Journal of Mathematical Research with Applications
Nov., 2012, Vol. 32, No. 6, pp. 715–722
DOI:10.3770/j.issn:2095-2651.2012.06.011
Http://jmre.dlut.edu.cn

The Second Critical Exponent for a Fast Diffusion Equation with Potential

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Abstract This paper considers a fast diffusion equation with potential $u_t = \Delta u^m - V(x)u^m + u^p$ in $\mathbb{R}^n \times (0,T)$, where $1-\frac{2}{\alpha m+n} < m \le 1$, $p>1$, $n \ge 2$, $V(x) \sim \frac{\omega}{|x|^2}$ with $\omega \ge 0$ as $|x| \to \infty$, and α is the positive root of $\alpha m(\alpha m + n - 2) - \omega = 0$. The critical Fujita exponent was determined as $p_c = m + \frac{2}{\alpha m+n}$ in a previous paper of the authors. In the present paper, we establish the second critical exponent to identify the global and non-global solutions in their co-existence parameter region $p > p_c$ via the critical decay rates of the initial data. With $u_0(x) \sim |x|^{-a}$ as $|x| \to \infty$, it is shown that the second critical exponent $a^* = \frac{2}{p-m}$, independent of the potential parameter ω, is quite different from the situation for the critical exponent p_c.

Keywords the second critical exponent; fast diffusion equation; potential; global solutions; blow-up.

MR(2010) Subject Classification 35K59; 35B33

1. Introduction

In this paper, we investigate the second critical exponent to the fast diffusion equation with source and quadratically decaying potential

$$\begin{cases} u_t = \Delta u^m - V(x)u^m + u^p, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) \gneq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $1-\frac{2}{\alpha m+n} < m \le 1$, $p>1$, $n \ge 2$, $V(x) \sim \frac{\omega}{|x|^2}$ with $\omega \ge 0$ as $|x| \to \infty$, $\alpha > 0$ is an explicit parameter related to ω, determined by

$$\alpha m(\alpha m + n - 2) - \omega = 0$$

with $u_0(x)$ continuous and bounded.

It is well known that the Cauchy problem

$$u_t = \Delta u + u^p \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.2}$$

Received February 19, 2012; Accepted March 27, 2012
Supported by the National Natural Science Foundation of China (Grant No. 11171048).
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admits a critical exponent $p_c = 1 + \frac{2}{n}$, such that the solutions blow up in finite time for any nontrivial initial data whenever $1 < p \le p_c$, and there are both global solutions and non-global solutions if $p > p_c$ (see [3]) and [1, 8, 9] (for the critical case). From then on, the Fujita phenomena have been studied for a great deal of PDEs [2, 11]. To identify the global and non-global solutions in their co-existence region, the so-called second critical exponent was introduced by Lee and Ni [10] with $a^* = \frac{2}{p-1}$ for the Cauchy problem (1.2). That is to say, with initial data $u_0(x) = \lambda\varphi(x)$ and $p > p_c = 1 + \frac{2}{n}$, there exist constants $\mu, \Lambda, \Lambda_0 > 0$ such that the solutions blow up in finite time whenever $\liminf_{x\to\infty} |x|^{a^*}\varphi(x) > \mu > 0$, $\lambda > \Lambda$, and must be global if $a \ge a^*$ with $\limsup_{x\to\infty} |x|^a\varphi(x) < \infty$, $\lambda < \Lambda_0$.

For the nonlinear diffusion case

$$u_t = \Delta u^m + u^p \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.3}$$

with $m > 1$ or $\max\{0, 1-\frac{2}{n}\} < m < 1$, the critical exponent was proved as $p_c = m + \frac{2}{n}$ by Galaktionov et al. [4, 5], Mochizuki and Mukai [12], and Qi [16, 17]. The second critical exponent to (1.3) was obtained with $a^* = \frac{2}{p-m}$ by Mukai et al. [14] (for $1 < m < p$) and Guo [7] (for $(1-\frac{2}{n}) < m < 1$).

Recently, the Fujita phenomena for reaction-diffusion equations with potentials have been thoroughly studied as well. Zhang [19] studied the influence of the potential on the critical exponent, without considering the quadratically decaying potential, which was shown as $p_c = 1 + \frac{2}{\alpha+n}$ by Ishige [15] for (1.1) with $m = 1$, and $p_c = m + \frac{2}{\alpha m+n}$ for $1 - \frac{2}{\alpha m+n} < m < 1$ by the authors [18].

