa)x \geq ax$, and so $ax \in E_S$.

(1) f is a constant mapping on E_S. Indeed: Let $t \in E_S$. Then $f(e) = f(t)$ for every $e \in E_\Omega$. In fact, since S is left and right simple, we have $(St] = S$ and $(tS] = S$. Since $e \in S$, we have

$e \in (St]$ and $e \in (tS]$, so there exist $x, y \in S$ such that $e \leq xt$ and $e \leq ty$. Hence

$$e^2 = ee \leq (xt)(xt) = (xtx)t \text{ and } e^2 = ee \leq (ty)(ty) = t(yty),$$

and we have $(xtx, t) \in A_{e^2}$ and $(t, yty) \in A_{e^2}$. Since f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(e^2) &\geq ((f \circ S) \cap (S \circ f))(e^2) = \min\{(f \circ S)(e^2), (S \circ f)(e^2)\} \\ &= \min\{ \bigvee_{(p,q) \in A_{e^2}} \min\{f(p), S(q)\}, \bigvee_{(u,v) \in A_{e^2}} \min\{S(u), f(v)\}\} \\ &\geq \min\{\min\{f(t), S(yty)\}, \min\{S(xtx), f(t)\}\} \\ &= \min\{\min\{f(t), 1\}, \min\{1, f(t)\}\} = f(t). \end{aligned}$$

Since $e \in E_S$, it follows that $e^2 \geq e$ and f is a fuzzy quasi-ideal of S, and we have $f(e) \geq f(e^2)$. Thus $f(e) \geq f(t)$. On the other hand, since S is left and right simple and $e \in S$, we have $(St] = S$ and $(tS] = S$. Since $t \in E_S \subseteq S$, as in the previous case, we also have $f(t) \geq f(t^2) \geq f(e)$.

(2) f is a constant mapping on S. Indeed: Let $a \in S$. Then $f(t) = f(a)$ for every $t \in E_S$. In fact, since S is regular, there exists $x \in S$ such that $a \leq axa$. Then

$$(ax)^2 = (axa)x \geq ax \text{ and } (xa)^2 = x(axa) \geq xa,$$

which implies that $ax, xa \in E_S$. Then by (1) we have $f(ax) = f(t)$ and $f(xa) = f(t)$. Since $(ax)(axa) \geq axa \geq a$, and $(axa)(xa) \geq axa \geq a$, we have, $(ax, axa) \in A_a$ and $(axa, xa) \in A_a$. Since f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(a) &\geq ((f \circ S) \cap (S \circ f))(a) = \min\{(f \circ S)(a), (S \circ f)(a)\} \\ &= \min\{ \bigvee_{(p,q) \in A_a} \min\{f(p), S(q)\}, \bigvee_{(u,v) \in A_a} \min\{S(u), f(v)\}\} \\ &\geq \min\{\min\{f(ax), S(axa)\}, \min\{S(axa), f(xa)\}\} \\ &= \min\{\min\{f(t), 1\}, \min\{1, f(t)\}\} = f(t). \end{aligned}$$

On the other hand, since S is left and right simple, we have $(Sa] = S, (aS] = S$. Since $t \in S$, $t \in (Sa]$ and $t \in (aS]$. Then $t \leq ua$ and $t \leq av$ for some $u, v \in S$. Then $(u, a) \in A_t$ and $(a, v) \in A_t$. Since f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(t) &\geq ((f \circ S) \cap (S \circ f))(t) = \min\{(f \circ S)(t), (S \circ f)(t)\} \\ &= \min\{ \bigvee_{(y_1,z_1) \in A_t} \min\{f(y_1), S(z_1)\}, \bigvee_{(y_2,z_2) \in A_t} \min\{S(y_2), f(z_2)\}\} \\ &\geq \min\{\min\{f(a), S(v)\}, \min\{S(u), f(a)\}\} \\ &= \min\{\min\{f(a), 1\}, \min\{1, f(a)\}\} = f(a). \end{aligned}$$

Summarizing two cases above, we have shown that f is a constant function.

Conversely, let $a \in S$. Since the set $(Sa]$ is a left ideal of S, and so $(Sa]$ is a quasi-ideal of S. By Lemma 2.2, the characteristic function $f_{(Sa]}$ of $(Sa]$ is a fuzzy quasi-ideal of S. By hypothesis, $f_{(Sa]}$ is a constant function, that is, there exists $c \in \{0, 1\}$ such that $f_{(Sa]}(x) = c$ for every $x \in S$. Let $(aS] \subset S$ and t be an element of S such that $t \notin (aS]$. Then $f_{(aS]}(t) = 0$.

Also, since $a^2 \in (Sa]$, $f_{(Sa]}(a^2) = 1$, leading to a contradiction to the fact that $f_{(Sa]}$ is a constant function. Thus $(Sa] = S$. Similarly, we can prove that $(aS] = S$. Therefore, by Lemma 5.1, S is left and right simple.

An equivalence relation ρ on an ordered semigroup S is called congruence if $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$ for every $c \in S$. A congruence ρ on S is called semilattice congruence on S, if $(a, a^2) \in \rho$ and $(ab, ba) \in \rho$ for all $a, b \in S$ (see [17]). An ordered semigroup S is called a semilattice of left and right simple semigroups if there exists a semilattice congruence ρ on S such that the ρ-class $(x)_\rho$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$. Equivalently, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left and right simple subsemigroups of S such that:

(i) $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha, \beta \in Y, \alpha \neq \beta$;

(ii) $S = \bigcup_{\alpha\in Y} S_\alpha$;

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$.

