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The Least Eigenvalue of Graphs

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Abstract In this paper we investigate the least eigenvalue of a graph whose complement is connected, and present a lower bound for the least eigenvalue of such graph. We also characterize the unique graph whose least eigenvalue attains the second minimum among all graphs of fixed order.

Keywords graph; complement; adjacency matrix; least eigenvalue.

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1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V=V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E=E(G)$. The adjacency matrix of G is defined to be a matrix $A(G)=[a_{ij}]$ of order n, where $a_{ij}=1$ if v_i is adjacent to v_j, and $a_{ij}=0$ otherwise. Since $A(G)$ is real and symmetric, its eigenvalues are real and can be arranged as: $\lambda_1(G)\le\lambda_2(G)\le\cdots\le\lambda_n(G)$. The eigenvalues of $A(G)$ are referred to as the eigenvalues of G. The largest eigenvalue $\lambda_n(G)$, denoted by $\rho(G)$, is exactly the spectral radius of $A(G)$. The least eigenvalue $\lambda_1(G)$ is denoted by $\lambda_{\min}(G)$, and the corresponding eigenvectors are called the first eigenvectors of G.

There are a lot of results on the spectral radius of graphs [1, 2]. However, relative to the spectral radius, the least eigenvalue has received less attention. In the past decades, the main work on the least eigenvalue of a graph was concerned to its bounds. Recently, the problem of

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minimizing the least eigenvalues of graphs subject to one or more given parameters has received much attention. Bell et al. [3, 4] characterized the graphs with minimum least eigenvalue within graphs of fixed order and size. Fan et.al. [5] determined the unique graph with minimum least eigenvalue among all unicyclic graphs with fixed order. Further results on the least eigenvalue were focused on graphs with some invariants being fixed, like connectivity by Ye and Fan [6], number of cut vertices by Wang et al. [7, 8], vertex (edge) independence number, or cover number by Tan and Fan [9], or some specified classes of graphs, like unicyclic graphs with prescribed number of pendant vertices by Liu et al. [10], bicyclic graphs by Petrović et al. [11], etc.

For convenience, a graph is called minimizing (or the second minimizing) in a certain class if its least eigenvalue attains the minimum (or the second minimum) among all graphs in this class. Denote by $\mathscr{G}_n$ the set of graphs of order n. It was proved by Constantine [12] (also see [6]) that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the unique minimizing graph in $\mathscr{G}_n$, where $K_{p,q}$ denotes a complete bipartite graph with two parts having p, q vertices, respectively. Note that $K^c_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ $(= K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lceil \frac{n}{2} \rceil})$ is disconnected, where K_m denotes a complete graph of order m, and G^c denotes the complement of a graph G. So, a problem arises naturally what is the minimizing graph among all graphs of order n whose complements are connected. In addition, what is the second minimizing graph(s) in $\mathscr{G}_n$?

In this paper we will address ourselves to these problems, and study the least eigenvalue of graphs from their complements. Similar work has been done in [13] on characterizing the minimizing graph among the complements of trees with fixed order.

2. Main result

We begin with some definitions. Given a graph G of order n, a vector $X \in \mathbb{R}^n$ is called a function defined on G, if there is a 1-1 map φ from $V(G)$ to the entries of X; simply written $X_u = \varphi(u)$ for each $u \in V(G)$ (X_u is also called the value of u given by X). If X is an eigenvector of $A(G)$, then X is defined naturally on G, i.e., X_u is the entry of X corresponding to the vertex u. One can find that

$$X^T A(G) X = 2 \sum_{uv \in E(G)} X_u X_v, \tag{1}$$

and λ is an eigenvalue of G corresponding to the eigenvector X if and only if $X \neq 0$ and

$$\lambda X_v = \sum_{u \in N_G(v)} X_u, \text{ for each vertex } v \in V(G), \tag{2}$$

where $N_G(v)$ denotes the neighborhood of v in G. The equation (2) is called (λ, X)-eigenequation of G. In addition, for an arbitrary unit vector $X \in \mathbb{R}^n$,

$$\lambda_{\min}(G) \leq X^T A(G) X \tag{3}$$

with equality if and only if X is a first eigenvector of G.

