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Positive Solutions for a Second-Order Nonlinear Impulsive Singular Integro-Differential Equation with Integral Conditions in Banach Spaces

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Abstract The existence of positive solutions to a boundary value problem of second-order impulsive singular integro-differential equation with integral boundary conditions in a Banach space is obtained by means of fixed point theory. Moreover, an application is also given to illustrate the main result.

Keywords impulsive singular integro-differential equation; positive solution; Mönch fixed point theorem; measure of noncompactness.

MR(2010) Subject Classification 34B16; 34B18; 34B37

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years [1–3]. Processes which experience a sudden change of their state at certain moments arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. They can be described by impulsive differential equations in R^n and cannot be described using classical differential equations. But the corresponding theory for impulsive integro-differential equations in Banach spaces has yet to be fully developed. Due to the difficulties brought by singularities, there are few results for differential equations with singularities in Banach spaces [4–9]. In recent papers [4, 5], Guo obtained the existence of positive solutions for some nth-order nonlinear impulsive singular integro-differential equations in Banach spaces by using Schauder fixed point theorem.

Moreover, the boundary value problem with integral boundary conditions has been the subject of investigations along the line with impulsive differential equations because of their wide applicability in various fields such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [10–13] and the references therein. For more information about the general theory of integral equations

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and their relation with boundary value problems we refer to the book of Corduneanu [14] and Agarwal and O'Regan [15].

In [16], Boucherif investigated the existence of positive solutions to the following boundary value problem

$$\begin{cases} y''(t) = f(t, y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) = \int_0^1 g_0(s)y(s)\mathrm{d}s, \\ y(1) - by'(1) = \int_0^1 g_1(s)y(s)\mathrm{d}s, \end{cases}$$

in the scalar space, where $f : [0,1] \times R \to R$ is continuous, $g_0,\ g_1 : [0,1] \to [0,+\infty)$ are continuous and positive, a and b are nonnegative real parameters.

In [17], when nonlinearity f is continuous, by means of the fixed point index theory of strict set contraction operators, Lv et al studied the existence of multiple positive solutions of the following second-order impulsive differential equations with integral boundary conditions in a real Banach space E

$$\begin{cases} x'' = f(t, x, x', Tx, Sx), \quad t \in J,\ t \neq t_k, \\ \Delta x|_{t=t_k} = -I_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) - ax'(0) = \theta, \\ x(1) - bx'(1) = \int_0^1 g(s)x(s)\mathrm{d}s, \end{cases}$$

where $a + 1 > b > 1$, $J = [0,1]$, $J' = J \backslash \{t_1, \dots, t_m\}$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, θ denotes the zero element of Banach space E.

To the author's knowledge, few papers are available for the existence of positive solutions to impulsive singular integro-differential equation with integral boundary conditions in Banach spaces. Motivated by papers [4, 5, 16, 17], in this paper, we are concerned with the existence of positive solution for the following second-order impulsive integro-differential equations with integral boundary conditions in a real Banach space E

$$\begin{cases} x''(t) = f(t, x(t), x'(t), (Tx)(t), (Sx)(t)), \quad t \in J'_+, \\ \Delta x|_{t=t_k} = -I_{0k}(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta x'|_{t=t_k} = I_{1k}(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) - ax'(0) = x_0, \\ x(1) - bx'(1) = \int_0^1 g(s)x(s)\mathrm{d}s, \end{cases} \tag{1}$$

where $a + 1 > b > 1$, $J = [0,1]$, $J_+ = (0,1)$, $J'_+ = J_+ \backslash \{t_1, \dots, t_m\}$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, f may be singular at $t = 0, 1$ and $x = \theta$ or $x' = \theta$, I_{ik} $(i = 0, 1)$ may be singular at $x = \theta$ or $x' = \theta$, θ denotes the zero element of Banach space E. By singularity, we mean that $\|f(t, x_0, x_1, x_2, x_3)\| \to \infty$ as $t \to 0^+$ (1^-) or $x_i \to \theta$ $(i = 0, 1)$. T and S are the linear operators defined as follows

$$(Tx)(t) = \int_0^t k(t,s)x(s)\mathrm{d}s, \quad (Sx)(t) = \int_0^1 h(t,s)x(s)\mathrm{d}s,$$

in which $k \in C[D, R_+]$, $h \in C[D_0, R_+]$, $D = \{(t,s) \in J \times J : t \geq s\}$, $D_0 = \{(t,s) \in J \times J : 0 \leq t, s \leq 1\}$, $R_+ = [0, +\infty)$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$, $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.

2. Preliminaries and several lemmas

Let $PC[J, E] = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+)$ exists for $k = 1, 2, 3, \ldots, m\}$ and $PC^1[J, E] = \{x \in PC[J, E] : x'(t)$ is continuous at $t \neq t_k$, and $x'(t_k^+)$, $x'(t_k^-)$ exist for $k = 1, 2, 3, \ldots, m\}$. Clearly, $PC[J, E]$ is a Banach space with the norm $\|x\|_{PC} = \sup_{t \in J} \|x(t)\|$ and $PC^1[J, E]$ is also a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Let P be a normal cone in E with normal constant N which defines a partial ordering in E by $x \leq y$. If $x \leq y$ and $x \neq y$, we write $x < y$. Let $P_+ = P \backslash \{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, we refer to [18].

In what follows, we always assume that $x_0 \in -P \backslash \{\theta\}$, $-x_0 \geq x^*$, $x^* \in P_+$. Let $P_{0\lambda} = \{x \in P : x \geq \lambda x^*\}(\lambda > 0)$. Obviously, $P_{0\lambda} \subset P_+$ for any $\lambda > 0$. By a positive solution of BVP (1), we mean a map $x \in PC^1[J, E] \cap C^2[J_+', E]$ such that $x^{(i)}(t) > \theta$ $(i = 0, 1)$ for $t \in J$ and $x(t)$ satisfies (1).

Let α, α_{PC^1} be the Kuratowski measure of non-compactness in E and $PC^1[J, E]$, respectively. For details on the definition and properties of the measure of noncompactness, the reader is referred to [19]. We set $J_1 = [0, t_1]$, $J_k = (t_{k-1}, t_k]$ $(k = 2, 3, \ldots, m)$, $R_+ = [0, +\infty)$, $R^+ = (0, +\infty)$, $u = \int_0^1 (a + s) g(s) \mathrm{d}s$. For notational simplicity, denote

$$D_0 = \frac{a}{a + 1 - b - u} \cdot \int_0^1 g(\tau) \frac{b - 1 + \tau}{a + 1 - b} \mathrm{d}\tau + \frac{b - 1}{a + 1 - b},$$

$$D_1 = \frac{1}{a + 1 - b - u} \cdot \int_0^1 g(\tau) \frac{b - 1 + \tau}{a + 1 - b}\, d\tau + \frac{1}{a + 1 - b},\ \lambda^* = \min\{D_0, D_1\}.$$

Throughout this paper, we make the following assumptions.

