

Spectral Characterization of Generalized Cocktail-Party Graphs

Jianfeng WANG^{1,*}, Qiongxiang HUANG²

1. *Department of Mathematics, Qinghai Normal University, Qinghai 810008, P. R. China;*
2. *College of Mathematics and System Science, Xinjiang University, Xinjiang 830046, P. R. China*

Abstract In the paper, we prove that all generalized cocktail-party graphs with order at least 23 are determined by their adjacency spectra.

Keywords adjacency matrix; spectral characterization; generalized line graph; generalized cocktail-party graphs.

MR(2010) Subject Classification 05C50

1. Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). All notions on graphs that are not defined here can be found in [4]. For a graph $G = (V(G), E(G))$, let $n(G) = n$, $m(G)$, $\ell(G)$ and $A = A(G)$ be respectively the order, size, line graph and adjacency matrix of G . For some vertex $v_i \in V(G)$, let $d_i = d(v_i)$ stand for the degree of v_i . We denote the characteristic polynomial $\det(\lambda I - A)$ of G by $\phi(G, \lambda)$ or simply $\phi(G)$. The adjacency spectrum of G , denoted by $\text{Spec}(G)$, is the multiset of eigenvalues of $A(G)$. Since $A(G)$ is symmetric, its eigenvalues are real and we set $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \lambda_n(G)$. The maximum eigenvalue $\lambda_1(G)$ of G is called the spectral radius (or index) of G and it is often denoted by $\rho(G)$.

Two graphs G and H are said to be A -cospectral if the corresponding adjacency spectra are the same. A graph is said to be determined by the A -spectrum (or simply a DAS-graph) if there is no other non-isomorphic graph A -cospectral to it, i.e., $\phi(G) = \phi(H)$ implies $G \cong H$. The background of the question “which graphs are determined by their spectrum?” originates from Chemistry (in 1956, Günthadr and Primas [15] raised this question in the context of Hückel’s theory). For additional remarks on the topic we refer the readers to [11, 12].

Some other notations and terminology are also needed. Let $\Delta(G)$ be the maximum degree of a graph G . Let $G_1 \cup G_2$ denote the disjoint union of graphs G_1 and G_2 , and kG_1 the disjoint

Received May 31, 2011; Accepted October 30, 2011

Supported by the National Natural Science Foundation of China (Grant No.10961023), National Natural Science Foundation of China for Young Scholars (Grant No.11101232), Natural Science Foundation of Qinghai Province for Young Scholars (Grant No.2011-Z-929Q) and Scientific Research Innovation Program of Qinghai Normal University.

* Corresponding author

E-mail address: jfwang4@yahoo.com.cn (Jianfeng WANG); huangqx@xju.edu.cn (Qiongxiang HUANG)

union of k copies of G_1 . The join (or complete product) $G_1 \nabla G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2 . As usual, let P_n , C_n , K_n and $K_{a_1, a_2, \dots, a_k}$ ($a_1 + a_2 + \dots + a_k = n$) denote the path, the cycle, the complete graph and the complete k -multipartite graph of order n , respectively.

Cvetković, Doob, Simić [5] defined a generalized cocktail-party graph, denoted by GCP , as a complete graph with some independent edges removed. A special case of this graph is the well-known cocktail-party graph $CP(k)$ obtained from K_{2k} by removing k disjoint edges. Hoffman [16] introduced the generalized line graph as follows: for any graph H with n vertices $v_1, v_2, \dots, v_n$ and any non-negative integers $a_1, a_2, \dots, a_n$, then the generalized line graph $L(H; a_1, a_2, \dots, a_n)$ is the graph consisting of disjoint copies of $\ell(H)$ and $CP(a_i)$ together with additional edges joining a vertex in $\ell(H)$ with a vertex in $CP(a_i)$ if the vertex in $\ell(H)$ corresponding to an edge in H has v_i as an end-vertex ($i = 1, 2, \dots, n$). It is well-known that the generalized line graphs are related to the following famous and important theorem:

Theorem 1.1 ([2]) *Let G be a connected graph with least eigenvalue at least -2 . Then either G is a generalized line graph or G can be represented by vectors in the root system E_8 .*

Graphs with least eigenvalue at least -2 have been studied since the very beginnings of the theory of graphs spectra. Much information on this field can be found in the books [1, 3, 10, 14].