It is easily seen that $A(G^c) = \mathbf{J} - \mathbf{I} - A(G)$, where $\mathbf{J}, \mathbf{I}$ respectively denote the all-one square matrix and the identity matrix both of suitable sizes. So for any vector X,

$$X^T A(G^c)X = X^T(\mathbf{J} - \mathbf{I})X - X^T A(G)X. \tag{4}$$

We introduce an important graph $G(p,q)$ of order $p+q$ ($p \geq q \geq 1$), which is obtained from two disjoint complete graphs K_p and K_q by joining one vertex of K_p and one vertex of K_q; see Figure 1.

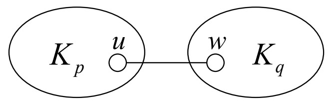

Figure 1 The graph $G(p,q)$

Lemma 1 *Given a positive integer n ($n \geq 4$), for any positive integers p,q such that $p \geq q \geq 1$ and $p+q=n$,*

$$\lambda_{\min}(G(p,q)^c) \geq \lambda_{\min}(G(\lceil n/2 \rceil, \lfloor n/2 \rfloor)^c),$$

with equality if and only if $p = \lceil n/2 \rceil, q = \lfloor n/2 \rfloor$.

Proof Let $G(p,q)$ be as in Figure 1 with the edge uw between K_p and K_q. First assume that $q \geq 2$. Note that $G(p,q)^c = K_{p,q} - uw$. Then $G(p,q)^c$ is connected and non-complete, and hence $\lambda_{\min}(G(p,q)^c) =: \lambda_1 < -1$. Let X be a first eigenvector of $G(p,q)^c$. By (2), all vertices in $V(K_p)$ except u have the same values given by X, say X_1, and all vertices in $V(K_q)$ except w have the same values, say X_4. Let $X_u =: X_2, X_w =: X_3$. Also by (2), we have

$$\begin{cases} \lambda_1 X_1 = (q-1)X_4 + X_3, \\ \lambda_1 X_2 = (q-1)X_4, \\ \lambda_1 X_3 = (p-1)X_1, \\ \lambda_1 X_4 = (p-1)X_1 + X_2. \end{cases} \tag{5}$$

Transform (5) into a matrix equation $(B - \lambda_1 \mathbf{I})X' = 0$, where $X' = (X_1, X_2, X_3, X_4)^T$ and

$$B = \begin{bmatrix} 0 & 0 & 1 & q-1 \\ 0 & 0 & 0 & q-1 \\ p-1 & 0 & 0 & 0 \\ p-1 & 1 & 0 & 0 \end{bmatrix}.$$

We have

$$f(\lambda; p, q) := \det(B - \lambda \mathbf{I}) = \lambda^4 - (pq-1)\lambda^2 + (p-1)(q-1) \tag{6}$$

and

$$f(\lambda; p, q) - f(\lambda; p-1, q+1) = (p-q-1)(\lambda^2 - 1).$$

In addition, λ_1 is the least root of $f(\lambda; p, q)$.

If $p \geq q+2$, we have $f(\lambda; p, q) - f(\lambda; p-1, q+1) > 0$ when $\lambda < -1$. In particular $f(\lambda_1; p-1, q+1) < 0$ as $f(\lambda_1; p, q) = 0$, which implies

$$\lambda_{\min}(G(p,q)^c) > \lambda_{\min}(G(p-1,q+1)^c) > \cdots > \lambda_{\min}(G(\lceil n/2 \rceil, \lfloor n/2 \rfloor)^c).$$

