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On the Congruence $\sigma(n) \equiv 1 \pmod{n}$

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Abstract Let $k \geq 2$ be an integer, and let $\sigma(n)$ denote the sum of the positive divisors of an integer n. We call n a quasi-multiperfect number if $\sigma(n) = kn + 1$. In this paper, we give some necessary properties of them.

Keywords quasiperfect number; quasi-multiperfect number.

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1. Introduction

For a positive integer n, let $\phi(n)$, $\omega(n)$ and $\sigma(n)$ denote the Euler function of n, the number of distinct prime factors of n and the sum of the positive divisors of n, respectively. Let a be an integer. C. Pomerance [8] proved that $S(a) = \{n : \sigma(n) \equiv a \pmod{n}\}$ has density 0. It is known that $\sigma(n) = n+1$ if and only if n is prime, thus we only need to consider $\sigma(n) = kn+1$ for $k \geq 2$. We call n a quasi-multiperfect(QM) number if $\sigma(n) = kn + 1$ with $k \geq 2$. In particular, we call n a quasiperfect number if $\sigma(n) = 2n + 1$ and n a quasi-triperfect(QT) number if $\sigma(n) = 3n + 1$. Up to now, no quasiperfect numbers are known, but necessary properties of them are described in many papers [1–7].

In this paper, we obtain the following results:

Theorem 1 *If n is QM and odd, then $\omega(n) \geq 7$. If n is QM and even, then $\omega(n) \geq 3$.*

Theorem 2 *If n is an even QM and $\omega(n) = 3$, then n is QT and $n = 2^{\alpha_1}3^{\alpha_2}p^2$, where α_1, α_2 are even,*

$$p = \Big[\frac{2^{\alpha_1+1}3^{\alpha_2+1}}{2^{\alpha_1+1} + 3^{\alpha_2+1} - 1}\Big],\ p > \sqrt{2^{\alpha_1+1}},\ p > \sqrt{3^{\alpha_2+1}}.$$

In fact,

$$\frac{2^{\alpha_1+1}3^{\alpha_2+1}}{2^{\alpha_1+1} + 3^{\alpha_2+1} - 1} - \frac{1}{2} < p < \frac{2^{\alpha_1+1}3^{\alpha_2+1}}{2^{\alpha_1+1} + 3^{\alpha_2+1} - 1}.$$

Corollary 3 *If n is an even QM and $\omega(n) = 3$, then $n > 10^{100}$.*

Conjecture 4 There is no even QM with three different divisors.

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2. Proofs

Lemma 1 ([5, Theorem 3]) *If n is a quasiperfect number, then $\omega(n) \geq 7$.*

Lemma 2 *If n is QT and $n = 2^{\alpha_1}3^{\alpha_2}p^{\alpha_3}$, then $\alpha_i' s$ are even for $1 \leq i \leq 3$.*

Proof Since n is QT, we have $\sigma(n) \equiv 1 \pmod 3$. If α_1 is odd, then $\sigma(2^{\alpha_1}) = 2^{\alpha_1+1} - 1 \equiv 0 \pmod 3$, thus $\sigma(n) \equiv 0 \pmod 3$, a contradiction. Hence α_1 is even. Since n is QT, we have

$$\sigma(n) = \sigma(2^{\alpha_1}3^{\alpha_2}p^{\alpha_3}) = 2^{\alpha_1}3^{\alpha_2+1}p^{\alpha_3} + 1 \equiv 1 \pmod 2.$$

That is,

$$\sigma(n) = (1 + 2 + \cdots + 2^{\alpha_1})(1 + 3 + \cdots + 3^{\alpha_2})(1 + p + \cdots + p^{\alpha_3}) \equiv 1 \pmod 2.$$

We have $(\alpha_2 + 1)(\alpha_3 + 1) \equiv 1 \pmod 2$, hence α_2 and α_3 must be even. This completes the proof of Lemma 2. $\square$

Proof of Theorem 1 Assume that $\sigma(n) = kn + 1$ and n is odd. If $\omega(n) \leq 6$, then we have

$$2 \leq k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} < 3.$$

Thus $k = 2$, n is a quasiperfect number. By Lemma 1, we have $\omega(n) \geq 7$, a contradiction.

Assume that $\sigma(n) = kn + 1$ and n is even, if $\omega(n) = 1$ or 2, then we have

$$2 \leq k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \leq \frac{2}{1} \cdot \frac{3}{2} = 3.$$

Thus $k = 2$, n is a quasiperfect number. By a result of Cattaneo [2], n must be an odd square, a contradiction. This completes the proof of Theorem 1. $\square$

Proof of Theorem 2 Since n is an even QM and $\omega(n) = 3$, we may assume that $n = 2^{\alpha_1}p_1^{\alpha_2}p_2^{\alpha_3}$, where $3 \leq p_1 < p_2$. If $p_1 \geq 5$, then

$$2 \leq k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \leq \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{7}{6} < 3,$$

thus $k = 2$, n is a quasiperfect number. By Lemma 1, this is impossible. Hence $p_1 = 3$ and

$$2 \leq k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \leq \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} = 3.75.$$

Then $\sigma(n) = 2n + 1$ or $\sigma(n) = 3n + 1$. By Lemma 1, we have $\sigma(n) = 3n + 1$, thus n is QT and $n = 2^{\alpha_1}3^{\alpha_2}p^{\alpha_3}$. Hence we have

$$(2^{\alpha_1+1} - 1)(3^{\alpha_2+1} - 1)(p^{\alpha_3+1} - 1) = 2^{\alpha_1+1}3^{\alpha_2+1}p^{\alpha_3}(p - 1) + 2(p - 1). \tag{1}$$