It is an interesting problem to find which graphs with least eigenvalue at least -2 are A -cospectral graphs or DAS-graphs. Here we mention some known results. An exceptional graph is a connected graph with least eigenvalue at least -2 which is not a generalized line graph. Cvetković and Lepović [6, 7] studied the phenomenon of A -cospectrality in generalized line graphs and in exceptional graphs. For the regular DAS-graphs with least eigenvalue at least -2 , van Dam and Haemers [11] gave an almost complete answer (see their Theorem 8). Further results on A -cospectral graphs may be found in Section 4.2 of [10]. However, for the non-regular case, van Dam and Haemers [11] stated that the following question remains open.

Problem 1.1 Which non-regular graphs with least eigenvalue at least -2 are DAS-graphs?

The rest of the paper is organized as follows: In Section 2 we cite some results of graphs with least eigenvalue at least -2 and define an important graph invariant which will be helpful in proving our main results. In Section 3 we investigate the spectral characterization of generalized cocktail-party graph.

2. Basic results and an invariant of graphs with least eigenvalue at least -2

Lemma 2.1 ([4]) *Let G_i be an r_i -regular graph of order n_i ($i = 1, 2$). Then*

$$\phi(G_1 \nabla G_2, \lambda) = \frac{\phi(G_1, \lambda)\phi(G_2, \lambda)}{(\lambda - r_1)(\lambda - r_2)} [(\lambda - r_1)(\lambda - r_2) - n_1 n_2].$$

Doob and Cvetković [13] characterized all connected graphs with the least eigenvalue greater than -2 in the theorem below:

Theorem 2.1 ([13]) *Let G be a connected graph with $\lambda_n(G) > -2$. Then*

- (i) $G \in \mathcal{G}_1 = \{\ell(T) | T \text{ is a tree}\}$;
- (ii) $G \in \mathcal{G}_2 = \{L(T; 1, 0, \dots, 0) | T \text{ is a tree}\}$;
- (iii) $G \in \mathcal{G}_3 = \{\ell(H) | H \text{ is an odd-unicyclic graph}\}$;
- (iv) $G \in \mathcal{G}_4 = \{20 \text{ graphs with order 6 that are represented in } E_6\}$;
- (v) $G \in \mathcal{G}_5 = \{110 \text{ graphs with order 7 that are represented in } E_7\}$;
- (vi) $G \in \mathcal{G}_6 = \{443 \text{ graphs with order 8 that are represented in } E_8\}$.

For convenience, set $\mathcal{L} = \{G | G \text{ is a connected graph and } \lambda_n(G) \geq -2\}$, $\mathcal{L}^+ = \{G | G \text{ is a connected graph and } \lambda_n(G) > -2\}$ and $\mathcal{L}^0 = \{G | G \text{ is a connected graph and } \lambda_n(G) = -2\}$. Clearly, $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^0$ and $\mathcal{L}^+ = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$. A graph in $\mathcal{L}$ ($\mathcal{L}^+$ or $\mathcal{L}^0$) is called an $\mathcal{L}$ -graph ($\mathcal{L}^+$ -graph or $\mathcal{L}^0$ -graph). For the $\mathcal{L}^+$ -graphs we have the following:

Theorem 2.2 ([1]) *Let G be a $\mathcal{L}^+$ -graph with order n . Then*

$$\det(2I + A(G)) = \begin{cases} n+1, & \text{if } G \in \mathcal{G}_1; \\ 4, & \text{if } G \in \mathcal{G}_2 \cup \mathcal{G}_3; \\ 3, & \text{if } G \in \mathcal{G}_4; \\ 2, & \text{if } G \in \mathcal{G}_5; \\ 1, & \text{if } G \in \mathcal{G}_6. \end{cases}$$