Next assume that $q = 1$. Then $G(p,q)^c$ is a union of a star on $n-1$ vertices and an isolated vertex. So $\lambda_{\min}(G(n-1,1)^c) = -\sqrt{n-2}$. Since

$$f(\lambda_{\min}(G(n-1,1)^c); n-2, 2) = f(-\sqrt{n-2}; n-2, 2) = -(n-3)^2 < 0,$$

we have $\lambda_{\min}(G(n-1,1)^c) > \lambda_{\min}(G(n-2,2)^c)$. The result follows by the above discussion. $\square$

Remark In the proof of Lemma 1, we can consider the spectral radius instead in the case of $p \geq q \geq 2$. Noting that $G^c(p,q)$ is bipartite, we have $\lambda_{\min}(G^c(p,q)) = -\rho(G^c(p,q))$. We also find $\rho(G^c(p,q)) > 1$ by interlacing theorem, and $\rho(G^c(p,q))$ is the largest root of the polynomial in (6) by a similar discussion. In addition, the eigenvector corresponding to $\rho(G^c(p,q))$ can be chosen (entrywise) positive since $G^c(p,q)$ is connected and $A(G^c(p,q))$ is irreducible. Therefore the first eigenvector of $G(p,q)^c$ contains no zero entries in this case by the eigenvector property of bipartite graphs.

Theorem 2 *Let G be a connected graph of order $n \geq 4$. Then*

$$\lambda_{\min}(G^c) \geq \lambda_{\min}(G(\lceil n/2 \rceil, \lfloor n/2 \rfloor)^c)$$

with equality if and only if $G = G(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$.

Equivalently, among the complements of connected graphs of order $n \geq 4$, the complement of $G(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$ is the unique minimizing graph.

Proof If $G \neq K_n$, then G^c contains at least one edge, and hence $\lambda_{\min}(G^c) \leq -1$. So we can assume $G \neq K_n$ since $\lambda_{\min}(K_n^c) = 0$.

Let X be a unit first eigenvector of G^c. Then X contains both positive entries and negative entries. Let $V^+ = \{v : X_v \geq 0\}$ and $V^- = \{v : X_v < 0\}$ with cardinalities p and $q = n-p$, respectively. We may assume $p \geq q \geq 1$; otherwise we consider the eigenvector $-X$.

Now by adding all possible edges within $G[V^+]$, we get a complete graph K_p, and then $X^TA(G[V^+])X \leq X^TA(K_p)X$, where $G[U]$ denotes the subgraph of G induced on the vertices of a subset $U \subseteq V(G)$. Similarly, by adding all possible edges with $G[V^-]$, we get K_q, and then $X^TA(G[V^-])X \leq X^TA(K_q)X$. Since G is connected, there exists at least one edge, say uw, between $G[V^+]$ and $G[V^-]$. Consequently, we have

$$\begin{aligned} X^TA(G)X &= X^TA(G[V^+])X + X^TA(G[V^-])X + 2\sum_{vv' \in E(V^+,V^-)} X_v X_{v'} \\ &\leq X^TA(K_p)X + X^TA(K_q)X + 2X_uX_w \\ &= X^TA(G(p,q))X, \end{aligned} \tag{7}$$

where $E(V^+, V^-)$ denotes the set of edges of G joining one vertex of V^+ and one of V^-. By (4) and Lemma 1, we have

$$\lambda_{\min}(G^c) = X^T A(G^c)X \geq X^T A(G(p,q)^c)X \geq \lambda_{\min}(G(p,q)^c) \geq \lambda_{\min}(G(\lceil n/2\rceil, \lfloor n/2\rfloor)^c). \quad (8)$$