The present paper aims to investigate the second critical exponent for the problem (1.1) with obtaining $a^* = \frac{2}{p-m}$. It is interesting that, differently from the situation for the critical Fujita exponent p_c, the second critical exponent a^* is independent of the quadratically decaying potential.

Denote by $C_b(\mathbb{R}^n)$ the space of all bounded continuous functions in $\mathbb{R}^n$, and let

$$\mathbb{I}_a = \Big\{\varphi(x) \in C_b(\mathbb{R}^n) |\, \varphi(x) \ge 0, \liminf_{|x|\to\infty} |x|^a\varphi(x) > 0\Big\},$$

$$\mathbb{I}^a = \Big\{\varphi(x) \in C_b(\mathbb{R}^n) | \varphi(x) \ge 0, \limsup_{|x|\to\infty} |x|^a\varphi(x) < \infty\Big\}.$$

The main result of the paper is the following two theorems.

Theorem 1 *Let $p > p_c = m + \frac{2}{\alpha m+n}$, $n \ge 2$, $V(x) \le \frac{\omega}{|x|^2}$ for $|x|$ large, the initial data $u_0 = \lambda\varphi(x)$, $\varphi(x) \in \mathbb{I}_a$ for $\lambda > 0$. If $a \in (0, a^*)$, or $a \ge a^*$ with λ large enough, then the solution of (1.1) blows up in finite time.*

Theorem 2 *Let $p > p_c = m + \frac{2}{\alpha m+n}$, $V(x) \ge \frac{\omega}{|x|^2}$ in $\mathbb{R}^n \setminus \{0\}$, $u_0 = \lambda\varphi(x)$ with $\varphi(x) = |x|^\alpha\psi(x) \in \mathbb{I}^a$ and $\lambda > 0$. If $a > a^*$, then there exist $\lambda_0, C_1 > 0$ such that the solution u is global in time and satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \le C_1 t^{-\frac{a}{2-a(1-m)}} \quad \text{for all } t > 0$$

whenever $\lambda \in (0, \lambda_0)$.

We will prove the two theorems in Sections 2 and 3, respectively.

2. Non-global solutions

In this section, we deal with the blow-up of solutions to prove Theorem 1.

Proof of Theorem 1 Let $U(r,t)$ be a radial solution to (1.1) with initial data $0 \le U_0(r) = \lambda\bar{\varphi}(r) \le u_0(x) = \lambda\varphi(x)$, $\bar{\varphi}(r) \in \mathbb{I}_a$, $\bar{\varphi}'(r) \le 0$. Similarly to [18], we introduce a series of transformations. Set $U(r,t) = r^\alpha v(r,t)$, $v(r,t) = \xi(\rho,t)$ with

$$\rho = \rho(r) = \frac{2}{\alpha(1-m)+2} r^{\frac{\alpha(1-m)+2}{2}}. \tag{2.1}$$

Then due to $V(x) \le \frac{\omega}{|x|^2}$ for $|x|$ large, we have

$$\xi_t \ge (\xi^m)_{\rho\rho} + \frac{N-1}{\rho}(\xi^m)_\rho + C_0\rho^q\xi^p, \quad \rho > \rho_0,\ t \in (0,T),$$

for ρ_0 large enough, with

$$\xi(\rho,0) = \lambda\big(\frac{\alpha(1-m)+2}{2}\rho\big)^{-\frac{2\alpha}{\alpha(1-m)+2}}\bar{\varphi}\big(\big(\frac{\alpha(1-m)+2}{2}\rho\big)^{\frac{2}{\alpha(1-m)+2}}\big), \quad \rho > \rho_0,$$

where

$$N = \frac{2\alpha+2\alpha m+2n}{\alpha(1-m)+2}, \quad C_0 = \Big[\frac{\alpha(1-m)+2}{2}\Big]^{\frac{2\alpha(p-1)}{\alpha(1-m)+2}}, \quad q = \frac{2\alpha(p-1)}{\alpha(1-m)+2}. \tag{2.2}$$