Definition 5.4 ([17]) *A subsemigroup F of an ordered semigroup S is called a filter of S if (1) $a, b \in S, ab \in F \;\Rightarrow\; a \in F$ and $b \in F$; (2) $a \in F$, $S \ni c \geq a \;\Rightarrow\; c \in F$.*

We denote by $N(a)$ the filter of S generated by a ($a \in S$), and by " $\mathcal{N}$ " the equivalence relation on S defined by $a\mathcal{N}b$ if and only if $N(a) = N(b)$. $\mathcal{N}$ is a semilattice congruence on S (see [17]).

A subset T of an ordered semigroup S is called completely semiprime if for every $a \in S$ such that $a^2 \in T$ we have $a \in T$ (see [9]).

Lemma 5.5 ([3]) *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) $(x)_\mathcal{N}$ is a left (resp., right) simple subsemigroup of S, for every $x \in S$;

(2) Every left (resp., right) ideal of S is a right (resp., left) ideal of S and completely semiprime.

Lemma 5.6 ([3]) *An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left and right simple semigroups if and only if $(A] = A$ and $(AB] = (BA]$ for all quasi-ideals A, B of S.*

Lemma 5.7 ([5]) *An ordered semigroup $(S, \cdot, \leq)$ is regular if and only if $(RL] = R \cap L$ for every right ideal R and left ideal L of S.*

Theorem 5.8 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is a semilattice of left and right simple semigroups if and only if for every fuzzy quasi-ideal f of S, we have*

$$f(a) = f(a^2) \text{ and } f(ab) = f(ba) \text{ for all } a, b \in S.$$

Proof $\Longrightarrow$. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left and right simple subsemigroups of S such that:

(i) $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha, \beta \in Y, \alpha \neq \beta$;

(ii) $S = \cup_{\alpha\in Y} S_\alpha$;

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$.

Let f be a fuzzy quasi-ideal of S. Then we have

(1) Let $a \in S$. Then $f(a) = f(a^2)$. Indeed: By Theorem 4.2, it suffices to prove that S is strongly regular. Since $a \in S = \cup_{\alpha \in Y} S_\alpha$, there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left and right simple, we have $S_\alpha = (S_\alpha a]$ and $S_\alpha = (aS_\alpha]$. Then we have $(aS_\alpha] = (a(S_\alpha a]] = (aS_\alpha a]$. Since $a \in S_\alpha$, we have $a \in (aS_\alpha a]$, then there exists $x \in S_\alpha$ such that $a \leq axa$. Since $x \in (aS_\alpha a]$, there exists $y \in S_\alpha$ such that $x \leq aya$. Thus

$$a \leq axa \leq a(aya)a = a^2ya^2 \in a^2S_\alpha a^2 \subseteq a^2Sa^2 \subseteq (a^2S] \cap (Sa^2],$$

which implies that S is strongly regular.

(2) Let $a, b \in S$. Then $f(ab) = f(ba)$. Indeed: By (1), we have $f(ab) = f((ab)^2) = f((ab)^4)$. Moreover, we have

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \subseteq (Q(aba)Q(babab)] \\
&= (Q(babab)Q(aba)] \text{ (by Lemma 5.6)} \\
&= ((babab \cup (bababS \cap Sbabab)](aba \cup (abaS \cap Saba)]] \\
&= ((babab \cup (bababS \cap Sbabab))(aba \cup (abaS \cap Saba))] \\
&\subseteq ((babab \cup bababS)(aba \cup Saba)] \\
&\subseteq ((baS)(Sba)] = ((baS](Sba]] \\
&= (baS] \cap (Sba] \text{ (by Lemmas 5.2 and 5.7)}.
\end{aligned}$$

Then there exist $x, y \in S$ such that $(ab)^4 \leq (ba)x$ and $(ab)^4 \leq y(ba)$, i.e., $(ba, x) \in A_{(ab)^4}$ and $(y, ba) \in A_{(ab)^4}$. Since f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned}
f((ab)^4) &\geq ((f \circ S) \cap (S \circ f))((ab)^4) = \min\{(f \circ S)((ab)^4), (S \circ f)((ab)^4)\} \\
&= \min\{\bigvee_{(p,q) \in A_{(ab)^4}} \min\{f(p), S(q)\}, \bigvee_{(u,v) \in A_{(ab)^4}} \min\{S(u), f(v)\}\} \\
&\geq \min\{\min\{f(ba), S(x)\}, \min\{S(y), f(ba)\}\} \\
&= \min\{\min\{f(ba), 1\}, \min\{1, f(ba)\}\} = f(ba).
\end{aligned}$$

It thus follows that $f(ab) = f((ab)^4) \geq f(ba)$. In a similar way, we can prove that $f(ba) \geq f(ab)$.

$\Longleftarrow$. Since $\mathcal{N}$ is a semilattice congruence on S, by Lemma 5.5, it suffices to prove that every left (resp., right) ideal of S is an ideal of S and completely semiprime. Let L be a left ideal of S and hence a quasi-ideal of S. By Lemma 2.2, f_L is a fuzzy quasi-ideal of S. Let $a \in L, b \in S$. Then, by hypothesis, we have $f_L(ab) = f_L(ba) = 1$, which implies that $ab \in L$. Thus $LS \subseteq L$ and if $a \in L, S \ni b \leq a$, then $b \in L$. Thus L is a right ideal of S. Hence L is an ideal of S. Let $x \in S$ such that $x^2 \in L$. Then $x \in L$. Indeed: Since L is a quasi-ideal of S, by Lemma 2.2, f_L is a fuzzy quasi-ideal of S. By hypothesis, $f_L(x^2) = f_L(x)$. Since $x^2 \in L$, we have $f_L(x^2) = 1$. Thus we have $f_L(x) = 1$, and $x \in L$. Hence L is completely semiprime. In a similar way we can prove that every right ideal of S is an ideal and completely semiprime.

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