(H$_0$) $g \in L^1[0, 1]$ is nonnegative, and $u \in [0, a + 1 - b)$;

(H$_1$) $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P \times P, P]$ for any $\lambda > 0$ and there exist $a, b, c \in L[J_+, R_+]$ and $h \in C[R^+ \times R^+ \times R_+ \times R_+, R_+]$ such that

$$\|f(t, x_0, x_1, x_2, x_3)\| \leq a(t) + b(t) h(\|x_0\|, \|x_1\|, \|x_2\|, \|x_3\|), \quad \forall\ t \in J_+, x_0, x_1 \in P_{0\lambda^*}, x_2, x_3 \in P,$$

and

$$\frac{\|f(t, x_0, x_1, x_2, x_3)\|}{c(t)(\|x_0\| + \|x_1\| + \|x_2\| + \|x_3\|)} \to 0 \text{ as } x_0, x_1 \in P_{0\lambda^*}, x_2, x_3 \in P,$$

$$\|x_0\| + \|x_1\| + \|x_2\| + \|x_3\| \to \infty,$$

uniformly for $t \in J_+$, and

$$\int_0^1 a(t) \mathrm{d}t = a^* < \infty,\ \int_0^1 b(t) \mathrm{d}t = b^* < \infty,\ \int_0^1 c(t) \mathrm{d}t = c^* < \infty.$$

(H$_2$) $I_{ik} \in C[P_{0\lambda} \times P_{0\lambda}, P]$ for any $\lambda > 0$ $(i = 0, 1; k = 1, 2, \ldots, m)$ and there exist $F_i \in C[R^+ \times R^+, R_+]$ and constants η_{ik}, γ_{ik} $(i = 0, 1; k = 1, 2, \ldots, m)$ such that

$$\|I_{ik}(x_0, x_1)\| \leq \eta_{ik} F_i(\|x_0\|, \|x_1\|), \quad \forall x_0, x_1 \in P_{0\lambda^*},$$

and

$$\frac{\|I_{ik}(x_0,x_1)\|}{\gamma_{ik}(\|x_0\|+\|x_1\|)} \to 0 \ \text{ as } x_0,x_1 \in P_{0\lambda^*},\ \|x_0\|+\|x_1\| \to \infty$$

uniformly for $k=1,2,\dots,m$ $(i=0,1)$. We write

$$\eta_i^* = \sum_{k=1}^m \eta_{ik}, \quad \gamma_i^* = \sum_{k=1}^m \gamma_{ik}.$$

(H$_3$) For any $t\in J_+, R>0$ and countable bounded sets $V_i \subset C[J,P^*_{0\lambda^*R}]$ $(i=0,1), V_2,V_3 \subset C[J,P^*_R]$, there exist $L_i(t)\in L[J,R_+]$ $(i=0,1,2,3)$ and positive numbers d_{ikj} $(i,j=0,1;k=1,2,\dots,m)$ such that

$$\alpha(f(t,V_0(t),V_1(t),V_2(t),V_3(t))) \le \sum_{i=0}^3 L_i(t)\alpha(V_i(t)),$$

$$\alpha(I_{ik}(V_0(t),V_1(t))) \le d_{ik0}\alpha(V_0(t)) + d_{ik1}\alpha(V_1(t), \quad i=0,1;k=1,2,\dots,m,$$

where $P^*_{0\lambda^*R} = \{x\in P: x\ge \lambda^* x^*, \|x\|\le R\}$ and $P^*_R=\{x\in P:\|x\|\le R\}$, $d_i = \sum_{k=1}^m\sum_{j=0}^1 d_{ikj}$.

Hereafter, we write $Q=\{x\in PC^1[J,P]: x^{(i)}(t)\ge \lambda^*x^*, \forall t\in J, i=0,1\}$. Evidently, Q is a closed convex set in $PC^1[J,E]$.

Lemma 1 *Let (H$_1$) and (H$_2$) hold. Then $x\in PC^1[J,E]\cap C^2[J'_+,E]$ is a solution to (1) if and only if $x\in PC^1[J,E]\cap C^2[J'_+,E]$ is a solution to the following impulsive integral equation:*

$$\begin{aligned} x(t) = &\int_0^1 H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{k=1}^m H_1(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \\ &\sum_{k=1}^m H_2(t,t_k)I_{0k}(x(t_k),x'(t_k)) - \Big(\frac{a+t}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \\ &\frac{b-1+t}{a+1-b}\Big)x_0, \end{aligned} \tag{2}$$

where

$$H_1(t,s) = G_1(t,s) + \frac{a+t}{a+1-b-u}\int_0^1 G_1(\tau,s)g(\tau)\mathrm{d}\tau,$$

$$H_2(t,s) = G_2(t,s) + \frac{a+t}{a+1-b-u}\int_0^1 G_2(\tau,s)g(\tau)\mathrm{d}\tau,$$

$$G_1(t,s) = \begin{cases} \dfrac{1}{a+1-b}(a+t)(b+s-1), & t\le s,\\ \dfrac{1}{a+1-b}(a+s)(b+t-1), & s\le t,\end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{a+t}{a+1-b}, & t\le s,\\ \dfrac{b+t-1}{a+1-b}, & s\le t.\end{cases}$$

Proof Necessity. Suppose that $x \in PC^1[J,E] \cap C^2[J'_+,E]$ is a solution to problem (1). For $t \in J$, integrating (1) from 0 to t, we have

$$x'(t) = x'(0) + \int_0^t f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{0<t_k<t} I_{1k}(x(t_k),x'(t_k)).$$

Integrating again, we can get

$$\begin{aligned} x(t) =& x(0) + tx'(0) + \int_0^t (t-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \sum_{0<t_k<t} (t-t_k)I_{1k}(x(t_k),x'(t_k)) - \sum_{0<t_k<t} I_{0k}(x(t_k),x'(t_k)). \end{aligned} \tag{3}$$

In particular,

$$x'(1) = x'(0) + \int_0^1 f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{0<t_k<1} I_{1k}(x(t_k),x'(t_k)),$$

and

$$\begin{aligned} x(1) =& x(0) + x'(0) + \int_0^1 (1-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \sum_{0<t_k<1} (1-t_k)I_{1k}(x(t_k),x'(t_k)) - \sum_{0<t_k<1} I_{0k}(x(t_k),x'(t_k)). \end{aligned}$$