Let $\{a, b\} = \{2^{\alpha_1+1}, 3^{\alpha_2+1}\}$ with $a > b \geq 8$. Then

$$(a - 1)(b - 1)(p^{\alpha_3+1} - 1) = abp^{\alpha_3}(p - 1) + 2(p - 1). \tag{2}$$

By (2) we have

$$p^{\alpha_3}\Big(p(a - 1)(b - 1) - (p - 1)ab\Big) = (a - 1)(b - 1) + 2(p - 1).$$

That is,

$$p^{\alpha_3}\Big(ab-(a+b-1)p\Big)=(a-1)(b-1)+2(p-1). \tag{3}$$

By (3) we have $ab-(a+b-1)p>0$, thus

$$p<\frac{ab}{a+b-1}. \tag{4}$$

By (4) we have $p<a$, $p<b$. By (2) and $p\nmid 2(p-1)$ we have $p\nmid b-1$. Hence $p\le b-2$. By (2) we have

$$b\big((a-1)(p^{\alpha_3+1}-1)-ap^{\alpha_3}(p-1)\big)=(a-1)(p^{\alpha_3+1}-1)+2(p-1)>0. \tag{5}$$

Thus

$$(p+2)\big((a-1)(p^{\alpha_3+1}-1)-ap^{\alpha_3}(p-1)\big)\le(a-1)(p^{\alpha_3+1}-1)+2(p-1). \tag{6}$$

Then

$$a\big((p+2)(p^{\alpha_3+1}-1)-(p+2)p^{\alpha_3}(p-1)-(p^{\alpha_3+1}-1)\big)\le(p+1)(p^{\alpha_3+1}-1)+2(p-1).$$

That is,

$$a(2p^{\alpha_3}-p-1)\le p^{\alpha_3+2}+p^{\alpha_3+1}+p-3.$$

Hence

$$\begin{aligned}a&\le\frac{p^{\alpha_3+2}+p^{\alpha_3+1}+p-3}{2p^{\alpha_3}-p-1}=\frac{p^2+p+\frac{p-3}{p^{\alpha_3}}}{2-\frac{p+1}{p^{\alpha_3}}}\\&\le\frac{p^2+p+\frac{p-3}{p^2}}{2-\frac{p+1}{p^2}}=\frac{p^4+p^3+p-3}{2p^2-p-1}<p^2.\end{aligned}$$

Then we have

$$\sqrt{a}<p<b. \tag{7}$$

By (3) and (7) we have

$$\begin{aligned}p^{\alpha_3}&\le p^{\alpha_3}\big(ab-(a+b-1)p\big)=(a-1)(b-1)+2(p-1)\\&\le(p^2-2)(p^2-3)+2(p-1)=p^4-5p^2+2p+4<p^4.\end{aligned}$$

Since α_3 is even and $\alpha_3>0$, we have $\alpha_3=2$.

By (3) and $\alpha_3=2$ we have

$$\begin{aligned}p^2\big(ab-(a+b-1)p\big)&=(a-1)(b-1)+2(p-1)\le(p^2-2)(b-1)+2(p-1)\\&=p^2(b-1)-2(b-p)<p^2(b-1).\end{aligned}$$

Thus $ab-(a+b-1)p<b-1$. So

$$p>\frac{ab}{a+b-1}-\frac{b-1}{a+b-1}>\frac{ab}{a+b-1}-\frac{1}{2}. \tag{8}$$

By (4) and (8), we have

$$\frac{2^{\alpha_1+1}3^{\alpha_2+1}}{2^{\alpha_1+1}+3^{\alpha_2+1}-1}-\frac{1}{2}<p<\frac{2^{\alpha_1+1}3^{\alpha_2+1}}{2^{\alpha_1+1}+3^{\alpha_2+1}-1},$$

thus

$$p = \Big[\frac{2^{\alpha_1+1}3^{\alpha_2+1}}{2^{\alpha_1+1}+3^{\alpha_2+1}-1}\Big].$$

This completes the proof of Theorem 2. □

Proof of Corollary 3 If n is an even QM with $\omega(n)=3$ and $n\le 10^{100}$, then by Theorem 2 we have $p>\sqrt{2^{\alpha_1+1}}$ and $p>\sqrt{3^{\alpha_2+1}}$, thus

$$\big(2^{\alpha_1}3^{\alpha_2}\big)^{3/2}\cdot\sqrt{6} < n = 2^{\alpha_1}3^{\alpha_2}p^2 \le 10^{100}.$$

Then we have $2^{\alpha_1}3^{\alpha_2} < (\frac{10^{100}}{\sqrt{6}})^{2/3}$, hence $\alpha_1\le 216$ and $\alpha_2\le 136$.

Let

$$a=2^{\alpha_1+1},\ b=3^{\alpha_2+1},\ p=\Big[\frac{ab}{a+b-1}\Big].$$

By (2) in the proof of Theorem 2, we have

$$(a+b-1)p^2-(a-1)(b-1)(p+1)+2=0.$$

Using Mathematica, it is easy to verify that if $\alpha_1\in\{2u:\ u=1,2,\dots,108\}$, $\alpha_2\in\{2v:\ v=1,2,\dots,68\}$, then

$$(a+b-1)p^2-(a-1)(b-1)(p+1)+2\ne 0,$$

a contradiction. This completes the proof of Corollary 3. □

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