If $\lambda_{\min}(G^c) = \lambda_{\min}(G(\lceil n/2\rceil, \lfloor n/2\rfloor)^c)$, then all the inequalities in (8) become equalities. So, $p = \lceil n/2\rceil \geq 2$ and $q = \lfloor n/2\rfloor \geq 2$ by Lemma 1, and X is a first eigenvector of $G(\lceil n/2\rceil, \lfloor n/2\rfloor)^c$ by (3). By the Remark after Lemma 1, X contains no zero entries, which implies $X_v > 0$ for all $v \in V^+$. Considering the equality $X^T A(G^c)X = X^T A(G(p,q)^c)X$ or equivalently $X^T A(G)X = X^T A(G(p,q))X$, by (7) we have

$$G[V^+] = K_p,\ G[V^-] = K_q,\ E(V^+, V^-) = \{uw\}.$$

So the graph G is exactly $G(p,q)$ with $p = \lceil n/2\rceil, q = \lfloor n/2\rfloor$, and the result follows. □

Corollary 3 *Let G be a graph of order $n \geq 4$. If G^c is connected, then*

$$\lambda_{\min}(G) \geq \lambda_{\min}(K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} - e) = -\sqrt{\frac{\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - 1 + \sqrt{(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - 3)^2 + 4n - 13}}{2}}$$

with equality if and only if $G = K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} - e$, where e is an arbitrary edge of $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$.

Proof Noting that $G(\lceil\frac{n}{2}\rceil, \lfloor\frac{n}{2}\rfloor)^c = K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} - e$, by Theorem 2 we have

$$\lambda_{\min}(G) = \lambda_{\min}((G^c)^c) \geq \lambda_{\min}(G(\lceil\frac{n}{2}\rceil, \lfloor\frac{n}{2}\rfloor)^c) = \lambda_{\min}(K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} - e).$$

The value $\lambda_{\min}(G(\lceil\frac{n}{2}\rceil, \lfloor\frac{n}{2}\rfloor)^c)$ can be obtained directly from (6). The equality case also follows from Theorem 2. □

Next we discuss the second minimizing graph(s) in $\mathscr{G}_n$. Suppose that G is the second minimizing graph in $\mathscr{G}_n$. If G^c is disconnected, then G is a connected graph obtained from two subgraphs, say F, H, by joining all possible edges between F and H. If G is bipartite, then $G = K_{p,q}$ for some p, q. Noting that $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the unique minimizing graph [6, 12], we have $G = K_{\lceil n/2\rceil+1,\lfloor n/2\rfloor-1} =: \mathbf{G}_1$. If G is non-bipartite, then by a result in [14], $G = K_{\lceil n/2\rceil,\lfloor n/2\rfloor} + e =: \mathbf{G}_2$, where e lies within $V(K_{\lceil n/2\rceil})$. If G^c is connected, then by Corollary 3, $G = K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} - e =: \mathbf{G}_3$, where e is an arbitrary edge of the graph $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. In the following we compare the least eigenvalues of $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$.

Lemma 4 *Suppose $n \geq 4$. If n is even, then $\lambda_{\min}(\mathbf{G}_1) < \lambda_{\min}(\mathbf{G}_2)$; if n is odd, then $\lambda_{\min}(\mathbf{G}_2) < \lambda_{\min}(\mathbf{G}_1)$.*

Proof It is known that $\lambda_{\min}(\mathbf{G}_1) = -\sqrt{(\lceil n/2\rceil + 1)(\lfloor n/2\rfloor - 1)}$, and hence it is the least root of the polynomial

$$f(\lambda; \mathbf{G}_1) := \lambda^2 - (\lceil n/2\rceil + 1)(\lfloor n/2\rfloor - 1).$$

As pointed in [14], $\lambda_{\min}(\mathbf{G}_2)$ is the least root of the polynomial

$$f(\lambda; \mathbf{G}_2) := \lambda^3 - \lambda^2 - \lceil n/2\rceil\lfloor n/2\rfloor\lambda + (\lceil n/2\rceil - 2)\lfloor n/2\rfloor.$$

Now we have

$$(\lambda - 1)f(\lambda; \mathbf{G}_1) - f(\lambda; \mathbf{G}_2) = (\lceil n/2\rceil - \lfloor n/2\rfloor + 1)\lambda + 3\lfloor n/2\rfloor - \lceil n/2\rceil - 1.$$

If n is even, then $(\lambda-1)f(\lambda;\mathbf{G}_1)-f(\lambda;\mathbf{G}_2)=\lambda+n-1>0$ when $\lambda>-(n-1)$. Since $\lambda_{\min}(\mathbf{G}_2)>-(n-1)$, we have $\lambda_{\min}(\mathbf{G}_1)<\lambda_{\min}(\mathbf{G}_2)$.