This, together with the boundary condition that $x(0) = ax'(0) + x_0$, yields

$$\begin{aligned} x'(0) =& \frac{1}{a+1-b}\Big(\int_0^1 (b+s-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \sum_{0<t_k<1} (b+t_k-1)I_{1k}(x(t_k),x'(t_k)) + \sum_{0<t_k<1} I_{0k}(x(t_k),x'(t_k)) + \\ & \int_0^1 g(s)x(s)\mathrm{d}s - x_0\Big), \end{aligned}$$

which implies that

$$\begin{aligned} x(t) =& \frac{a+t}{a+1-b}\Big(\int_0^1 (b+s-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \sum_{0<t_k<1} (b+t_k-1)I_{1k}(x(t_k),x'(t_k)) + \sum_{0<t_k<1} I_{0k}(x(t_k),x'(t_k)) + \\ & \int_0^1 g(s)x(s)\mathrm{d}s - x_0\Big) + x_0 + \int_0^t (t-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \sum_{0<t_k<t} (t-t_k)I_{1k}(x(t_k),x'(t_k)) - \sum_{0<t_k<t} I_{0k}(x(t_k),x'(t_k)) \\ =& \frac{1}{a+1-b}\int_0^t (a+s)(b+t-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \frac{1}{a+1-b}\int_t^1 (a+t)(b+s-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \\ & \frac{1}{a+1-b}\sum_{0<t_k<t} (a+t_k)(b+t-1)I_{1k}(x(t_k),x'(t_k)) + \end{aligned}$$

$$\frac{1}{a+1-b}\sum_{t\le t_k<1}(a+t)(b+t_k-1)I_{1k}(x(t_k),x'(t_k))+$$
$$\frac{1}{a+1-b}\sum_{0<t_k<t}(b+t-1)I_{0k}(x(t_k),x'(t_k))+$$
$$\frac{1}{a+1-b}\sum_{t\le t_k<1}(a+t)I_{0k}(x(t_k),x'(t_k))+\frac{a+t}{a+1-b}\int_0^1 g(s)x(s)\mathrm{d}s-$$
$$\frac{b-1+t}{a+1-b}x_0.$$

Thus,

$$x(t)=\int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\sum_{k=1}^m G_1(t,t_k)I_{1k}(x(t_k),x'(t_k))+$$
$$\sum_{k=1}^m G_2(t,t_k)I_{0k}(x(t_k),x'(t_k))+\frac{a+t}{a+1-b}\int_0^1 g(s)x(s)\mathrm{d}s-\frac{b-1+t}{a+1-b}x_0.$$

By (H_1), it is easy to see that the integral $\int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s$ is convergent. On the other hand,

$$\begin{aligned}\int_0^1 g(t)x(t)\mathrm{d}t=&\int_0^1 g(t)\Big(\int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\\&\sum_{k=1}^m G_1(t,t_k)I_{1k}(x(t_k),x'(t_k))-\frac{b-1+t}{a+1-b}x_0+\\&\sum_{k=1}^m G_2(t,t_k)I_{0k}(x(t_k),x'(t_k))+\frac{a+t}{a+1-b}\int_0^1 g(s)x(s)\mathrm{d}s\Big)\mathrm{d}t,\\=&\int_0^1\int_0^1 g(t)G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s\mathrm{d}t+\\&\int_0^1 g(t)\Big(\sum_{k=1}^m G_1(t,t_k)I_{1k}(x(t_k),x'(t_k))\Big)\mathrm{d}t-\int_0^1 g(t)\cdot\frac{b-1+t}{a+1-b}\mathrm{d}t\cdot x_0+\\&\int_0^1 g(t)\Big(\sum_{k=1}^m G_2(t,t_k)I_{0k}(x(t_k),x'(t_k))\Big)\mathrm{d}t+\int_0^1\frac{a+t}{a+1-b}g(t)\mathrm{d}t\int_0^1 g(t)x(t)\mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 g(s)x(s)\mathrm{d}s=&\frac{1}{1-\int_0^1\frac{a+s}{a+1-b}g(s)\mathrm{d}s}\Big(\int_0^1\Big(\int_0^1 G_1(\tau,s)g(\tau)\mathrm{d}\tau\Big)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\\&\int_0^1 g(\tau)(\sum_{k=1}^m G_1(\tau,t_k)I_{1k}(x(t_k),x'(t_k)))\mathrm{d}\tau-\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau\cdot x_0+\\&\int_0^1 g(\tau)(\sum_{k=1}^m G_2(\tau,t_k)I_{0k}(x(t_k),x'(t_k)))\mathrm{d}\tau\Big).\end{aligned}$$

Consequently, we have

$$x(t)=\int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+$$

$$\sum_{k=1}^{m} G_1(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^{m} G_2(t,t_k)I_{0k}(x(t_k),x'(t_k)) +$$
$$\frac{a+t}{a+1-b-u}\Big(\int_0^1 \Big(\int_0^1 G_1(\tau,s)g(\tau)\mathrm{d}\tau\Big) f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s +$$
$$\int_0^1 g(\tau)\Big(\sum_{k=1}^{m} G_1(\tau,t_k)I_{1k}(x(t_k),x'(t_k))\Big)\mathrm{d}\tau + \int_0^1 g(\tau)\Big(\sum_{k=1}^{m} G_2(\tau,t_k)I_{0k}(x(t_k),x'(t_k))\Big)\mathrm{d}\tau -$$
$$\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau\cdot x_0\Big) - \frac{b-1+t}{a+1-b}\cdot x_0$$
$$= \int_0^1 H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{k=1}^{m} H_1(t,t_k)I_{1k}(x(t_k),x'(t_k)) -$$
$$\Big(\frac{a+t}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \frac{b-1+t}{a+1-b}\Big)x_0 + \sum_{k=1}^{m} H_2(t,t_k)I_{0k}(x(t_k),x'(t_k)).$$

It is easy to see by (H_1) that the integral $\int_0^1 H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s$ is convergent.

Sufficiency. If $x\in PC^1[J,E]\cap C^2[J',E]$ is a solution of (2), then a direct differentiation of (2) yields, for $t\in J_k$ and $t\neq t_k$

$$x'(t) = \int_0^t \frac{a+s}{a+1-b} f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s +$$
$$\int_t^1 \frac{b+s-1}{a+1-b} f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s +$$
$$\sum_{0<t_k<t} \frac{a+t_k}{a+1-b} I_{1k}(x(t_k),x'(t_k)) + \sum_{t\le t_k<1} \frac{b+t_k-1}{a+1-b} I_{1k}(x(t_k),x'(t_k)) +$$
$$\frac{1}{a+1-b}\sum_{k=1}^{m} I_{0k}(x(t_k),x'(t_k)) + \frac{1}{a+1-b-u}$$
$$\Big(\int_0^1 \Big(\int_0^1 G_1(\tau,s)g(\tau)\mathrm{d}\tau\Big) f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s +$$
$$\int_0^1 g(\tau)(\sum_{k=1}^{m} G_1(\tau,t_k)I_{1k}(x(t_k),x'(t_k)))\mathrm{d}\tau +$$
$$\int_0^1 g(\tau)(\sum_{k=1}^{m} G_2(\tau,t_k)I_{0k}(x(t_k),x'(t_k)))\mathrm{d}\tau\Big) -$$
$$\Big(\frac{1}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \frac{1}{a+1-b}\Big)x_0.$$