Let w solve

$$\begin{cases} w_t = (w^m)_{\rho\rho} + \frac{N-1}{\rho}(w^m)_\rho + C_0\rho^q w^p, & (\rho,t) \in (\rho_0,\infty)\times(0,T), \\ w(\rho_0,t) = U|_{\rho=\rho_0}, & t \in (0,T), \\ w(\rho,0) = \lambda\big(\frac{\alpha(1-m)+2}{2}\rho\big)^{-\frac{2\alpha}{\alpha(1-m)+2}}\bar{\varphi}\big(\big(\frac{\alpha(1-m)+2}{2}\rho\big)^{\frac{2}{\alpha(1-m)+2}}\big), & \rho \in (\rho_0,\infty). \end{cases}$$

It suffices to prove the finite time blow-up of w when $\bar{\varphi}(r) \in \mathbb{I}_a$ for $0 < a < a^*$, or $a \ge a^*$ with λ large enough.

Set

$$S_\varepsilon(r) = \zeta_R(r)\mathrm{e}^{-\varepsilon(r-R)^2} \quad \text{in } (R,\infty), \tag{2.3}$$

with

$$\zeta_R(r) = \begin{cases} \frac{r-R}{r}, & N \ge 3, \\ \log r - \log R, & 2 \le N < 3. \end{cases}$$

By Lemmas 4.1 and 4.2 in [13] with a simple computation, we know that $S_\varepsilon \in C^2(R,\infty)$ and satisfies

$$\begin{aligned} & S_\varepsilon'' + \frac{N-1}{r}S_\varepsilon' \ge -2(N+2)\varepsilon S_\varepsilon \quad \text{in } (R,\infty), \\ & S_\varepsilon(R) = S_\varepsilon(\infty) = 0, \\ & S_\varepsilon > 0 \quad \text{in } (R,\infty) \text{ with } \int_R^\infty S_\varepsilon(r) r^{N-1}\mathrm{d}r < \infty. \end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0}\varepsilon^{N/2}\int_R^\infty S_\varepsilon(r)r^{N-1}\mathrm{d}r=\pi^{\frac{N}{2}},\quad N\geq 3, \tag{2.4}$$

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{N}{2}}[\log\varepsilon^{-\frac{1}{2}}]^{-1}\int_R^\infty S_\varepsilon(r)r^{N-1}\mathrm{d}r=\pi^{\frac{N}{2}},\quad 2\leq N<3. \tag{2.5}$$

Since $\bar\varphi(r)\in\mathbb{I}_a$, there are $L,R_1>0$ such that $\bar\varphi(r)\geq Lr^{-a}$ for all $r\geq R_1$, and hence $\bar\varphi\big((\frac{\alpha(1-m)+2}{2}\rho)^{\frac{2}{\alpha(1-m)+2}}\big)\geq L(\frac{\alpha(1-m)+2}{2}\rho)^{-\frac{2a}{\alpha(1-m)+2}}$ for all $\rho\geq\rho_1$ with ρ_1 large enough. Set $\rho_2=\max\{\rho_0,\rho_1\}$, and denote

$$\psi_\varepsilon(\rho)=C_\varepsilon S_\varepsilon(\rho)\quad\text{in }(\rho_2,\infty), \tag{2.6}$$

with

$$C_\varepsilon=\Big(\int_{\rho_2}^\infty S_\varepsilon(\rho)\rho^{N-1}\mathrm{d}\rho\Big)^{-1}. \tag{2.7}$$

Define

$$y(t)=\int_{\rho_2}^\infty w(\rho,t)\psi_\varepsilon(\rho)\rho^{N-1}\mathrm{d}\rho.$$