The following contents of this section first appeared in [19, 20]. In order to the fullness of this paper, we will state it again. Cvetković and Lepović [8] adopted the nomenclature from lattice theory and defined

$$d_G = (-1)^n \phi(G, -2)$$

as the discriminant of an $\mathcal{L}$ -graph G . Additionally, for an $\mathcal{L}$ -graph G they obtained an important graph invariant named star value and showed that its formula is

$$S = \frac{(-1)^n}{(n-k)!} \phi^{(n-k)}(G, \lambda - 2) = (-1)^n \Pi_G(0) = \prod_{i=1}^k (\lambda_i + 2),$$

where $\phi^{(p)}(x)$ denotes the p -th derivative function of $\phi(x)$, $\Pi_G(\lambda) = \prod_{i=1}^k (\lambda - (\lambda_i + 2))$ (it is called the principal polynomial of G (see [8])) and $\lambda_1, \lambda_2, \dots, \lambda_k$ are the eigenvalues greater than -2 of G . For the discriminant and the star value of G , we have the following conclusion. If G is an $\mathcal{L}^0$ -graph, then $d_G = 0 < S$. On the other hand, it is easy to see that $\phi(G, \lambda - 2) = \lambda^{n-k} \Pi_G(\lambda)$ and then $d_G = S$ if G is an $\mathcal{L}^+$ -graph.

Now we synthesize the above facts into the following definition and give a visualized notation to the star value:

Definition 2.1 *Let G be an $\mathcal{L}$ -graph of order n and $\Pi_G(\lambda)$ the principal polynomial of G . Then the star value of G is defined as*

$$\star(G) = \begin{cases} (-1)^n \Pi_G(0), & \text{if } G \in \mathcal{L}^0; \\ (-1)^n \phi(G, -2), & \text{if } G \in \mathcal{L}^+. \end{cases}$$

The following corollary is an immediate consequence of Definition 2.1:

Corollary 2.1 Let $G = \cup_{i=1}^k G_i$ and H be two $\mathcal{L}$ -graphs. Then

- (i) $\star(G) = \bigcup_{i=1}^k \star(G_i)$;
- (ii) if G and H are A -cospectral, then $\star(G) = \star(H)$.

Corollary 2.1(ii) indicates that the star value of a graph is a graph invariant determined by the spectrum. The readers will see that it will play an important role in studying the spectral characterization of graphs with least eigenvalue at least -2 .

Note that $\phi(G, -2) = \det(-2I - A(G)) = (-1)^n \det(2I + A(G))$, where n is the order of G . Hence, $\det(2I + A(G)) = (-1)^n \phi(G, -2)$ and so the following corollary follows from Theorem 2.2 and Definition 2.1:

Corollary 2.2 Let G be an $\mathcal{L}^+$ -graph with order n . Then

$$\star(G) = \begin{cases} n+1, & \text{if } G \in \mathcal{G}_1; \\ 4, & \text{if } G \in \mathcal{G}_2 \cup \mathcal{G}_3; \\ 3, & \text{if } G \in \mathcal{G}_4; \\ 2, & \text{if } G \in \mathcal{G}_5; \\ 1, & \text{if } G \in \mathcal{G}_6. \end{cases}$$

3. Spectral characterization of generalized cocktail-party graphs

In what follows we will directly use a well-known fact that if G and H are A -cospectral graphs, then they respectively share the same numbers of order, size and closed walks of any length.

Note that any vertex in GCP is of degree $n-1$ or $n-2$. The following lemma indicates the reason that we adopt the notation $GCP(n, k)$ instead of GCP .

Lemma 3.1 A graph G with order n is a GCP iff $G = K_{n-2k} \nabla CP(k)$, where $k \geq 1$.

Proof The sufficiency follows from the fact that $CP(k)$ is a $(2k-2)$ -regular graph. For the necessity, by the definition of GCP we know that G is a bidegreed graph with vertex degree $n-1$ or $n-2$. Suppose that G has $n-t$ vertices of degree $n-1$. Clearly, $t \geq 2$. Thus, $G = K_{n-t} \nabla H$, where H is a $(t-2)$ -regular graph. Since the size $m(H) = t(t-2)/2$ is an integer, then t is even. Set $t = 2k$. Note that the order $n(H) = t$ and H is $(t-2)$ -regular. Therefore, H is obtained from K_t by removing t pairwise disjoint edges. So, $H = CP(k)$.