Thus,

$$x'(t) = \int_0^1 H_1'(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s +$$
$$\sum_{k=1}^{m} H_1'(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^{m} H_2'(t,t_k)I_{0k}(x(t_k),x'(t_k)) -$$

$$\Big(\frac{1}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{1}{a+1-b}\Big)x_0,$$

where

$$H_1'(t,s)=G_1'(t,s)+\frac{1}{a+1-b-u}\int_0^1 G_1(\tau,s)g(\tau)\mathrm{d}\tau,$$

$$H_2'(t,s)=\frac{1}{a+1-b}+\frac{1}{a+1-b-u}\int_0^1 G_2(\tau,s)g(\tau)\mathrm{d}\tau,$$

$$G_1'(t,s)=\begin{cases}\dfrac{b+s-1}{a+1-b}, & t\le s,\\ \dfrac{a+s}{a+1-b}, & s\le t.\end{cases}$$

Differentiating above, we see

$$x''(t)=f(t,x(t),x'(t),(Tx)(t),(Sx)(t)).$$

Clearly,

$$\Delta x|_{t=t_k}=-I_{0k}(x(t_k),x'(t_k)),\quad \Delta x'|_{t=t_k}=I_{1k}(x(t_k),x'(t_k)),$$

$$x(0)-ax'(0)=x_0,\quad x(1)-bx'(1)=\int_0^1 g(s)x(s)\mathrm{d}s.$$

The proof is completed. □

Lemma 2 ([17]) *For $t,s\in[0,1]$, we have*

$$\frac{a(b-1)}{a+1-b}\le G_1(t,s)\le\frac{(a+1)b}{a+1-b},$$

$$\frac{b-1}{a+1-b}\le G_2(t,s)\le\frac{a+1}{a+1-b},$$

$$\frac{b-1}{a+1-b}\le G_1'(t,s)\le\frac{a+1}{a+1-b}.$$

Lemma 3 ([17]) *For $t,s\in[0,1]$, there exist positive constants $m_i,\overline{m}_i$ $(i=1,2,3,4)$ such that*

$$m_1=\frac{a(b-1)}{a+1-b}+\frac{a^2(b-1)u_1}{u_2}\le H_1(t,s)\le\frac{(a+1)b}{a+1-b}+\frac{(a+1)^2bu_1}{u_2}=m_2,$$

$$\overline{m}_1=\frac{b-1}{a+1-b}+\frac{a(b-1)u_1}{u_2}\le H_2(t,s)\le\frac{a+1}{a+1-b}+\frac{(a+1)^2u_1}{u_2}=\overline{m}_2,$$

$$m_3=\frac{b-1}{a+1-b}+\frac{a(b-1)u_1}{u_2}\le H_1'(t,s)\le\frac{a+1}{a+1-b}+\frac{(a+1)bu_1}{u_2}=m_4,$$

$$\overline{m}_3=\frac{1}{a+1-b}+\frac{(b-1)u_1}{u_2}\le H_2'(t,s)\le\frac{1}{a+1-b}+\frac{(a+1)u_1}{u_2}=\overline{m}_4,$$

where

$$u_1=\int_0^1 g(s)\mathrm{d}s,\quad u_2=(a+1-b-u)(a+1-b).$$

We shall reduce BVP (1) to an impulsive integral equation in E. To this end, we first consider operator A defined by

$$(Ax)(t)=\int_0^1 H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\sum_{k=1}^m H_1(t,t_k)I_{1k}(x(t_k),x'(t_k))+$$

$$\sum_{k=1}^{m} H_2(t,t_k)I_{0k}(x(t_k),x'(t_k)) - \Big(\frac{a+t}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \frac{b-1+t}{a+1-b}\Big)x_0. \tag{4}$$

Lemma 4 *If conditions* (H_0)–(H_2) *are satisfied, then operator* A *defined by (4) is a continuous operator from* Q *into* Q.

Proof Let

$$\varepsilon_0 = \min\Big\{\frac{1}{4m_2c^*(2+k^*+h^*)}, \frac{1}{4m_4c^*(2+k^*+h^*)}\Big\}, \tag{5}$$

where

$$k^* = \max\{k(t,s) : (t,s) \in D\}, \quad h^* = \max\{h(t,s) : (t,s) \in D_0\},$$

and

$$r = \frac{\lambda^*\|x^*\|}{N} > 0. \tag{6}$$

By virtue of condition (H_1), there exists an $R > r$ such that

$$\|f(t,x_0,x_1,x_2,x_3)\|\| \le \varepsilon_0 c(t)(\|x_0\|+\|x_1\|+\|x_2\|+\|x_3\|),$$

$$\forall\ t \in J_+, x_0, x_1 \in P_{0\lambda^*}, x_2, x_3 \in P, \|x_0\|+\|x_1\|+\|x_2\|+\|x_3\| > R,$$

and

$$\|f(t,x_0,x_1,x_2,x_3)\| \le a(t) + Mb(t),$$

$$\forall\ t \in J_+, x_0, x_1 \in P_{0\lambda^*}, x_2, x_3 \in P, \|x_0\|+\|x_1\|+\|x_2\|+\|x_3\| \le R,$$

where

$$M = \max\{h(x_0,x_1,x_2,x_3) : r \le x_i \le R\ (i=0,1), 0 \le x_2 \le R, 0 \le x_3 \le R\}.$$

Hence

$$\|f(t,x_0,x_1,x_2,x_3)\| \le \varepsilon_0 c(t)(\|x_0\|+\|x_1\|+\|x_2\|+\|x_3\|) + a(t) + Mb(t),$$

$$\forall\ t \in J_+, x_0, x_1 \in P_{0\lambda^*}, x_2, x_3 \in P. \tag{7}$$

On the other hand, let

$$\overline{\varepsilon}_0 = \min\{\frac{1}{16\overline{m}_2\gamma_0{}^*}, \frac{1}{16\overline{m}_4\gamma_0{}^*}\},\ \overline{\varepsilon}_1 = \min\{\frac{1}{16m_2\gamma_1{}^*}, \frac{1}{16m_4\gamma_1{}^*}\}. \tag{8}$$