By Hölder's inequality,

$$\begin{aligned}y'(t)&=\int_{\rho_2}^\infty w^m\rho^{N-1}\big(\psi_\varepsilon''+\frac{N-1}{\rho}\psi_\varepsilon'\big)\mathrm{d}\rho+C_0\int_{\rho_2}^\infty\rho^{N+q-1}\psi_\varepsilon w^p\mathrm{d}\rho\\&\geq-2(N+2)\varepsilon y^m(t)+C_0\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{-(p-1)}y^p(t)\\&=y^p(t)\left(C_0\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{-(p-1)}-2(N+2)\varepsilon y^{-(p-m)}(t)\right),\end{aligned}$$

and thus

$$y'(t)\geq\frac{C_0}{2}\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{-(p-1)}y^p(t) \tag{2.8}$$

provided

$$y^{-(p-m)}(t)\leq\frac{C_0\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{-(p-1)}}{4(N+2)\varepsilon},$$

or equivalently,

$$y(t)\geq C_0^{-\frac{1}{p-m}}\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{\frac{p-1}{p-m}}\big(4(N+2)\varepsilon\big)^{\frac{1}{p-m}},$$

which is ensured by

$$y(0)\geq C_0^{-\frac{1}{p-m}}\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{\frac{p-1}{p-m}}\big(4(N+2)\varepsilon\big)^{\frac{1}{p-m}}. \tag{2.9}$$

We deduce from (2.8) that

$$y(t)\geq\Big(y^{-(p-1)}(0)-\frac{C_0}{2}\Big(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho\Big)^{-(p-1)}(p-1)t\Big)^{-\frac{1}{p-1}},$$

and hence, $y(t)\to\infty$ as $t\to T=\frac{2}{C_0(p-1)}(\int_{\rho_2}^\infty\rho^{N-1-\frac{q}{p-1}}\psi_\varepsilon\mathrm{d}\rho)^{p-1}y^{-(p-1)}(0)$. This concludes that w blows up in finite time for large initial data required by (2.9).

Now we verify that the condition (2.9) is valid under either $a \in (0, a^*)$, or $a \geq a^*$ with λ large. By (2.3) and (2.6),

$$\begin{aligned} y(0) &= \int_{\rho_2}^{\infty} w(\rho,0)\psi_\varepsilon(\rho)\rho^{N-1}\mathrm{d}\rho \\ &\geq \lambda L\Big(\frac{\alpha(1-m)+2}{2}\Big)^{-\frac{2(a+\alpha)}{\alpha(1-m)+2}} C_\varepsilon \int_{\rho_2}^{\infty} \rho^{N-1-\frac{2(a+\alpha)}{\alpha(1-m)+2}}\zeta_{\rho_2}(\rho)\mathrm{e}^{-\varepsilon(\rho-\rho_2)^2}\mathrm{d}\rho \\ &= \lambda L\Big(\frac{\alpha(1-m)+2}{2}\Big)^{-\frac{2(a+\alpha)}{\alpha(1-m)+2}} C_\varepsilon \varepsilon^{-\frac{N}{2}+\frac{a+\alpha}{\alpha(1-m)+2}} \int_{\sqrt{\varepsilon}\rho_2}^{\infty} \tau^{N-1-\frac{2(a+\alpha)}{\alpha(1-m)+2}}\zeta_{\rho_2}\big(\frac{\tau}{\sqrt{\varepsilon}}\big)\mathrm{e}^{-(\tau-\sqrt{\varepsilon}\rho_2)^2}\mathrm{d}\tau. \end{aligned} \tag{2.10}$$

On the other hand, (2.9) is equivalent to

$$\begin{aligned} y(0) &\geq C_0^{-\frac{1}{p-m}} C_\varepsilon^{\frac{p-1}{p-m}} \Big(\int_{\rho_2}^{\infty} \rho^{N-1-\frac{q}{p-1}}\zeta_{\rho_2}(\rho)\mathrm{e}^{-\varepsilon(\rho-\rho_2)^2}\mathrm{d}\rho\Big)^{\frac{p-1}{p-m}} \big(4(N+2)\varepsilon\big)^{\frac{1}{p-m}} \\ &= B_0 C_\varepsilon^{\frac{p-1}{p-m}} \varepsilon^{-\frac{N(p-1)}{2(p-m)}+\frac{q}{2(p-m)}+\frac{1}{p-m}} \Big(\int_{\sqrt{\varepsilon}\rho_2}^{\infty} \tau^{N-1-\frac{q}{p-1}}\zeta_{\rho_2}\big(\frac{\tau}{\sqrt{\varepsilon}}\big)\mathrm{e}^{-(\tau-\sqrt{\varepsilon}\rho_2)^2}\mathrm{d}\tau\Big)^{\frac{p-1}{p-m}} \end{aligned} \tag{2.11}$$