If n is odd, then $(\lambda-1)f(\lambda;\mathbf{G}_1)-f(\lambda;\mathbf{G}_2)=2\lambda+n-3<0$ when $\lambda<-(n-3)/2$. Since $\lambda_{\min}(\mathbf{G}_1)=-\frac{\sqrt{n^2-9}}{2}<-\frac{(n-3)}{2}$, we have $\lambda_{\min}(\mathbf{G}_2)<\lambda_{\min}(\mathbf{G}_1)$. $\Box$

Corollary 5 *Let $G\neq K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ be a connected graph of order $n\geq 4$, whose complement is disconnected. If n is even, then $\lambda_{\min}(G)\geq\lambda_{\min}(K_{\lceil n/2\rceil+1,\lfloor n/2\rfloor-1})$ with equality if and only if $G=K_{\lceil n/2\rceil+1,\lfloor n/2\rfloor-1}$. If n is odd, then $\lambda_{\min}(G)\geq\lambda_{\min}(K_{\lceil n/2\rceil,\lfloor n/2\rfloor}+e)$ with equality if and only if $G=K_{\lceil n/2\rceil,\lfloor n/2\rfloor}+e$, where e is an arbitrary edge within $V(K_{\lceil n/2\rceil})$.*

Equivalently, among the complement of disconnected graphs of order $n\geq 4$ except $K_{\lceil n/2\rceil}\cup K_{\lfloor n/2\rfloor}$, the complement of $K_{\lceil n/2\rceil+1}\cup K_{\lfloor n/2\rfloor-1}$ (for even n) or $(K_{\lceil n/2\rceil}-e)\cup K_{\lfloor n/2\rfloor}$ (for odd n) is the unique minimizing graph, where e is an arbitrary edge of $K_{\lceil n/2\rceil}$.

Proof Noting that $(K_{\lceil n/2\rceil+1,\lfloor n/2\rfloor-1})^c=K_{\lceil n/2\rceil+1}\cup K_{\lfloor n/2\rfloor-1}$ and $(K_{\lceil n/2\rceil,\lfloor n/2\rfloor}+e)^c=(K_{\lceil n/2\rceil}-e)\cup K_{\lfloor n/2\rfloor}$, we get the result from Lemma 4 and the foregoing discussion. $\Box$

Lemma 6 *Suppose $n\geq 4$. If n is even, then $\lambda_{\min}(\mathbf{G}_3)<\lambda_{\min}(\mathbf{G}_1)$; if n is odd, then $\lambda_{\min}(\mathbf{G}_2)<\lambda_{\min}(\mathbf{G}_3)$.*

Proof By Lemma 1, $\lambda_{\min}(\mathbf{G}_3)$ is the least root of the polynomial

$$f(\lambda,\mathbf{G}_3):=\lambda^4-(\lfloor n/2\rfloor\lceil n/2\rceil-1)\lambda^2+(\lfloor n/2\rfloor-1)(\lceil n/2\rceil-1).$$

If n is even, then $\lambda_{\min}(\mathbf{G}_1)=-\frac{\sqrt{n^2-4}}{2}$. By a little calculation, $f(\lambda_{\min}(\mathbf{G}_1),\mathbf{G}_3)<0$, which implies $\lambda_{\min}(\mathbf{G}_3)<\lambda_{\min}(\mathbf{G}_1)$.