From Lemma 3.1 it is easy to see $GCP = GCP(n, k) = L(K_{1, n-2k}; k, 0, \dots, 0)$ which is a generalized line graph. Next we give some details for its index and least eigenvalue. Let $\rho_{n,k} = \rho(GCP(n, k))$ and $\lambda_{n,k} = \lambda_n(GCP(n, k))$. The following lemma is needed.

Lemma 3.2 ([4]) Let G be an r -regular graph with order n and $\overline{G}$ its complement. Then

$$\phi(\overline{G}, \lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} \phi(G, -\lambda - 1).$$

Lemma 3.3 Let $k \geq 1$. Then

- (i) $GCP(n, k)$ has exactly one positive eigenvalue $\rho_{n,k}$.
- (ii) $\rho_{n,k} = \frac{n-3+\sqrt{(n-3)^2+8(n-k-1)}}{2} \geq n-2$ with equality if and only if $n = 2k$.
- (iii) $\lambda_{n,k} \geq -2$ with equality if and only if $k > 1$.

Proof By Lemma 3.2 we get

$$\phi(CP(k)) = \lambda^k(\lambda + 2)^{k-1}(\lambda - 2k + 2).$$

From Lemma 2.1 it follows that

$$\begin{aligned} \phi(GCP(n, k)) &= \phi(K_{n-2k} \nabla CP(k)) \\ &= \frac{\phi(K_{n-2k})\phi(CP(k))}{(\lambda - n + 2k + 1)(\lambda - 2k + 2)} [(\lambda - n + 2k + 1)(\lambda - 2k + 2) - 2k(n - 2k)] \\ &= \lambda^k(\lambda + 1)^{n-2k-1}(\lambda + 2)^{k-1}[\lambda^2 - (n - 3)\lambda - 2(n - k - 1)] \end{aligned}$$

which shows that (iii) obviously holds. Let $h_1(\lambda)$ denote the above quadratic factor. It is easy to see that $\rho_{n,k}$ is the largest root of $h_1(\lambda) = 0$. Thus, $\rho_{n,k} = \frac{n-3+\sqrt{(n-3)^2+8(n-k-1)}}{2}$. Since $n - 2k \geq 0$, we have $8(n - k - 1) \geq 4n - 8$ and so $\rho_{n,k} \geq \frac{n-3+\sqrt{(n-3)^2+4n-8}}{2} = n - 2$ with equality if and only if $n = 2k$. Thus, (ii) holds. Let λ be another root of $h_1(\lambda) = 0$. Then $\lambda \cdot \rho_{n,k} = -2(n - k - 1)$ and thus λ is negative and (i) follows.

Remark 3.1 For $n = 2k$, then $G = CP(k)$ which has been shown to be a DAS-graph [11]. Thus, we always set $n - 2k > 0$ in this subsection and so $\rho_{n,k} > n - 2$.

Lemma 3.4 ([17]) A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.

Note that from Lemma 3.3(i) and Lemma 3.4 we know $GCP(n, k)$ is also a complete $(n - k)$ -multipartite graph. Actually, $GCP(n, k) = K_{1^{n-2k}, 2^k}$, where a^b means that b is the number of parts of cardinality a . The following observation follows readily from Lemma 3.4:

Lemma 3.5 Let G be a graph A -cospectral with $GCP(n, k)$. Then G is the union of some isolated vertices and a complete multipartite graph.

Lemma 3.6 No two non-isomorphic generalized cocktail-party graphs are A -cospectral.

Proof Assume by way of contradiction that $GCP(n', k')$ and $GCP(n, k)$ are A -cospectral. Then $n' = n$ and $\rho_{n', k'} = \rho_{n, k} = \rho_{n, k}$. From Lemma 3.3(i) it follows that

$$\frac{n - 3 + \sqrt{(n - 3)^2 + 8(n - k' - 1)}}{2} = \rho_{n, k'} = \rho_{n, k} = \frac{n - 3 + \sqrt{(n - 3)^2 + 8(n - k - 1)}}{2},$$

which leads to $k' = k$ and thus $GCP(n', k') \cong GCP(n, k)$.

To prove the following theorem, we need the notion of a maximal exceptional graph: every exceptional graph is a subgraph of (a least) one such graph.