with $B_0 = C_0^{-\frac{1}{p-m}}(4(N+2))^{\frac{1}{p-m}}$. We have by (2.4), (2.5) and (2.7) that

$$C_\varepsilon \int_{\sqrt{\varepsilon}\rho_2}^{\infty} \tau^{N-1-\frac{2(a+\alpha)}{\alpha(1-m)+2}}\zeta_1(\tau)\mathrm{e}^{-\tau^2}\mathrm{d}\tau \geq C_1\varepsilon^{\frac{N}{2}}$$

$$C_\varepsilon^{\frac{p-1}{p-m}}\Big(\int_{\sqrt{\varepsilon}\rho_2}^{\infty} \tau^{N-1-\frac{q}{p-1}}\zeta_{\rho_2}\big(\frac{\tau}{\sqrt{\varepsilon}}\big)\mathrm{e}^{-(\tau-\sqrt{\varepsilon}\rho_2)^2}\mathrm{d}\tau\Big)^{\frac{p-1}{p-m}} \leq C_2\varepsilon^{\frac{N(p-1)}{2(p-m)}}$$

with some $C_1, C_2 > 0$ independent of ε. Since $q = \frac{2\alpha(p-1)}{\alpha(1-m)+2}$, we have

$$\varepsilon^{\frac{a+\alpha}{\alpha(1-m)+2}} \gg \varepsilon^{\frac{q}{2(p-m)}+\frac{1}{p-m}} \quad \text{as} \quad \varepsilon \to 0,$$

provided $0 < a < a^* = \frac{2}{p-m}$. If $a = a^*$, the right hand sides of (2.10) and (2.11) share the same order of ε. So, we can arrive at (2.9) by letting λ be large enough. If $a > a^*$, for any fixed ε, there exists $\lambda_\varepsilon > 0$, such that (2.9) holds whenever $\lambda > \lambda_\varepsilon$. $\square$

3. Global solutions

In this section, we will show that the solutions must be global and decay to zero as $t \to \infty$, if $a > a^*$ with λ small, where $u_0(x) = \lambda\varphi(x)$ with $\varphi(x) = |x|^\alpha\psi(x) \in \mathbb{I}^a$.

It suffices to consider the case of $a^* < a < \alpha m + n$. The conclusion for $a \geq \alpha m + n$ can be proved by comparison. Introduce the following auxiliary problem

$$\begin{cases} W_\tau = (W^m)_{\rho\rho} + \frac{N-1}{\rho}(W^m)_\rho, & (\rho, t) \in [0,\infty)\times(0,T), \\ W(\rho,0) = M\rho^{-b}, & \rho \geq 0 \end{cases} \tag{3.1}$$

with N defined in (2.2), $b = \frac{2(a+\alpha)}{\alpha(1-m)+2}$. The condition $p > p_c$ with $a^* < a < \alpha m + n$ implies $b < N$. It was known by [6] that the problem (3.1) admits the self-similar solution

$$W = \tau^{-\beta b} f(\rho\tau^{-\beta}), \quad \beta = \frac{1}{2-(1-m)b} = \frac{\alpha(1-m)+2}{4-2a(1-m)}, \tag{3.2}$$

with $f(\eta)$ satisfying

$$\begin{cases} (f^m)'' + \frac{N-1}{\eta}(f^m)' + \beta\eta f' + \beta b f = 0, \quad \eta > 0, \\ f'(0) = 0, \quad \lim_{\eta\to\infty} \eta^b f(\eta) = M. \end{cases}$$

It is easy to verify that there exists $M_0 > 0$ such that

$$\|\eta^{\frac{2\alpha}{\alpha(1-m)+2}} f(\eta)\|_{L^\infty} \le \|(\eta+1)^{-\frac{2a}{\alpha(1-m)+2}}\big((\eta+1)^b f\big)\|_{L^\infty} \le M_0. \tag{3.3}$$

Lemma 3.1 *Let $a^* < a < \alpha m + n$, $M_1, \tau_0 > 0$, and $(h(t), \tau(t))$ solve the ODE system*

$$\begin{cases} h'(t) = \lambda^{p-1} M_1^{p-1} (\lambda^{m-1}\tau(t) + \tau_0)^{-\frac{a(p-1)}{2-a(1-m)}} h^p(t), & t > 0, \\ \tau'(t) = h^{m-1}(t), & t > 0, \\ h(0) = 1, \quad \tau(0) = 0. \end{cases} \tag{3.4}$$

Then there is $\lambda_0 > 0$ such that $h(t)$ is bounded in $[0, \infty)$ whenever $\lambda \in [0, \lambda_0)$.