We see that, by condition (H_2), there exists an $R_1 > r$ such that

$$\|I_{ik}(x_0,x_1)\| \le \overline{\varepsilon}_i\gamma_{ik}(\|x_0\|+\|x_1\|), \forall\ x_0, x_1 \in P_{0\lambda^*},$$
$$\|x_0\|+\|x_1\| > R_1\ (i=0,1),\ k=1,2,\ldots,m,$$

and

$$\|I_{ik}(x_0,x_1)\| \le \eta_{ik}M_1, \forall\ x_0, x_1 \in P_{0\lambda^*}, \|x_0\|+\|x_1\| \le R_1\ (i=0,1),\ k=1,2,\ldots,m,$$

where

$$M_1 = \max\{F_i(x_0,x_1) : r \le x_0, x_1 \le R_1, i=0,1\}.$$

Hence

$$\|I_{ik}(x_0,x_1)\|\le\overline{\varepsilon}_i\gamma_{ik}(\|x_0\|+\|x_1\|)+\eta_{ik}M_1,\ \forall x_0,x_1\in P_{0\lambda^*},\ i=0,1;k=1,2,\ldots,m. \tag{9}$$

Let $x\in Q$. We have by (7)

$$\|f(t,x(t),x'(t),(Tx)(t),(Sx)(t))\|\le\varepsilon_0c(t)(2+k^*+h^*)\|x\|_{PC^1}+a(t)+Mb(t),\quad\forall t\in J_+, \tag{10}$$

which together with (5), (H$_1$) and Lemma 3 implies that

$$\begin{aligned}&\int_0^1\|H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|\mathrm{d}s\\&\le m_2\int_0^1\|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|\mathrm{d}s\le\frac14\|x\|_{PC^1}+a^*+Mb^*.\end{aligned} \tag{11}$$

On the other hand, by (8), (9) and Lemma 3, we have

$$\sum_{k=1}^m\|H_1(t,t_k)I_{1k}(x(t_k),x'(t_k))\|\le\frac18\|x\|_{PC^1}+m_2\eta_1^*M_1, \tag{12}$$

$$\sum_{k=1}^m\|H_2(t,t_k)I_{0k}(x(t_k),x'(t_k))\|\le\frac18\|x\|_{PC^1}+\overline{m}_2\eta_0^*M_1. \tag{13}$$

It follows from (4), (11)–(13) that

$$\begin{aligned}\|(Ax)(t)\|\le&\frac14\|x\|_{PC^1}+a^*+Mb^*+\frac18\|x\|_{PC^1}+m_2\eta_1^*M_1+\frac18\|x\|_{PC^1}+\overline{m}_2\eta_0^*M_1+\\&\Big(\frac{a+1}{a+1-b-u}\cdot\int_0^1g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{b}{a+1-b}\Big)\|x_0\|\\=&\frac12\|x\|_{PC^1}+a^*+Mb^*+m_2\eta_1^*M_1+\overline{m}_2\eta_0^*M_1+\\&\Big(\frac{a+1}{a+1-b-u}\cdot\int_0^1g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{b}{a+1-b}\Big)\|x_0\|.\end{aligned} \tag{14}$$

Differentiating (4), we get

$$\begin{aligned}(Ax)'(t)=&\int_0^1H_1'(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\\&\sum_{k=1}^mH_1'(t,t_k)I_{1k}(x(t_k),x'(t_k))+\sum_{k=1}^mH_2'(t,t_k)I_{0k}(x(t_k),x'(t_k))-\\&\Big(\frac{1}{a+1-b-u}\cdot\int_0^1g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{1}{a+1-b}\Big)x_0,\end{aligned} \tag{15}$$

which implies that

$$\begin{aligned}\|(Ax)'(t)\|\le&\frac14\|x\|_{PC^1}+a^*+Mb^*+\frac18\|x\|_{PC^1}+m_4\eta_1^*M_1+\frac18\|x\|_{PC^1}+\overline{m}_4\eta_0^*M_1+\\&\Big(\frac{1}{a+1-b-u}\cdot\int_0^1g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{1}{a+1-b}\Big)\|x_0\|\\=&\frac12\|x\|_{PC^1}+a^*+Mb^*+m_4\eta_1^*M_1+\overline{m}_4\eta_0^*M_1+\\&\Big(\frac{1}{a+1-b-u}\cdot\int_0^1g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{1}{a+1-b}\Big)\|x_0\|.\end{aligned} \tag{16}$$

By (14) and (16), we obtain that $Ax \in PC^1[J,P]$ and

$$\|Ax\|_{PC^1} \le \frac{1}{2}\|x\|_{PC^1} + \gamma, \tag{17}$$

where

$$\begin{aligned}\gamma =& a^* + Mb^* + (m_2+m_4)\eta_1^* M_1 + (\overline{m}_2 + \overline{m}_4)\eta_0^* M_1 + \\ &\Big(\frac{a+1}{a+1-b-u} \cdot \int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \frac{b+1}{a+1-b}\Big)\|x_0\|.\end{aligned}$$

On the other hand, (4) and (15) imply that

$$\begin{aligned}(Ax)(t) \ge& -\Big(\frac{a}{a+1-b-u} \cdot \int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \frac{b-1}{a+1-b}\Big)x_0 \\ \ge& \lambda^* x^*, \quad \forall t \in J,\end{aligned} \tag{18}$$

and

$$\begin{aligned}(Ax)'(t) \ge& -\Big(\frac{1}{a+1-b-u} \cdot \int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau + \frac{1}{a+1-b}\Big)x_0 \\ \ge& \lambda^* x^*, \quad \forall t \in J.\end{aligned} \tag{19}$$

So, by (18) and (19) we see that $Ax \in Q$. Thus, we have proved that A maps Q into Q and (17) holds.

Finally, we show that A is continuous. Let $x_n, \bar{x} \in Q, \|x_n - \bar{x}\|_{PC^1} \to 0$ $(n \to \infty)$. Then $r = \sup_n \|x_n\|_{PC^1} < \infty$ and $\|\bar{x}\|_{PC^1} \le r$. It is easy to get, by (4) and (15) that

$$\begin{aligned}\|Ax_n - A\bar{x}\|_{PC^1} \le& (m_2+m_4)\int_0^1 \|f(s, x_n(s), x_n'(s), (Tx_n)(s), (Sx_n)(s)) - \\ & f(s, \bar{x}(s), \bar{x}'(s), (T\bar{x})(s), (S\bar{x})(s))\|\mathrm{d}s + \\ & (m_2+m_4)\Big(\sum_{i=1}^m \|I_{1k}(x_n(t_k), x_n'(t_k)) - I_{1k}(\bar{x}(t_k), \bar{x}'(t_k))\|\Big) + \\ & (\overline{m}_2 + \overline{m}_4)\Big(\sum_{i=1}^m \|I_{0k}(x_n(t_k), x_n'(t_k)) - I_{0k}(\bar{x}(t_k), \bar{x}'(t_k))\|\Big).\end{aligned} \tag{20}$$