Theorem 3.1 For $n \geq 23$, $GCP(n, k)$ is a DAS-graph.

Proof If $k = 0$, then $GCP(n, 0) = K_n$ which has been proved to be a *DAS*-graph [11]. Now set $k \geq 1$. Let G be any graph A -cospectral to $GCP(n, k)$. Then $n(G) = n$, $\rho(G) = \rho_{n,k}$ and $\lambda_n(G) = \lambda_{n,k}$. From the known fact that $\Delta(G) \geq \rho(G)$, we obtain by Lemma 3.3(i) that $\Delta(G) > n - 2$, and so $\Delta(G) = n - 1$. Hence, G is a connected graph and therefore G is a multipartite graph by Lemma 3.5. From Lemma 3.3(iii) it follows that $\lambda_n(G) = \lambda_{n,k} \geq -2$. Consequently, G is one of the graphs described in Theorem 1.1.

Claim 3.1 G cannot be represented by vectors in the root system E_8 .

Proof Assume by contradiction that G is such a graph. In order to show the claim, we need an important result due to Cvetković, Lepović, Rowlinson and Simić [9], which is that all the maximal exceptional graphs are determined. Such a graph has order 22, 28, 29, 30, 31, 32, 33, 34 or 36. Denote by ρ_n the maximal index of a maximal exceptional graph of order n . So $\rho(G) \leq \rho_n$. From Table 1 [9] we get the following table:

n	22	28	29	30	31	32	33	34	36
ρ_n	14	17	19	18.9282	19	19.2111	19.5498	20	21

Table 1 The index of maximal exceptional graphs

It follows that an exceptional graph has index at most 21. By Remark 3.1 we have $\rho_{n,k} > n - 2$. Hence, for $n \geq 23$ we get $\rho_{n,k} > 21 \geq \rho_n \geq \rho(G)$, a contradiction.

So, Claim 3.1 shows that G is a generalized line graph. Without loss of generality, set $G = L(H; a_1, a_2, \dots, a_n)$ and $V(G) = V(\ell(H)) \cup V(CP(a_1)) \cup \dots \cup V(CP(a_n))$. From the definition of generalized line graph it follows that $V(\ell(H)) \cap V(CP(a_i)) = \emptyset$ ($1 \leq i \leq n$) and $V(CP(a_i)) \cap V(CP(a_j)) = \emptyset$ ($1 \leq i \neq j \leq n$). Since G is a complete multipartite graph and $CP(a_i) = K_{2^{a_i}}$ ($1 \leq i \leq n$), $V(CP(a_i))$ can be partitioned into $C_{i1} \cup C_{i2} \cup C_{ia_i}$, where each cell C_{ij} contains exactly two vertices ($1 \leq j \leq a_i$ and $1 \leq i \leq n$).

Claim 3.2 At most one of $a_1, a_2, \dots, a_n$ is not equal to 0.

Proof Assume that $a_i \neq 0$ and $a_j \neq 0$ ($1 \leq i \neq j \leq n$). Let $u \in C_{ik} \subset V(CP(a_i))$ ($1 \leq k \leq a_i$) and $v \in C_{jl} \subset V(CP(a_j))$ ($1 \leq l \leq a_j$). Since $V(CP(a_i)) \cap V(CP(a_j)) = \emptyset$, we have $C_{ik} \cap C_{jl} = \emptyset$ and so $uv \notin E(G)$. On the other hand, since u and v belong to different cells, u must be adjacent to v , a contradiction.

From Claim 3.2, without loss of generality, we can set $a_2 = \dots = a_n = 0$.

Claim 3.3 The A -cospectral graph G is equal to $GCP(n, a_1) = K_{n-2a_1} \nabla CP(a_1)$.

Proof Since G is a multipartite graph and $V(\ell(H)) \cap V(CP(a_1)) = \emptyset$, every vertex of $\ell(H)$ must be adjacent to each one of $CP(a_1)$, which shows that all vertices e_i ($1 \leq i \leq n(\ell(H))$) of $\ell(H)$ must be such that all edges e_i ($1 \leq i \leq m(H)$) have a common vertex, say v_1 , in the graph

H . Hence, H is a star and so $\ell(H)$ is a clique denoted by K_t . Since $n(G) = t + 2a_1 = n$, we have $t = n - 2a_1$ and thus $G = GCP(n, a_1)$.