Proof The local existence and uniqueness of solutions to (3.4) follow from the standard ODE theory. Since $h'(t) > 0$ for $t > 0$, the solution $h(t) \ge 1$ can be continued whenever $h(t)$ is finite.

Suppose $h(s)$ exists in $[0, t]$. Then $\tau(t) = \int_0^t h^{m-1}(s)ds$. Since $0 < m \le 1$, $h'(t) > 0$, we have

$$h^{m-1}(t)t \le \tau(t) \le h^{m-1}(0)t = t.$$

By (3.4), we know

$$\begin{aligned} h'(t) &\le \lambda^{p-1} M_1^{p-1} (\lambda^{m-1} h^{m-1}(t) t + \tau_0)^{-\frac{a(p-1)}{2-a(1-m)}} h^p(t) \\ &= \lambda^{p-1} M_1^{p-1} (\lambda^{m-1} t + h^{1-m}(t)\tau_0)^{-\frac{a(p-1)}{2-a(1-m)}} h^{p+\frac{a(p-1)(1-m)}{2-a(1-m)}} \\ &\le \lambda^{p-1} M_1^{p-1} (\lambda^{m-1} t + \tau_0)^{-\frac{a(p-1)}{2-a(1-m)}} h^{p+\frac{a(p-1)(1-m)}{2-a(1-m)}}, \end{aligned} \tag{3.5}$$

for $h(t) \ge 1$. Due to $a > a^* = \frac{2}{p-m}$, integrate (3.5) to get

$$\begin{aligned} 1 - h^{1-p-\frac{a(p-1)(1-m)}{2-a(1-m)}} &\le \Big(p + \frac{a(p-1)(1-m)}{2-a(1-m)} - 1\Big) \lambda^{p-1} M_1^{p-1} \int_0^t (\lambda^{m-1}s + \tau_0)^{-\frac{a(p-1)}{2-a(1-m)}} \mathrm{d}s \\ &\le \Big(p + \frac{a(p-1)(1-m)}{2-a(1-m)} - 1\Big) \frac{\lambda^{p-m} M_1^{p-1} \tau_0^{-(\frac{a(p-1)}{2-a(1-m)}-1)}}{\frac{a(p-1)}{2-a(1-m)} - 1}. \end{aligned} \tag{3.6}$$

Let $\lambda_0 > 0$ satisfy

$$\Big(p + \frac{a(p-1)(1-m)}{2-a(1-m)} - 1\Big) \frac{\lambda_0^{p-m} M_1^{p-1} \tau_0^{-(\frac{a(p-1)}{2-a(1-m)}-1)}}{\frac{a(p-1)}{2-a(1-m)} - 1} = 1.$$

And define

$$C_\lambda = \Big(p + \frac{a(p-1)(1-m)}{2-a(1-m)} - 1\Big) \frac{\lambda^{p-m} M_1^{p-1} \tau_0^{-(\frac{a(p-1)}{2-a(1-m)})-1}}{\frac{a(p-1)}{2-a(1-m)} - 1}, \tag{3.7}$$

$$h_\lambda = \Big(\frac{1}{1 - C_\lambda}\Big)^{\frac{1}{p+\frac{a(p-1)(1-m)}{2-a(1-m)}-1}} \tag{3.8}$$

for $\lambda \in (0, \lambda_0)$. It follows from (3.6)–(3.8) that $h(t) \le h_\lambda$. $\Box$

Proof of Theorem 2 Since $\varphi(x) = |x|^\alpha \psi(x) \in \mathbb{I}^a$, we get $\psi(x) \in \mathbb{I}^{a+\alpha}$. There is $D > 0$ such that

$$\psi(x) \le D(1+|x|)^{-(a+\alpha)} \quad \text{for all } x \in \mathbb{R}^n.$$

Without loss of generality, assume $\psi(x)$ is radial with $r = |x|$. Let M in (3.1) be large so that $M > D\big(\frac{2}{\alpha(1-m)+2}\big)^b$. With ρ defined in (2.1), we have

$$W(\rho(r), 0) = M\rho^{-b} = M\big(\frac{2}{\alpha(1-m)+2}\big)^{-b} r^{-(a+\alpha)} > D(1+r)^{-(a+\alpha)} \ge \psi(r).$$