It is clear,

$$\begin{aligned}&f(t, x_n(t), x_n'(t), (Tx_n)(t), (Sx_n)(t)) \to f(t, \bar{x}(t), \bar{x}'(t), (T\bar{x})(t), (S\bar{x})(t)) \\ &\qquad \text{as } n \to \infty, \quad \forall\ t \in J_+,\end{aligned} \tag{21}$$

and, by (10),

$$\begin{aligned}&\|f(t, x_n(t), x_n'(t), (Tx_n)(t), (Sx_n)(t)) - f(t, \bar{x}(t), \bar{x}'(t), (T\bar{x})(t), (S\bar{x})(t))\| \\ &\le 2\varepsilon_0 c(t)(2 + k^* + h^*)r + 2a(t) + 2Mb(t) = \sigma(t) \in L[J_+, R_+].\end{aligned} \tag{22}$$

It follows from (21) and (22) and the dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^1 \|f(s, x_n(s), x_n'(s), (Tx_n)(s), (Sx_n)(s)) - f(s, \bar{x}(s), \bar{x}'(s), (T\bar{x})(s), (S\bar{x})(s))\|\mathrm{d}s = 0. \tag{23}$$

It is clear,

$$I_{ik}(x_n(t_k),x_n'(t_k))\to I_{ik}(\bar{x}(t_k),\bar{x}'(t_k)) \text{ as } n\to\infty,\ i=0,1;\ k=1,2,\ldots,m.$$

So,

$$\lim_{n\to\infty}\Big(\sum_{i=1}^{m}\|I_{ik}(x_n(t_k),x_n'(t_k))-I_{ik}(\bar{x}(t_k),\bar{x}'(t_k))\|\Big)=0. \tag{24}$$

It follows from (20), (23) and (24) that $\|Ax_n-A\bar{x}\|_{PC^1}\to 0$ as $n\to\infty$, and the continuity of A is proved. □

Lemma 5 *If $W\subset PC^1[J,E]$ is bounded and the elements of W' are equicontinuous on each J_k $(k=1,2,\ldots,m)$. Then $\alpha_{PC^1}(W)=\max\{\sup_{t\in J}\alpha(W(t)),\sup_{t\in J}\alpha(W'(t))\}$.*

Lemma 6 *Let H be a countable set of strongly measurable function $x:J\to E$ such that there exists an $M\in L[J,R_+]$ such that $\|x\|\le M(t)$ a.e., $t\in J$ for all $x\in H$. Then $\alpha(H(t))\in L[J,R_+]$ and*

$$\alpha\Big(\Big\{\int_J x(t)\mathrm{d}t : x\in H\Big\}\Big)\le 2\int_J \alpha(H(t))\mathrm{d}t.$$

3. Main result

Theorem 1 *If conditions (H$_0$)–(H$_3$) are satisfied, and*

$$2g_1^*\int_0^1\Big(\sum_{i=0}^{1}L_i(s)+L_2(s)k^*+L_3(s)h^*\Big)\mathrm{d}s+g_1^*\sum_{k=1}^{m}(d_{1k0}+d_{1k1})+g_2^*\sum_{k=1}^{m}(d_{0k0}+d_{0k1})<1, \tag{25}$$

in which $g_1^*=\max\{m_2,m_4\}$, $g_2^*=\max\{\overline{m}_2,\overline{m}_4\}$. Then BVP(1) has a positive solution $\bar{x}\in PC^1[J,E]\cap C^2[J_+',E]$ satisfying $(\bar{x})^{(i)}(t)\ge\lambda^*x^*$ for $t\in J$ $(i=0,1)$.

Proof By Lemma 4, operator A defined by (4) is a continuous operator from Q into Q, and, by Lemma 1, we need only to show that A has a fixed point $\bar{x}$ in Q. Choose $R>2\gamma$, and let $Q_1=\{x\in Q:\|x\|_{PC^1}\le R\}$. Obviously, Q_1 is a bounded closed convex set in space $PC^1[J,E]$. It is easy to see that Q_1 is not empty since $\omega(t)=-(\frac{a+t}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{b-1+t}{a+1-b})x_0\in Q_1$. It follows from (17) that $x\in Q_1$ implies $Ax\in Q_1$, i.e., A maps Q_1 into Q_1. Now, we are in a position to show that $A(Q_1)$ is relatively compact. Let $V=\{x_n:n=1,2,\ldots\}\subset Q_1$ satisfying $V\subset\overline{co}\{\{x_0\}\cup(AV)\}$ for some $x_0\in Q_1$. Then $\|x_n\|_{PC^1}\le R$ $(n=1,2,3,\ldots)$. We have, by (15),

$$\begin{aligned}(Ax_n)'(t)=&\int_0^1 H_1'(t,s)f(s,x_n(s),x_n'(s),(Tx_n)(s),(Sx_n)(s))\mathrm{d}s+\\&\sum_{k=1}^{m}H_1'(t,t_k)I_{1k}(x_n(t_k),x_n'(t_k))+\sum_{k=1}^{m}H_2'(t,t_k)I_{0k}(x_n(t_k),x_n'(t_k))-\\&\Big(\frac{1}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{1}{a+1-b}\Big)x_0,\\&\forall t\in J,\quad n=1,2,3,\ldots,\end{aligned} \tag{26}$$

and so, by virtue of (10) and Lemma 3, we get that

$$
\begin{aligned}
&\|(Ax_n)'(t_2)-(Ax_n)'(t_1)\|\\
&\le \int_{t_1}^{t_2}\frac{a+s}{a+1-b}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\\
&\quad\int_{t_1}^{t_2}\frac{b+s-1}{a+1-b}f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\\
&\quad\frac{2}{a+1-b-u}\cdot\Big(\int_{t1}^{t_2}\Big(\int_0^1 G_1(\tau,s)g(\tau)\mathrm{d}\tau\Big)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s\Big)\\
&\le\Big(\frac{a+1+b}{a+1-b}+\frac{2}{a+1-b-u}\cdot\frac{(a+1)b}{a+1-b}\int_0^1 g(\tau)\mathrm{d}\tau\Big)\cdot\\
&\quad\Big(\varepsilon_0(2+k^*+h^*)R\int_{t_1}^{t_2}c(s)\mathrm{d}s+\int_{t_1}^{t_2}(a(s)+Mb(s))\mathrm{d}s\Big),\\
&\quad\forall\ t_1,t_2\in J_k, t_2>t_1, k=1,2,\ldots,m; n=1,2,3,\ldots,
\end{aligned}
$$

which implies that $\{(Ax_n)'(t)\}$ $(n=1,2,3,\ldots)$ is equicontinuous on each J_k $(k=1,2,\ldots,m)$. It is clear, $\{Ax_n\}$ $(n=1,2,3,\ldots)\subset Q_1\subset PC^1[J,E]$ is bounded. By Lemma 5, we have

$$
\alpha_{PC^1}(AV)=\max\{\sup_{t\in J}\alpha((AV)^{(i)}(t)):i=0,1\},\tag{27}
$$

where $AV=\{Ax_n:n=1,2,\ldots\}$, and $(AV)^{(i)}(t)=\{(Ax_n)^{(i)}(t):n=1,2,\ldots\}$. By (2), we have

$$
\begin{aligned}
(Ax_n)(t)=&\int_0^1 H_1(t,s)f(s,x_n(s),x_n'(s),(Tx_n)(s),(Sx_n)(s))\mathrm{d}s-\\
&\Big(\frac{a+t}{a+1-b-u}\cdot\int_0^1 g(\tau)\frac{b-1+\tau}{a+1-b}\mathrm{d}\tau+\frac{b-1+t}{a+1-b}\Big)x_0+\\
&\sum_{k=1}^m H_1(t,t_k)I_{1k}(x_n(t_k),x_n'(t_k))+\sum_{k=1}^m H_2(t,t_k)I_{0k}(x_n(t_k),x_n'(t_k)).
\end{aligned}\tag{28}
$$