If n is odd, then $f(\lambda;\mathbf{G}_3)-(\lambda+1)f(\lambda;\mathbf{G}_2)=\lambda(2\lambda+n-1)$. Note that $f(-\frac{n-1}{2};\mathbf{G}_3)<0$, so $\lambda_{\min}(\mathbf{G}_3)<-\frac{n-1}{2}$. Now $f(\lambda;\mathbf{G}_3)-(\lambda+1)f(\lambda;\mathbf{G}_2)>0$ when $\lambda\leq\lambda_{\min}(\mathbf{G}_3)<-\frac{n-1}{2}$, which implies that $\lambda_{\min}(\mathbf{G}_2)<\lambda_{\min}(\mathbf{G}_3)$. $\Box$

By Theorem 2, Corollary 5 and Lemma 6, we identify the second minimizing graph over all graphs of order n.

Theorem 7 *Among all graphs of order n, the graph $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}-e$ (for even n) or $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}+e$ (for odd n) is the second minimizing graph, where e is an arbitrary edge of $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$ (for even n) or lies within $V(K_{\lceil n/2\rceil})$ (for odd n).*

References

[1] D. CVETKOVIĆ, P. ROWLINSON. *The largest eigenvalues of a graph: a survey*. Linear and Multilinear Algebra, 1990, **28**(1-2): 3–33.

[2] D. CVETKOVIĆ, M. DOOB, H. SACHS. *Spectra of Graphs, Theory and Applications (III)*. Johann Ambrosius Barth, Heidelberg, 1995.

[3] F. K. BELL, D. CVETKOVIĆ, P. ROWLINSON, et al. *Graphs for which the least eigenvalue is minimal (I)*. Linear Algebra Appl., 2008, **429**(1): 234–241.

[4] F. K. BELL, D. CVETKOVIĆ, P. ROWLINSON, et al. *Graphs for which the least eigenvalue is minimal (II)*. Linear Algebra Appl., 2008, **429**(8-9): 2168–2179.

[5] Yizheng FAN, Yi WANG, Yubin GAO. *Minimizing the least eigenvalues of unicyclic graphs with application*

to spectral spread. Linear Algebra Appl., 2008, **429**(2-3): 577–588.

[6] Miaolin YE, Yizheng FAN, Dong LIANG. *The least eigenvalue of graphs with given connectivity*. Linear Algebra Appl., 2009, **430**(4): 1375–1379.

[7] Yi WANG, Yun QIAN, Yizheng FAN. *On the least eigenvalue of graphs with cut vertices*. J. Math. Res. Exposition, 2010, **30**(6): 951–956.

[8] Yi WANG, Yizheng FAN. *The least eigenvalue of a graph with cut vertices*. Linear Algebra Appl., 2010, **433**(1): 19–27.

[9] Yingying TAN, Yizheng FAN. *The vertex (edge) independence number, vertex (edge) cover number and the least eigenvalue of a graph*. Linear Algebra Appl., 2010, **433**(4): 790–795.

[10] Ruifang LIU, Mingqing ZHAI, Jinglong SHU. *The least eigenvalues of unicyclic graph with n vertices and k pendant vertices*. Linear Algebra Appl., 2009, **431**(5-7): 657–665.

[11] M. PETROVIĆ, B. BOROVIĆANIN, T. ALEKSIĆ. *Bicyclic graphs for which the least eigenvalue is minimum*. Linear Algebra Appl., 2009, **430**(4): 1328–1335.

[12] G. CONSTANTINE. *Lower bounds on the spectra of symmetric matrices with nonnegative entries*. Linear Algebra Appl., 1985, **65**: 171–178.

[13] Yizheng FAN, Feifei ZHANG, Yi WANG. *The least eigenvalue of the complements of tree*. Linear Algebra Appl., 2011, **435**(9): 2150–2155.

[14] Yizheng FAN. *The least eigenvalue of a nonbipartite graph*. J. Anqing Teachers College Nat. Sci. Ed., 2009, **15**: 1–3. (in Chinese)