So, there is $\tau_0 \in (0,1)$ such that $\psi(r) < W(\rho(r), \tau_0)$. Denote $\zeta(\rho, \tau) = \lambda W(\rho, \lambda^{m-1}\tau + \tau_0)$ with $\lambda > 0$. Then ζ satisfies

$$\begin{cases} \zeta_\tau = (\zeta^m)_{\rho\rho} + \frac{N-1}{\rho}(\zeta^m)_\rho, & (\rho, t) \in [0,\infty) \times (0, T), \\ \zeta(x, 0) = \lambda W(\rho, \tau_0), & \rho \ge 0. \end{cases}$$

Set $\bar u = r^\alpha h(t) \zeta(\rho(r), \tau(t))$, with $(h(t), \tau(t))$ solving (3.4). It is easy to verify that

$$\begin{aligned} &\bar u_t - (\bar u^m)_{rr} - \frac{n-1}{r}(\bar u^m)_r + V(r)\bar u^m - \bar u^p \ge \bar u_t - (\bar u^m)_{rr} - \frac{n-1}{r}(\bar u^m)_r + \frac{\omega}{r^2}\bar u^m - \bar u^p \\ &\quad = r^\alpha \zeta(\rho, \tau)(h'(t) - r^{\alpha(p-1)} h^p(t) \zeta^{p-1}(\rho, \tau)), \\ &\bar u(r, 0) = r^\alpha \zeta(\rho(r), 0) = \lambda r^\alpha W\big(\rho(r), \tau_0\big) \ge \lambda r^\alpha \psi(r) = \lambda \varphi(x). \end{aligned}$$

To check $\bar u$ is a supersolution of (1.1), it suffices to show

$$h'(t) \ge (r^\alpha \zeta(\rho, \tau))^{p-1} h^p(t). \tag{3.9}$$

Actually, by (3.2) and (3.3),

$$\begin{aligned} &\|r^\alpha \zeta(\rho, \tau)\|_{L^\infty} \\ &\quad = \lambda\big(\frac{\alpha(1-m)+2}{2}\big)^{\frac{2\alpha}{\alpha(1-m)+2}} \Big\|\rho^{\frac{2\alpha}{\alpha(1-m)+2}} (\lambda^{m-1}\tau(t) + \tau_0)^{-\beta b} f(\rho(\lambda^{m-1}\tau(t) + \tau_0)^{-\beta})\Big\|_{L^\infty} \\ &\quad = \lambda\big(\frac{\alpha(1-m)+2}{2}\big)^{\frac{2\alpha}{\alpha(1-m)+2}} (\lambda^{m-1}\tau(t) + \tau_0)^{-\beta b + \frac{2\alpha\beta}{\alpha(1-m)+2}} \Big\|\eta^{\frac{2\alpha}{\alpha(1-m)+2}} f(\eta)\Big\|_{L^\infty} \\ &\quad \le \lambda M_1 (\lambda^{m-1}\tau(t) + \tau_0)^{-\frac{a}{2-a(1-m)}}, \end{aligned}$$

with $b = \frac{2(a+\alpha)}{\alpha(1-m)+2}$, $\beta = \frac{\alpha(1-m)+2}{4-2a(1-m)}$, $M_1 = M_0\big(\frac{\alpha(1-m)+2}{2}\big)^{\frac{2\alpha}{\alpha(1-m)+2}}$. By Lemma 3.1, there exists $\lambda_0 > 0$ such that (3.9) is true whenever $\lambda < \lambda_0$. Consequently, $\|u(r,t)\|_{L^\infty} \le \|\bar u(r,t)\|_{L^\infty} \le h_\lambda \|r^\alpha \zeta(\rho, \tau)\|_{L^\infty} \le C t^{-\frac{a}{2-a(1-m)}}$ for all $t > 0$ with $C = M_1(h_\lambda \lambda)^{\frac{2}{2-a(1-m)}}$. $\Box$

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