It follows from (26), (28), Lemmas 3 and 6 that

$$
\begin{aligned}
\alpha((AV)(t))\le&\Big[2m_2\int_0^1\Big(\sum_{i=0}^1 L_i(s)+L_2(s)k^*+L_3(s)h^*\Big)\mathrm{d}s+m_2\sum_{k=1}^m(d_{1k0}+d_{1k1})+\\
&\overline{m}_2\sum_{k=1}^m(d_{0k0}+d_{0k1})\Big]\cdot\alpha_{PC^1}(V),\quad\forall t\in J,
\end{aligned}\tag{29}
$$

$$
\begin{aligned}
\alpha((AV)'(t))\le&\Big[2m_4\int_0^1\Big(\sum_{i=0}^1 L_i(s)+L_2(s)k^*+L_3(s)h^*\Big)\mathrm{d}s+m_4\sum_{k=1}^m(d_{1k0}+d_{1k1})+\\
&\overline{m}_4\sum_{k=1}^m(d_{0k0}+d_{0k1})\Big]\cdot\alpha_{PC^1}(V),\quad\forall t\in J.
\end{aligned}\tag{30}
$$

Therefore,

$$
\alpha_{PC^1}(AV)\le\Big[2g_1^*\int_0^1\Big(\sum_{i=0}^1 L_i(s)+L_2(s)k^*+L_3(s)h^*\Big)\mathrm{d}s+g_1^*\sum_{k=1}^m(d_{1k0}+d_{1k1})+
$$

$$g_2^* \sum_{k=1}^{m} (d_{0k0} + d_{0k1})\Big] \cdot \alpha_{PC^1}(V). \tag{31}$$

On the other hand, $\alpha_{PC^1}(V) \le \overline{co}\{\{x_0\} \cup (AV)\} = \alpha_{PC^1}(AV)$. Then (25) and (31) imply $\alpha_{PC^1}(AV) = 0$, i.e., V is relatively compact in $PC^1[J, E]$. Hence, Mönch fixed point theorem guarantees that A has a fixed point $\overline{x}$ in Q_1 and the proof is completed. $\Box$

4. An example

Example 1 Consider the infinite system of scalar second-order impulsive singular integro-differential equation

$$\begin{cases} x_n''(t) = \dfrac{12}{n^2\sqrt{t}}\Big(3 + x_n(t) + x_{3n}'(t) + \dfrac{1}{2n^2 x_n(t)} + \dfrac{1}{8n^3 x_{3n}'(t)}\Big)^{\frac{1}{3}} + \\ \qquad \dfrac{1}{\sqrt{nt}}\Big(\displaystyle\int_0^t (1+ts) x_{n+2}(s)\mathrm{d}s\Big)^{\frac{1}{4}} + \dfrac{1}{n\sqrt{t}(1+t)} \cdot \\ \qquad \Big(\displaystyle\int_0^1 e^{-2s}\sin^2(t-s) x_n(s)\mathrm{d}s\Big)^{\frac{1}{5}} + \dfrac{1}{60e^{2t}}\ln(1 + x_n(t)),\ t \in J, t \ne t_1, \\ \Delta x_n|_{t_1=\frac{1}{2}} = -\dfrac{1}{n^5}\Big(x_{n+1}(\dfrac{1}{2}) + \dfrac{1}{x_n'(\frac{1}{2})}\Big)^{\frac{1}{2}}, \\ \Delta x_n'|_{t_1=\frac{1}{2}} = \dfrac{1}{n^9}\Big(x_n(\dfrac{1}{2}) + \dfrac{1}{x_{2n}'(\frac{1}{2})}\Big)^{\frac{1}{3}}, \\ x_n(0) - 4x_n'(0) = -\dfrac{1}{n}, \\ x_n(1) - 2x_n'(1) = \displaystyle\int_0^1 s x_n(s)\mathrm{d}s, \quad n = 1, 2, 3, \ldots. \end{cases} \tag{32}$$

Conclusion Infinite system (32) has a positive solution $\{x_n(t)\}$ satisfying $x_n(t) \ge \frac{3}{4n}$, $x_n'(t) \ge \frac{3}{4n}$ for $t \in [0, 1]$ $(n = 1, 2, 3, \ldots)$.

Proof Let $E = c_0 = \{x = (x_1, \ldots, x_n, \ldots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$ and $P = \{x = (x_1, \ldots, x_n, \ldots) \in c_0 : x_n \ge 0, n = 1, 2, 3, \ldots\}$. Then P is a normal cone in E and infinite system (32) can be regarded as a BVP of the form (1). In this situation, $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$, $z = (z_1, \ldots, z_n, \ldots)$, $w = (w_1, \ldots, w_n, \ldots)$, $g(s) = s$, $a = 4$, $b = 2$, $m = 1$, $k(t, s) = (1 + ts)$, $h(t, s) = e^{-2s}\sin^2(t - s)$, $x_0 = (-1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots)$, $f = (f_1, \ldots, f_n, \ldots)$ and $I_{ik} = (I_{ik1}, \ldots, I_{ikn}, \ldots)$ $(i = 0, 1)$, in which

$$\begin{aligned} f_n(t, x, y, z, w) =& \frac{12}{n^2\sqrt{t}}\Big(3 + x_n(t) + y_{3n}(t) + \frac{1}{2n^2 x_n(t)} + \frac{1}{8n^3 y_{3n}(t)}\Big)^{\frac{1}{3}} + \\ & \frac{1}{\sqrt{nt}}\sqrt[4]{z_{n+2}(t)} + \frac{1}{n\sqrt{t}(1+t)}\sqrt[5]{w_n(t)} + \frac{1}{60e^{2t}}\ln(1 + x_n(t)), \end{aligned} \tag{33}$$

and

$$I_{0kn}(x, y) = \frac{1}{n^5}\Big(x_{n+1} + \frac{1}{y_n}\Big)^{\frac{1}{2}}, \tag{34}$$

$$I_{1kn}(x, y) = \frac{1}{n^9}\Big(x_n + \frac{1}{y_{2n}}\Big)^{\frac{1}{3}}. \tag{35}$$