From Claim 3.3 and Lemma 3.6, we obtain that $G \cong GCP(n, k)$ and we are done.

Acknowledgement The first author would like to thank Prof. Peter Rowlinson for checking the English and pointing out a small mistake in the original manuscript. The authors are also grateful to the referees whose suggestions improved the final form of this manuscript.

References

- [1] A. E. BROUWER, A. M. COHEN, A. NEUMAIER. *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
- [2] P. J. CAMERON, J. M. GOETHALS, J. J. SEIDEL, et al. *Line graphs, root systems, and elliptic geometry*. J. Algebra, 1976, **43**(1): 305–327.
- [3] P. J. CAMERON, J. H. VAN LINT. *Designs, Graphs, Codes and Their Links*. Cambridge University Press, Cambridge, 1991.
- [4] D. CVETKOVIĆ, M. DOOB, H. SACHS. *Spectra of Graphs. Theory and Applications (III)*. Johann Ambrosius Barth, Heidelberg, 1995.
- [5] D. CVETKOVIĆ, M. DOOB, S. K. SIMIĆ. *Generalized line graphs*. J. Graph Theory 1981, **5**: 385–399.
- [6] D. CVETKOVIĆ, M. LEPOVIĆ. *A table of cospectral graphs with least eigenvalue at least -2* . <http://www.mi.sanu.ac.yu/projects/results1389.htm>
- [7] D. CVETKOVIĆ, M. LEPOVIĆ. *Sets of cospectral graphs with least eigenvalue -2 and some related results*. Bull. Cl. Sci. Math. Nat. Sci. Math., 2004, **29**: 85–102.
- [8] D. CVETKOVIĆ, M. LEPOVIĆ. *Cospectral graphs with least eigenvalue at least -2* . Publ. Inst. Math. (Beograd) (N.S.), 2005, **78**(92): 51–63.
- [9] D. CVETKOVIĆ, M. LEPOVIĆ, P. ROWLINSON, et al. *The maximal exceptional graphs*. J. Combin. Theory Ser. B, 2002, **86**(2): 347–363.
- [10] D. CVETKOVIĆ, M. LEPOVIĆ, P. ROWLINSON, et al. *Spectral Generalizations of Line Graphs, On Graphs with Least Eigenvalue -2* . Cambridge University Press, Cambridge, 2004.
- [11] E. R. VAN DAM, W. H. HAEMERS. *Which graphs are determined by their spectra*. Linear Algebra Appl. 2003, **373**: 241–272.
- [12] E. R. VAN DAM, W. H. HAEMERS. *Developments on spectral characterizations of graphs*. Discrete Math., 2009, **309**(3): 576–586.
- [13] M. DOOB, D. CVETKOVIĆ. *On spectral characterizations and embeddings of graphs*. Linear Algebra Appl., 1979, **27**: 17–26.
- [14] C. GODSI, G. ROYLE. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.
- [15] H. H. GÜNTARD, H. PRIMAS. *Zusammenhang von Graphtheorie und Mo-Theotie von Molekeln mit Systemen konjugierter Bindungen*. Helv. Chim. Acta 1956, **39**: 1645–1653.
- [16] A. J. HOFFMAN. *Combinatorial Structures and their Applications*. Gordon and Breach, Science Publishers, New York-London-Paris, 1970.
- [17] J. H. SMITH, *Some Properties of the Spectrum of a Graph*. Gordon and Breach, New York, 1970.
- [18] Kexiang XU, Juan YAN, Baogang XU, et al. *Some results on graph products determined by their spectra*. J. Math. Res. Exposition, 2010, **30**(2): 227–232.
- [19] Jianfeng WANG, Yufa SHEN, Qiongxiang HUANG. *Notes on graphs with least eigenvalue at least -2* . Electronic Journal of Linear Algebra, 2012, **23**: 387–396.
- [20] Jianfeng WANG, Shuning SHI. *The line graphs of lollipop graphs are determined by their spectra*. Linear Algebra Appl., 2012, **436**(7): 2630–2637.