Let $x^* = -x_0$. Then $P_{0\lambda} = \{x = (x_1, \ldots, x_n, \ldots) : x_n \ge \frac{\lambda}{n}, n = 1, 2, \ldots\}$ for $\lambda > 0$. By direct computation, we have $u = \frac{7}{3}$, $u_1 = \frac{1}{2}$, $u_2 = 2$, $k^* < 2$, $h^* = 1$, $m_2 = \frac{95}{6}$, $\overline{m}_2 = \frac{95}{12}$, $m_4 = \frac{25}{6}$, $\overline{m}_4 = \frac{19}{12}$, $D_0 = 2$, $D_1 = \frac{3}{4}$, $\lambda^* = \frac{3}{4}$, $g_1^* = \frac{95}{6}$, $g_2^* = \frac{95}{12}$. It is clear, (H_0) holds for $u = \frac{7}{3} \in [0, a+1-b) = [0, 3)$. Obviously, $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P \times P, P]$, $I_{ik} \in C[P_{0\lambda} \times P_{0\lambda}, P]$ for any $\lambda > 0$ ($i = 0, 1; k = 1$). Noticing that $e^{2t} > \sqrt{t}$ ($t > 0$), for $t \in J_+$, $x, y \in P_{0\lambda^*}$, $z, w \in P$, we have, by (33),

$$\|f(t,x,y,z,w)\| \leq \frac{12}{\sqrt{t}}\Big\{\big(\frac{25}{6} + \|x\| + \|y\|\big)^{\frac{1}{3}} + \|z\|^{\frac{1}{4}} + \|w\|^{\frac{1}{5}} + \ln(1 + \|x\|)\Big\}. \tag{36}$$

So, (H_1) is satisfied for $a(t) = 0$, $b(t) = c(t) = \frac{12}{\sqrt{t}}$ and

$$h(u_0, u_1, u_2, u_3) = \big(\frac{25}{6} + u_0 + u_1\big)^{\frac{1}{3}} + u_2^{\frac{1}{4}} + u_3^{\frac{1}{5}} + \ln(1 + u_0).$$

On the other hand, for $x \in P_{0\lambda^*}$, $y \in P_{0\lambda^*}$, we have, by (34) and (35) that

$$\|I_{0k}(x,y)\| \leq (\frac{4}{3} + \|x\|)^{\frac{1}{2}}, \quad \|I_{1k}(x,y)\| \leq (\frac{8}{3} + \|x\|)^{\frac{1}{3}},$$

which imply that condition (H_2) is satisfied for

$$F_0(u_0, u_1) = (\frac{4}{3} + u_0)^{\frac{1}{2}}, \quad F_1(u_0, u_1) = (\frac{8}{3} + u_0)^{\frac{1}{3}}$$

and $\eta_{0k} = \eta_{1k} = \gamma_{0k} = \gamma_{1k} = 1$.

Next, we check condition (H_3). Let $f^1 = \{f_1^1, f_2^1, \ldots, f_n^1, \ldots\}$, $f^2 = \{f_1^2, f_2^2, \ldots, f_n^2, \ldots\}$, where

$$\begin{aligned} f_n^1(t,x,y,z,w) =& \frac{12}{n^2\sqrt{t}}\big(3 + x_n + y_{3n} + \frac{1}{2n^2 x_n} + \frac{1}{8n^3 y_{3n}}\big)^{\frac{1}{3}} + \\ & \frac{1}{\sqrt{nt}}\sqrt[4]{z_{n+2}} + \frac{1}{n\sqrt{t}(1+t)}\sqrt[5]{w_n}, \end{aligned} \tag{37}$$

$$f_n^2(t,x,y,z,w) = \frac{1}{60e^{2t}}\ln(1 + x_n). \tag{38}$$

Let $t \in J_+$ and $R > 0$ be given and $\{z^{(m)}\}$ be any sequence in $f^1(t, P^*_{0\lambda^* R}, P^*_{0\lambda^* R}, P^*_R, P^*_R)$, where $z^{(m)} = (z_1^{(m)}, \ldots, z_n^{(m)}, \ldots)$. By (37), we have

$$0 \leq z_n^{(m)} \leq \frac{12}{\sqrt{nt}}\Big[\big(\frac{25}{6} + 2R\big)^{\frac{1}{3}} + \sqrt[4]{R} + \sqrt[5]{R}\Big], \quad n, m = 1, 2, 3, \ldots. \tag{39}$$

So, $\{z_n^{(m)}\}$ is bounded and by the diagonal method together with the method of constructing subsequence, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$\{z_n^{(m)}\} \to \overline{z}_n \quad \text{as } i \to \infty, \quad n = 1, 2, 3, \ldots, \tag{40}$$

which implies by (39)

$$0 \leq \overline{z}_n \leq \frac{12}{\sqrt{nt}}\Big[\big(\frac{25}{6} + 2R\big)^{\frac{1}{3}} + \sqrt[4]{R} + \sqrt[5]{R}\Big], \quad n = 1, 2, 3, \ldots. \tag{41}$$

Hence $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n, \ldots) \in c_0$. It is easy to see from (39)–(41) that

$$\|z^{(m_i)} - \overline{z}\| = \sup_n |z_n^{(m_i)} - \overline{z}_n| \to 0 \text{ as } i \to \infty.$$

Thus, we have proved that $f^1(t, P^*_{0\lambda^* R}, P^*_{0\lambda^* R}, P^*_R, P^*_R)$ is relatively compact in c_0. Similarly, by (34) and (35), we can show that $I_{ik}(P^*_{0\lambda^* R}, P^*_{0\lambda^* R})$ ($i = 0, 1$) are relatively compact in c_0. For any $t \in J_+, R > 0, x, \overline{x} \in D \subset P^*_{0\lambda^* R}$, we have by (38)

$$|f_n^2(t,x,y,z,w) - f_n^2(t,\overline{x},\overline{y},\overline{z},\overline{w})| = \frac{1}{60e^{2t}(1+t)}|\ln(1+x_n) - \ln(1+\overline{x}_n)| \le \frac{1}{60e^{2t}}\frac{|x_n - \overline{x}_n|}{1+\xi_n}, \tag{42}$$

where ξ_n is between x_n and $\overline{x}_n$. By (42), we get

$$\|f^2(t,x,y,z,w) - f^2(t,\overline{x},\overline{y},\overline{z},\overline{w})\| \le \frac{1}{60e^{2t}}\|x - \overline{x}\|,\ x,\ \overline{x} \in D. \tag{43}$$

Thus, we have shown that (H_3) holds for $L_0(t) = \frac{1}{60e^{2t}}$, $L_i(t) = 0$ $(i = 1,2,3)$, $d_{ikj} = 0$ $(i,j = 0,1; k = \frac{1}{2})$. It is not difficult to see that (25) is also satisfied. Hence, our conclusion follows from Theorem 1. $\square$

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